Pure gauge theory on the lattice

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第三届格点数值量子场论训练营
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References

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Introduction to gauge theories

Lagrangian formalism

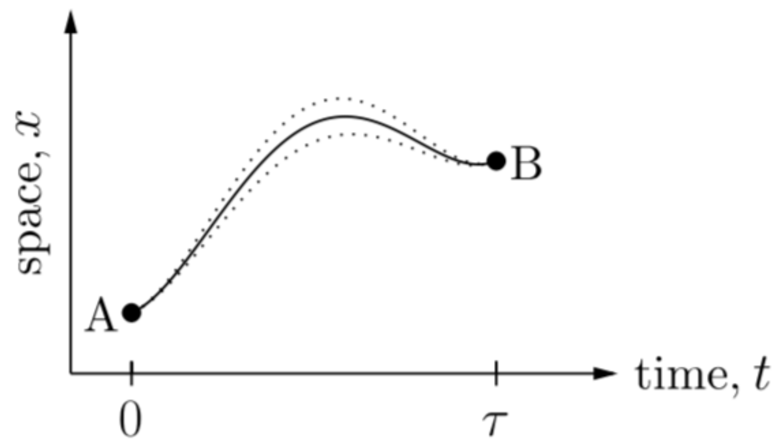
- In classical mechanics, for a system consisting of a set of point masses, we have

$$L(q_i, \dot{q}_i) = T - V, \qquad S = \int dt\, L$$

$$\text{Example}: \quad L = \frac{1}{2} m\dot{q}^2 - V(q), \qquad S = \int_{t_1}^{t_2} dt\, L$$

- Classical trajectory is determined by the requirement that the action is stationary

$$\delta S = 0$$



- Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$

$$\text{Example}: \quad m\ddot{q} = -\,\partial V/\partial q$$

Lagrangian formalism

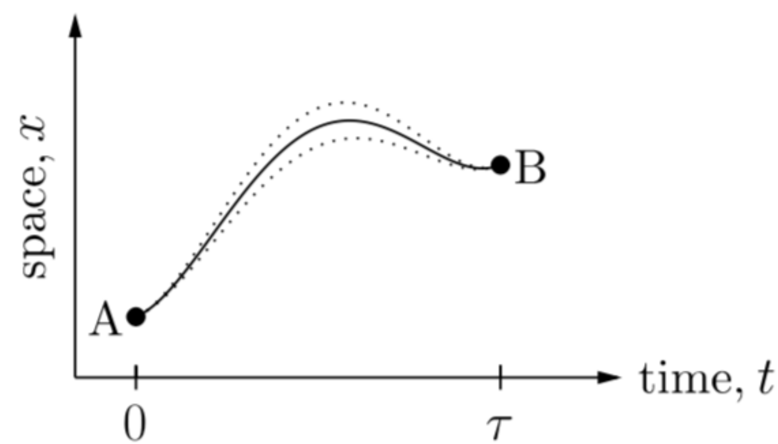
- In classical mechanics, for a system consisting of a set of point masses, we have

$$L(q_i, \dot{q}_i) = T - V, \qquad S = \int dt\, L$$

Homework: What if the Lagrangian contains higher-order derivatives?

- Classical trajectory is determined by the requirement that the action is stationary

$$\delta S = 0$$



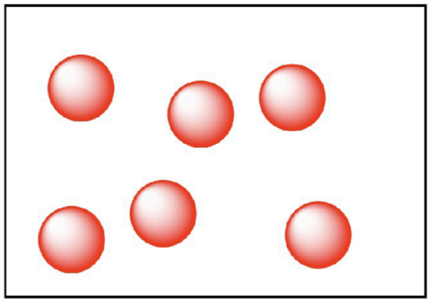
- Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$

Example : $m\ddot{q} = -\partial V/\partial q$

Lagrangian formalism

- Concept of classical fields:



$\rightarrow$

- For a continuum system with infinite # of dof's and Lorentz symmetry, we can do the replacement

$$q_i \to \phi(x), \qquad \dot{q}_i \to \partial_\mu \phi(x)$$

- The action has the form

$$S = \int dt\, L = \int d^4x \mathscr{L}(\phi, \partial_\mu \phi), \qquad L = \int d^3\overrightarrow{x} \mathscr{L}$$

- Equation of motion

$$\delta S = 0 \to \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi)} = \frac{\partial L}{\partial \phi}$$

Lagrangian formalism

- Example 1:

$$\mathscr{L} = \mathscr{T} - \mathscr{V} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - \frac{1}{2}m^2\phi^2$$

- Equation of motion

$$(\partial^\mu \partial_\mu + m^2)\phi(x) = 0$$

which is nothing but the relativistic energy-mass relation

- Example 2:

