Power Accuracy in the Lattice Calculations of Parton Distributions

Rui Zhang （张睿）

University of Maryland

第二届中国格点量子色动力学研讨会

上海交通大学， 2022年10月9日

Collaborators： Xiangdong Ji, Yushan Su, Jack Holligan

Outline

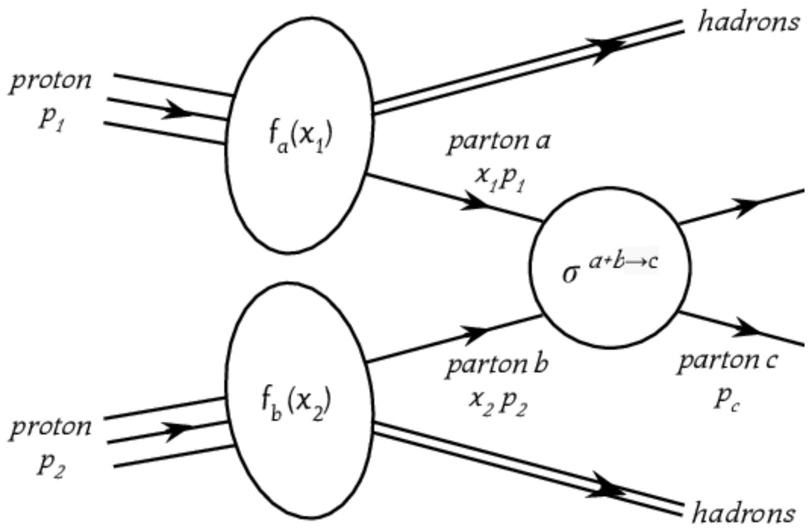
- ❑Calculating Parton Distribution Functions on Lattice
  - ❑Parton Distribution Functions
  - ❑Large Momentum Effective Theory and quasi-PDFs
- ❑Power Accuracy in the PDF Extraction from quasi-PDFs
  - ❑Renormalization of quasi-PDFs
  - ❑Renormalons in quasi-PDFs
  - ❑Fixed Order Extraction of Renormalons
- ❑Leading Renormalon Resummation
- ❑Conclusions

Parton distribution functions

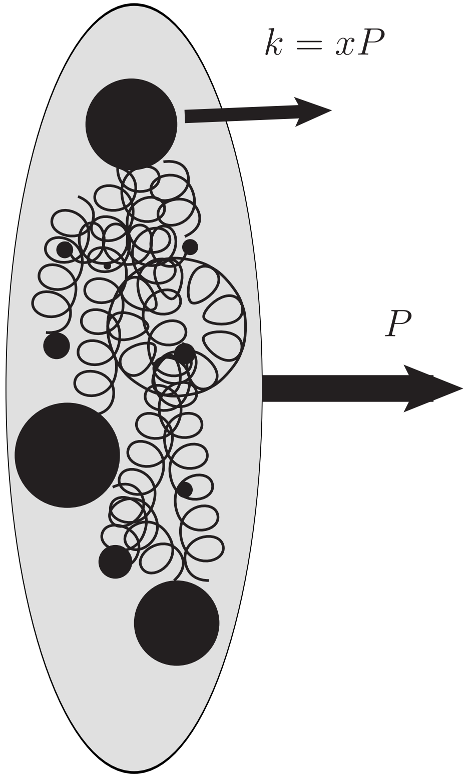
- The cross section of hadron scattering can be factorized into

$$\sigma_{hh'\to c+X}(s) = \int dx_1 dx_2 f_{a/h}(x_1) f_{b/h'}(x_2)\sigma_{ab\to c}(x_1 x_2 s)$$

-



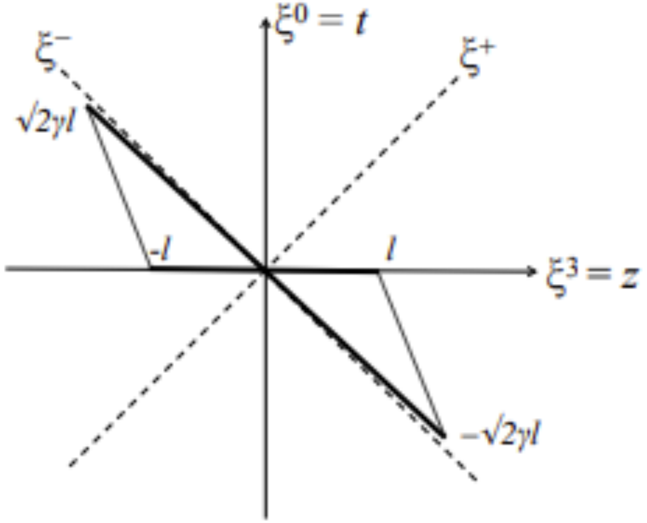
- Physical meaning: $f_{j/h}(x)$ number density to find parton j out of hadron h with momentum fraction x
- A universal distribution function in various processes on the hadron colliders
- Non-perturbative