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4}\lambda\phi^4$$

the last term has the form of Higgs potential, and is important for spontaneous symmetry breaking

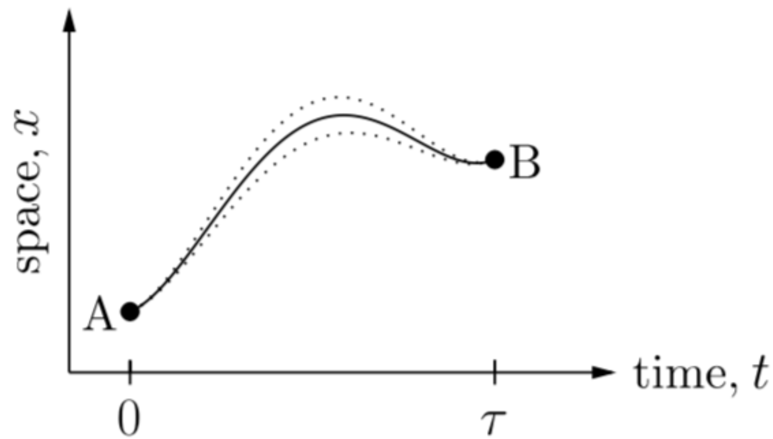
Lagrangian formalism

- Quantum theory (path integral perspective):

- In quantum mechanics, for a particle propagating from A to B all paths are allowed, and have to be summed up, but with a weight factor $e^{iS/\hbar}$

$$\langle q_B, t_B | q_A, t_A \rangle =$$

$$N \int \mathscr{D}q \exp\left[i \int_{t_A}^{t_B} L(q, \dot{q}) dt \right]$$



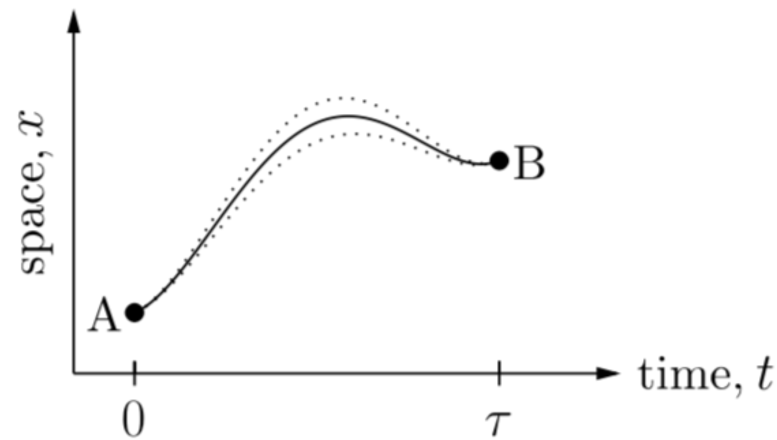
- Contribution of different paths cancels out except near the stationary phase leads to the classical trajectory

Lagrangian formalism

- Quantum theory (path integral perspective):

- In quantum field theory, for a particle (antiparticle) propagating from A to B all paths are allowed, and have to be summed up, but with a weight factor $e^{iS/\hbar}$

$$\langle T\phi(x_B)\phi(x_A)\rangle =$$
$$N\int \mathcal{D}\phi\, \phi(x_B)\phi(x_A)\, \exp\left[i\int d^4x \mathcal{L}(\phi)\right]$$



- This functional integral is complex and strongly oscillating, difficult to give it a satisfactory mathematical meaning

Euclidean formulation

- This can be resolved by going to imaginary time or to Euclidean spacetime

$$t \to -it_E, \qquad \exp[iS] \to \exp[-S_E]$$

- The Euclidean path integral becomes real and bounded from above, if the potential is bounded from below

- Numerical calculations and theoretical analysis become much easier, similarity with statistical physics

- Sufficient to extract most physical information, can also be analytically continued back to real time (Minkowski spacetime) if needed (for analytic calculations)

Euclidean formulation

- In analogy with statistical physics, physical observables are evaluated as

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}\phi \, O \, \exp\left[-S_E \right]$$

- The partition function

$$Z = \int \mathcal{D}\phi \, \exp\left[-S_E \right]$$

- Example 1:

$$S_E = \int d^4 x_E \left[\frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 \right]$$

- From here on, the discussions will be in Euclidean spacetime

Gauge theories: Quantum electrodynamics

- Electric and magnetic fields are described by a 4-vector

$$A^\mu = (\varphi, \overrightarrow{A})$$

- The field strength is

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

- A QED Lagrangian shall contain both electrons and photons, we can begin with

$$\mathscr{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi$$

- It leads to free EOMs for electrons and photons, the former is Dirac equation, the relativistic analogue of Schroedinger equation in quantum mechanics
- How does the interaction enter?