Definition: $$f_{q/h}(x,\mu) = \int \frac{d\xi^-}{4\pi} e^{-ix\xi^-\cdot P^+} \langle N|\bar{q}(\xi^-)\gamma^+ W(\xi^-,0) q(0)|N\rangle$$

Quasi-PDF

- PDF defined on the lightcone: $f_{q/h}(x,\mu) = \int \frac{d\xi^-}{4\pi} e^{-ix\xi^- \cdot P^+} \langle N|\bar{q}(\xi^-)\gamma^+ W(\xi^-,0) q(0)|N\rangle$
- Euclidean nature of lattice: lightcone direction becomes the origin
- Large Momentum Effective Theory: Calculate the momentum distribution at finite P, then match to the light cone. Ji,PRL.2013
- Quasi-PDF
  - $\tilde{q}(x) = \int \frac{dz}{2\pi} e^{-ixP_z z} \langle P|\bar{\psi}(z)W(z,0)\gamma^t \bar{\psi}(0)|P\rangle$
- Same IR behavior, different UV behavior
- Matching to lightcone PDF:
  - $\tilde{q}(x,P_z) = \int_{-1}^{1} \frac{dy}{|y|} q(y) C(x,y,P_z,\mu) + O\left(\frac{\Lambda_{QCD}^2}{x^2 P_z^2}, \frac{\Lambda_{QCD}^2}{(1-x)^2 P_z^2}\right)$?



Xiong, et al., PRD (2014) , Ma, et al., PRD (2018)
Izubuchi, et al., PRD (2018) , Ji, et al., RMP (2021)

Lattice calculation of PDF

1. Calculate the matrix element $h^B = \langle P \left| \bar{\psi}(z) W(z,0) \gamma^t \bar{\psi}(0) \right| P \rangle$
2. Renormalization $h^R(a, z, P_z) = Z(a, z) h^B(a, z, P_z)$
3. Extrapolation to continuum $h^R(a, z, P_z) \rightarrow h^R(z, P_z)$
4. Matching to lightcone PDF
   - $M^{\overline{MS}}(\mu, z, P_z) = \int d\nu \, C(\nu, P_z, \mu) h^R(\nu z, P_z)$
5. Inverse Fourier Transformation
   - $f(\mu, x) = \int \frac{dz}{2\pi} e^{-ixP_z z} M^{\overline{MS}}(\mu, z, P_z)$

exchangeable

Renormalization of quasi-PDF on lattice

- Wilson line $W(z,0)$ self energy: linear divergence $e^{-\delta m\, z}$
  - Mass counter term: $\delta m = \frac{1}{a}\sum \alpha^{n+1} r_n = \frac{m_{-1}}{a} ? = \frac{m_{-1}}{a} + m_0$ ➡Ambiguity of $O(\Lambda_{QCD})$
- Hybrid scheme: only subtract linear divergence at long distance
  - $z < z_s$: Ratio scheme/other schemes
  - $z > z_s$: $Z(a,z)= Z(a,z_s)e^{-\delta m|z-z_s|}$
  - Scheme conversion to $\overline{MS}$ is always perturbative

X. Ji, et.al, NPB 2021

- Self-Renormalization:
  - Extract $Z(a,z)$ from $P = 0$ matrix elements
  - Obtain the m_0 and m_{-1} from fit to lattice data
  - Apply the same parameters to renormalize large P data

LPC, NPB 2021

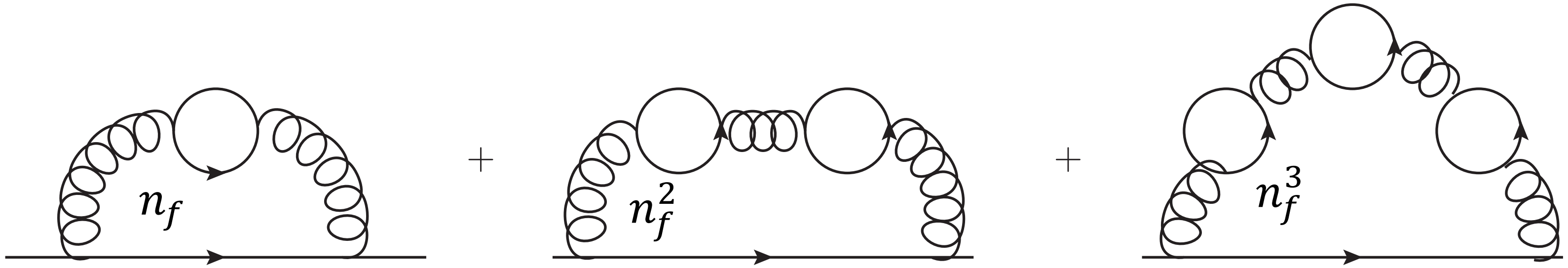
- The linear divergence in the renormalization constant is ambiguous.

Renormalon ambiguity in pert series

- If we expand the heavy quark on-shell mass in perturbation series :
  - $m_{OS} = m_{\overline{MS}} + \delta m = m_{\overline{MS}} + \sum_{i=0} \alpha^{i+1}(\mu) r_i,\ r_i \sim \mu N_m i!\, b^i$
  - The perturbation series diverge for any $\alpha(\mu)$, no well-defined sum
  - Borel Sum: $\int_0^\infty du\ e^{-u/\alpha} \sum_i \frac{r_i u^i}{i!} = \int_0^\infty du\ e^{-u/\alpha} \frac{N_m \mu}{1-ub}$ ⟹ pole at $u = \frac{1}{b}$: Renormalon
- Define a specific summation scheme (an integration path in the Borel plane): integrate above or below the pole, or combine them.

- Different summation schemes may differ by an ambiguity $O(\Lambda_{QCD})$.

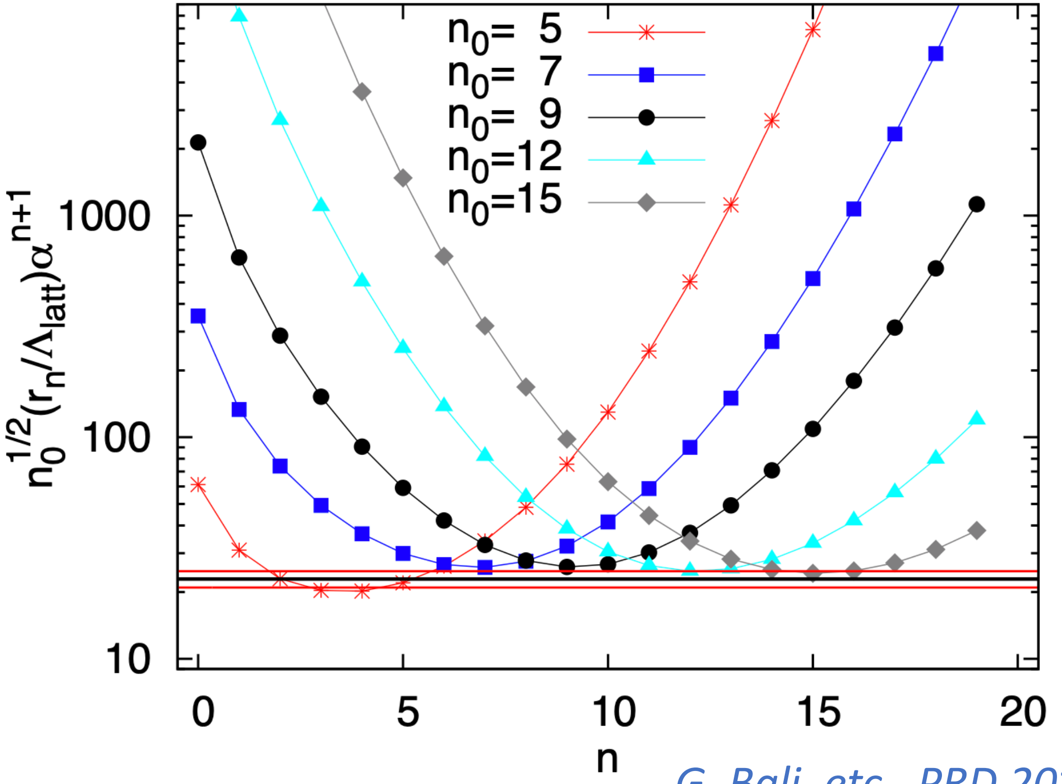
An ad hoc argument: large n_f (β_0) limit



- Bubble chain diagrams (t'Hooft):

$$m_{OS} = m_{\overline{MS}} + \alpha(\mu) \sum_n n! \left(\frac{\beta_0 \alpha(\mu)}{2\pi}\right)^n$$

- Verified by lattice simulations (quenched):
  - Same growth as predicted
  - Similar size of normalization constants



G. Bali, etc., PRD 2013

Relating lattice and $\overline{MS}$ calculations

- $\mathrm{P}_z = 0$ matrix elements (Wilson Coefficient C_0):
  - $M^{\overline{MS}}(z,\mu) = 1 + \alpha^{\overline{MS}}(\mu)(c^{(1)}_{\overline{MS},0} + c^{(1)}_{\overline{MS},1} \ln z\mu) + \alpha^2_{\overline{MS}}(\mu)\left(c^{(2)}_{\overline{MS},0} + c^{(2)}_{\overline{MS},1} \ln z\mu + c^{(2)}_{\overline{MS},2} \ln^2 z\mu\right) + \cdots$
  - $M^{lat}(z,\mu) = e^{-\delta m\, z}[1 + \alpha_{lat}(a)(c^{(1)}_{lat,0} + c^{(1)}_{lat,1} \ln\frac{z}{a}) + \alpha^2_{lat}(a)\left(c^{(2)}_{lat,0} + c^{(2)}_{lat,1} \ln\frac{z}{a} + c^{(2)}_{lat,2} \ln^2\frac{z}{a}\right) + \cdots]$
- Log terms become large when $z \to \infty$ or $z \to 0$
  - Renormalization group resummation (RGR)
- $M(\alpha(\mu), z) = M(\alpha(z^{-1}), z)\mathrm{e}^{I(\mu)}\ \mathrm{e}^{-I(z^{-1})}$
- RGR not only improves the convergence of pert series, but also factors out the renormalization scale and physical scale dependences

Relating lattice and $\overline{MS}$ calculations

- After RGR:
  - $M^{lat}(z,a) = e^{-\delta m\, z} f^{lat}(z, z^{-1}) e^{-I^{lat}(z^{-1})}\, e^{I^{lat}(a^{-1})}$
  - $M^{\overline{MS}}(z,\mu) = M^{\overline{MS}}(z, z^{-1}) e^{-I^{\overline{MS}}(z^{-1})}\, e^{I^{\overline{MS}}(\mu)}$
- $M^{\overline{MS}}(z, z^{-1}) e^{-I^{\overline{MS}}(z^{-1})}$ describes the dependence on z, thus is physical, and should be renormalization scale & scheme independent
- $M^{lat}(z,a) = e^{-\delta m\, z} e^{I^{lat}(a^{-1})} e^{I^{\overline{MS}}(\mu)} M^{\overline{MS}}(z, z^{-1}) e^{I^{\overline{MS}}(\mu) - I^{\overline{MS}}(z^{-1})}$

  Renormalization constants for all P_z

- Renormalons: introduced by renormalization, cancelled by matching
  - Different summation schemes introduces $O(z\Lambda_{QCD})$ corrections

Leading Power Accuracy

- $\tilde{q}(x, P_z) = \int_{-1}^{1} \frac{dy}{|y|} q(y) C(x, y, P_z, \mu) + O(\frac{\Lambda_{QCD}^2}{x^2 P_z^2}, \frac{\Lambda_{QCD}^2}{(1-x)^2 P_z^2})$?
- Different summation schemes of renormalization and perturbative matching introduces $\Delta m_0 z$ correction, and in momentum space:
  - $\tilde{q}(x, P_z) = \int_{-1}^{1} \frac{dy}{|y|} q(y) C(x, y, P_z, \mu) + O(\frac{\Delta m_0}{x|P_z|}) + O(\frac{\Lambda_{QCD}^2}{x^2 P_z^2}, \frac{\Lambda_{QCD}^2}{(1-x)^2 P_z^2})$
- To properly deal with the renormalon we need either:
  1. Calculate the series to very high orders (difficult on lattice, impossible in $\overline{MS}$)
  2. Use a phenomenological term $e^{-m_0^{eff} z}$ to account for summation scheme difference
- Approach 2 needs the renormalized ME matches the $\overline{MS}$ up to linear z:

$$\ln\left[M^{lat}(z, a) e^{m_{-1}\frac{z}{a}} e^{I^{lat}(a^{-1})}\right] - \ln\left[M^{\overline{MS}}(z) e^{-I^{\overline{MS}}(z^{-1})}\right] = m_0^{eff} z + O(z^2 \Lambda_{QCD}^2)$$

- m_0^{eff} should be stable at moderate z, corrections up to 30% at 0.2fm

Fixed-order extraction

- In lattice with very small a, we can define a sum to the minimal term, and fit to data.
- Extract the z dependence from lattice data: $e^{-\frac{m_{-1}}{a}z}$ $\quad e^{-m_0^{eff}z}M^{\overline{MS}}(z)e^{-I^{\overline{MS}}(z^{-1})}$

$$m_{\rm NLO}^{\rm lat}(z,a) = \exp\left[\frac{kz}{a\ln[a\Lambda_{\rm QCD}]} + g(z)\right]$$

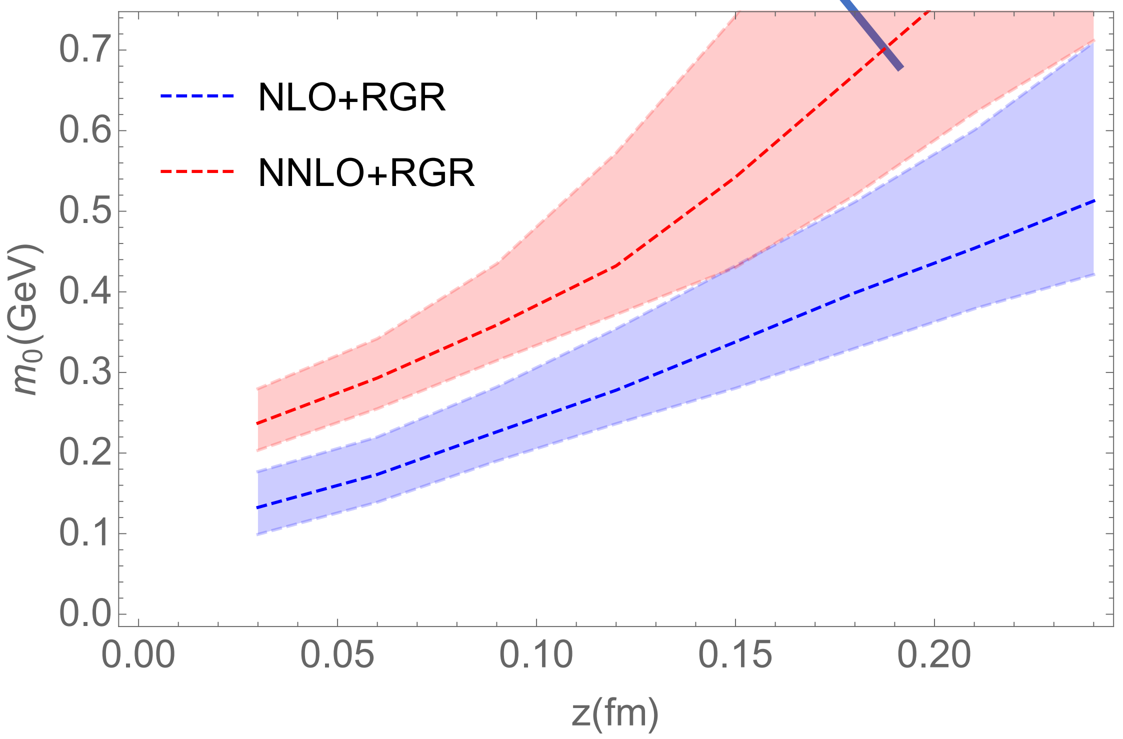
$$e^{I^{lat}(a^{-1})}$$

$$\exp\left[\frac{3C_F}{b_0}\ln[\ln[1/(a\Lambda_{\rm QCD})]] + \ln[1+\frac{d}{\ln(a\Lambda_{\rm QCD})}] + f_1(z)a\right]$$

- The m_0^{eff} shows no convergence
- Not linear z dependence

Error band: varying starting scale of RGR $0.8z^{-1}\sim 1.2z^{-1}$

Data from LPC, NPB 2021

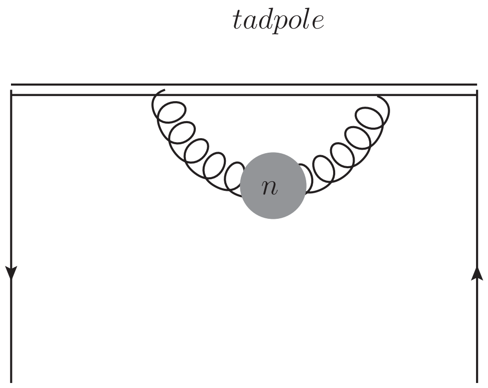


Problems and improvements

- Renormalon shows up at different orders for different z (higher order for smaller $\alpha(z^{-1})$), the fixed-order truncation introduces extra z dependence.
- What about separating the leading renormalon contributions, and resum them?
- z value won't affect the renormalon part too much
- The remaining part is now free of leading renormalon, thus must converge better (or diverge at higher orders)

Large β_0 renormalon poles in Quasi- PDF

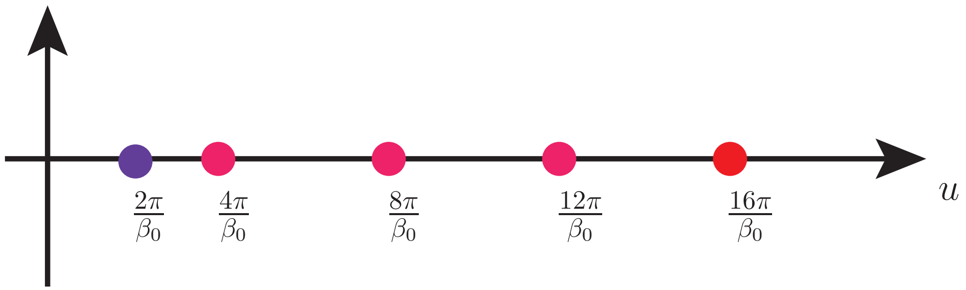
[V. Braun, et. al, PRD 2019](#)



- $u = \frac{4\pi}{\beta_0} w = \frac{2\pi}{\beta_0}$
  - From $\frac{1}{1-2w}$ term, tadpole diagram only
  - Linear divergence
  - Cancel the one in mass counter term

$$M_{\rm tp}(z,\mu)|_{\rm PV} = \int_{0,\rm PV}^{\infty} dw e^{-4\pi w/\alpha(\mu)\beta_0} \frac{2C_F}{\beta_0} \times \left(\frac{\Gamma(1-w) e^{\frac{5}{3}w} (z^2\mu^2/4)^w}{(1-2w)\Gamma(1+w)} - 1 \right) /w,$$

- $u = \frac{4\pi}{\beta_0}, \frac{8\pi}{\beta_0} \ldots$
  - From $\Gamma(1-w)$, all bubble chain diagrams
  - Power corrections
  - Cancel the higher twist effects

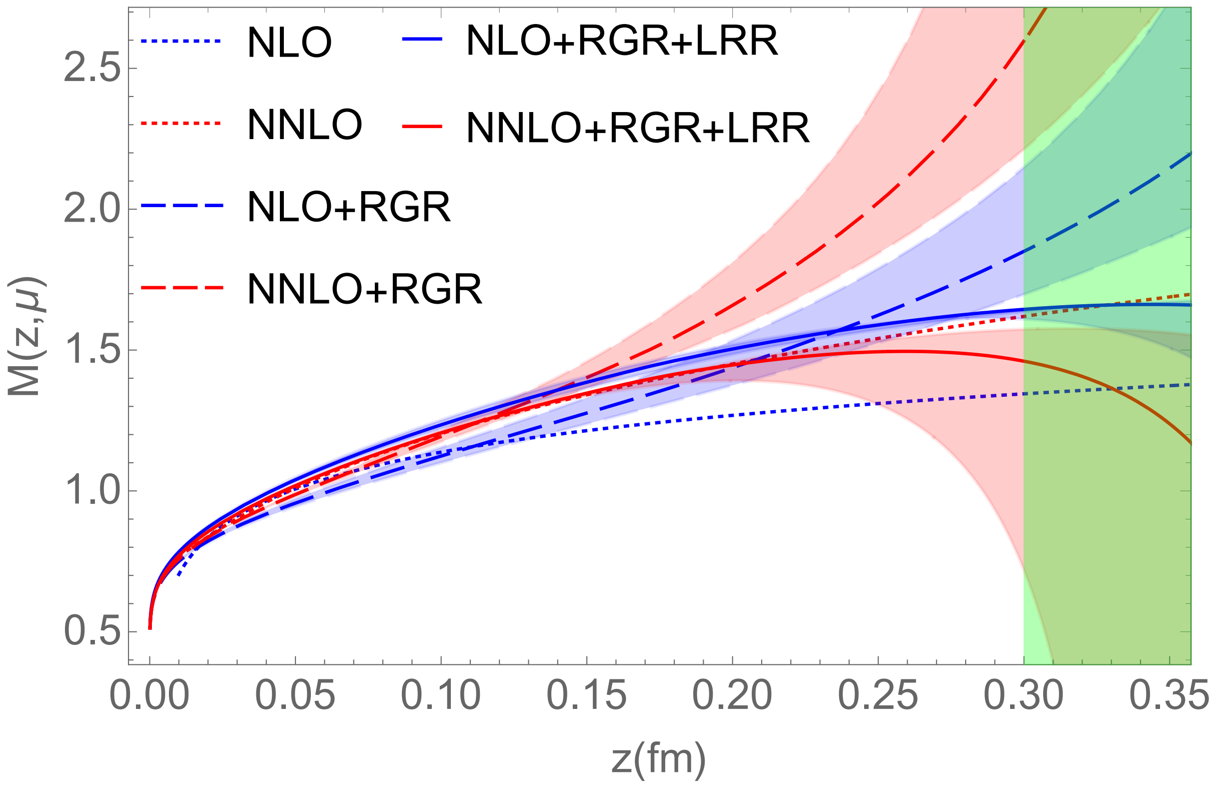


Leading Renormalon Resummation in $\overline{MS}$

- $M_{LRR}^{\overline{MS}}(z) = M_{tp}(z)|_{PV} + \sum M^{\overline{MS},(i)}(z) - M_{tp}^{(i)}(z)$
- We subtract $M_{tp}^{(i)}(z)$ from $M^{\overline{MS},(i)}(z)$ at fixed order to avoid overcounting such a contribution
- The subtracted series $M^{\overline{MS},(i)}(z) - M_{tp}^{(i)}(z)$ is now free of the leading renormalon thus converges better
- $M_{tp}(z)|_{PV}$ is insensitive to z

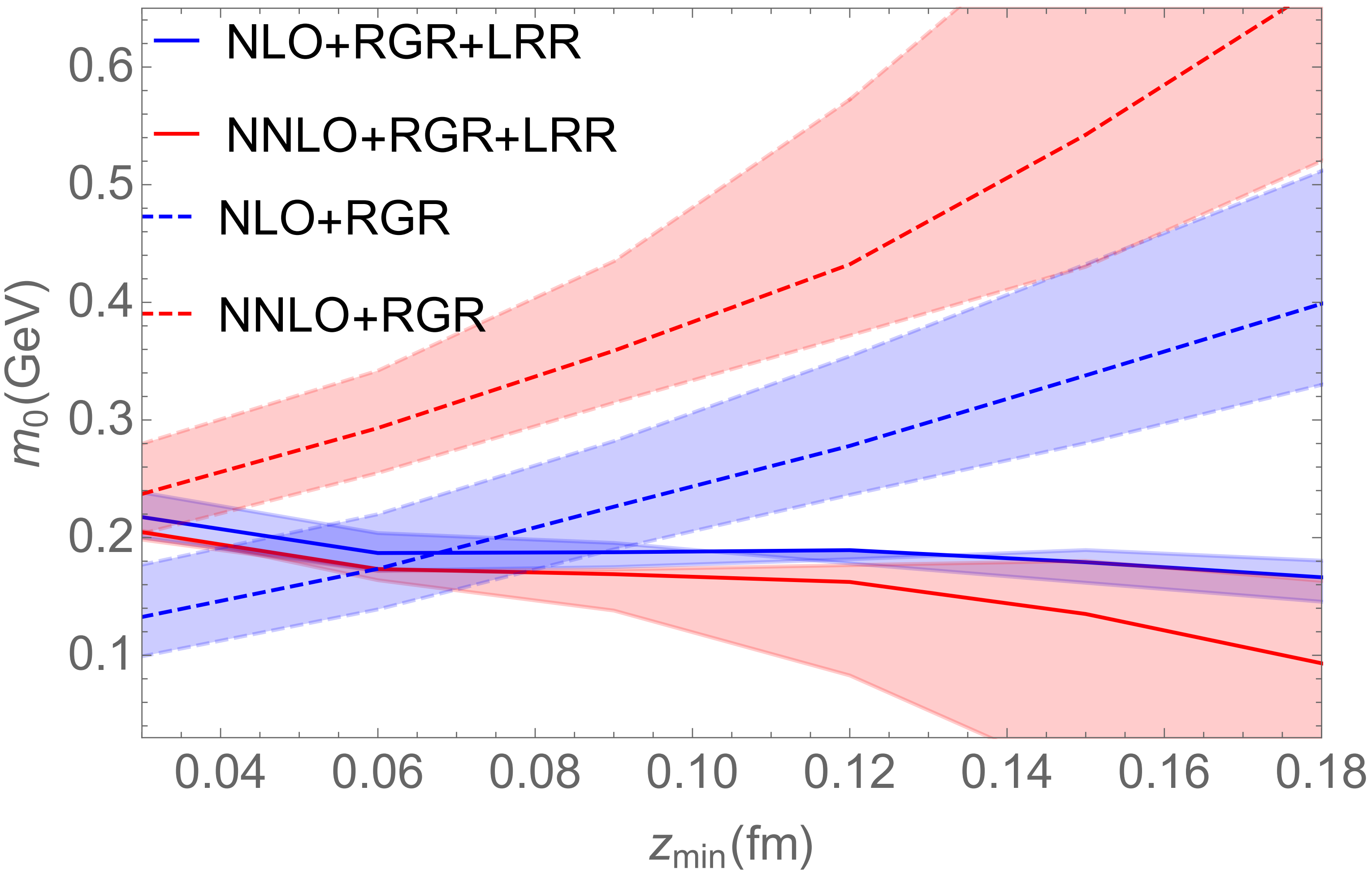
LRR Wilson Coefficients

- The P=0 matrix elements (C_0) after LRR+RGR:
- $M_{LRR}^{\overline{MS}}(z) = M_{tp}(z)|_{PV} + \sum M^{\overline{MS},(i)}(z) - M_{tp}^{(i)}(z)$
- The LRR+RGR results show milder z dependence and better convergence.



Comparison with the naive extraction

- A clear plateau near $z = 0.1fm$
- Better convergence when including higher order terms
- small z: discretization
- Large z: Landau pole



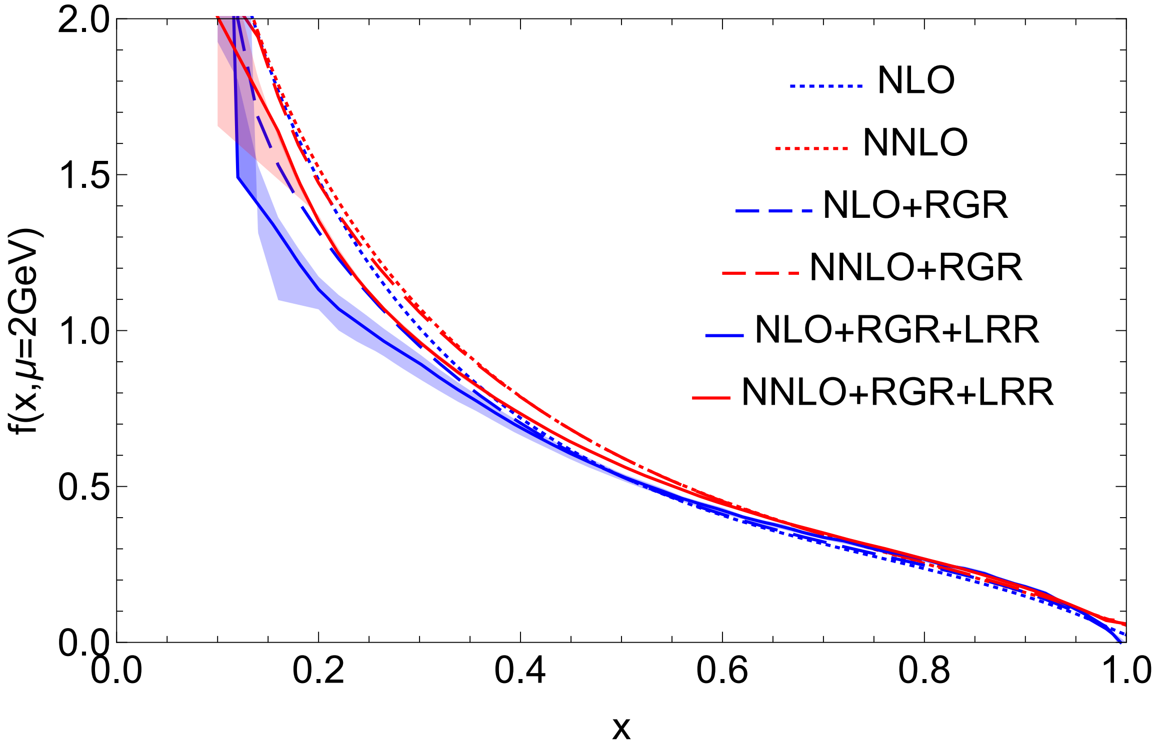
Consistent renormalization and matching

1. Renormalize the lattice bare matrix element with fitted m_0
2. Modify the matching kernel correspondingly
   1. Modification to the Wilson coefficients:

$$\Delta C_n^{\mathrm{LRR}}(z,\mu) = M_{\mathrm{tp}}(z,\mu)|_{\mathrm{PV}} - \sum_i M_{\mathrm{tp}}^{(i)}(z,\mu).$$

   2. Modification to the matching kernel:
      - RGR implemented in momentum space
        arxiv: [2209.01236](https://arxiv.org/abs/2209.01236)

$$\Delta C^{\mathrm{LRR}}(x/y,\mu,P_z) = \int \frac{p_z dz}{2\pi} e^{i(x/y-1)zp_z}$$
$$e^{(\mathcal{I}(\mu)-\mathcal{I}(z^{-1}))}\left[M_{\mathrm{tp}}(z,z^{-1})|_{\mathrm{PV}} - \sum_i (M_{\mathrm{tp}}^{(i)}(z,z^{-1}))\right].$$



Conclusion

- The renormalon ambiguity exists in the Wilson line self energy as well as the perturbative matching
- Different summation schemes lead to linear power corrections
- Fixed-order extraction of renormalon effect fails because of the fixed order truncation of renormalon series introduces extra z dependence
- Resumming the leading renormalons in the large β_0 limit significantly improves the stability and convergence of extracting m_0^{eff}
- A corresponding modification to the perturbative matching guarantees leading power accuracy

THANK YOU!