Gauge theories: Quantum electrodynamics

- The Lagrangian is invariant under a global symmetry transformation

$$\psi'(x) = e^{i\omega q}\psi(x), \quad \bar{\psi}'(x) = e^{-i\omega q}\bar{\psi}(x), \quad A'_\mu(x) = A_\mu(x)$$

with ω a constant and q the charge of the electron

- However, electron fields at different spacetime points shall be able to transform differently with

$$\omega = \omega(x)$$

then the global symmetry becomes local

- $m\bar{\psi}\psi$ is still invariant, but $\bar{\psi}\gamma^\mu\partial_\mu\psi$ is not. The Lagrangian can be made invariant if we replace

$$\partial_\mu\psi \rightarrow D_\mu\psi = (\partial_\mu - iqA_\mu)\psi \quad \text{with} \quad A'_\mu(x) = A_\mu(x) + \partial_\mu\omega(x)$$

Gauge theories: Quantum electrodynamics

- The invariant Lagrangian under local gauge symmetry transformation is

$$\mathscr{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \bar{\psi}\gamma^{\mu}(\partial_{\mu} - iqA_{\mu})\psi - m\bar{\psi}\psi$$

- Local gauge symmetry dictates interactions
- The field strength can also be written as

$$F^{\mu\nu} = D^{\mu}A^{\nu} - D^{\nu}A^{\mu}$$

- The gauge transformation phase factor

$$\Omega(x) = e^{i\omega(x)}$$

forms a group, the U(1) (1-dim. unitary) group. It has 1 dof, corresponds to 1 photon field

- We call this group an Abelian (commutative) group as

$$\Omega(x)\Omega(y) = \Omega(y)\Omega(x)$$

Gauge theories: Quantum electrodynamics

- The invariant Lagrangian under local gauge symmetry transformation is

$$\mathscr{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \bar{\psi}\gamma^{\mu}(\partial_{\mu} - iqA_{\mu})\psi - m\bar{\psi}\psi$$

Homework: Derive these equations by yourself.

- Lo
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Gauge theories: Quantum electrodynamics

- We can rewrite the gauge transformation as

$$\psi'(x) = \Omega(x)\psi(x), \quad \bar{\psi}'(x) = \Omega^*(x)\bar{\psi}(x), \quad A'_\mu(x) = A_\mu(x) + i\Omega(x)\partial_\mu\Omega^*(x)$$

- The covariant derivative acting on $\psi(x)$ transforms just like $\psi(x)$ itself

$$D'_\mu\psi'(x) = [\partial_\mu - iqA'_\mu(x)]\psi'(x) = \Omega(x)D_\mu\psi(x)$$

so that $\bar{\psi}'\gamma^\mu D'_\mu\psi' = \bar{\psi}\gamma^\mu D_\mu\psi$ is gauge invariant

- Natural to ask: What if the gauge transformation phase factor $\Omega(x)$ is not just a number?

Gauge theories: Quantum chromodynamics

- $\Omega(x)$ can be generalized to matrix-valued, which means we enlarge the symmetry group

- For example, the U(1) group in QED can be generalized to SU(3) (3-dim. special unitary), whose elements are 3x3 unitary matrices with determinant 1 ($3^2 - 1 = 8$ dofs)

$$\Omega(x) = e^{i\omega^k(x)t_k}$$

- $t_k,\ k = 1...8$ are a complete set of Hermitian traceless 3x3 matrices, also called generators of the group (in a given representation), correspond to 8 gauge fields - gluons
- Now $\Omega(x)\Omega(y) \neq \Omega(y)\Omega(x)$, it is called a non-Abelian group (Yang-Mills theory)

Gauge theories: Quantum chromodynamics

- A standard choice: $t_k = 1/2\lambda_k$ with (σ_i are Pauli matrices)

$$\lambda_i = \begin{pmatrix} \sigma_i & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ for } i = 1,2,3, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

- They satisfy

$$\mathrm{Tr}(t_k t_l) = \frac{1}{2}\delta_{kl}, \qquad [t_k, t_l] = i f_{klm} t_m$$

- f_{klm} are structure constants of the group, and are totally antisymmetric with respect to the interchange of any two indices

Gauge theories: Quantum chromodynamics

- Under the matrix-valued gauge transformation, the fermion field transforms as

$$\psi(x) \to \psi'(x) = \Omega(x)\psi(x), \quad \bar{\psi}(x) \to \bar{\psi}'(x) = \bar{\psi}(x)\Omega^\dagger(x)$$

- Which implies

$$\bar{\psi}(x)\gamma^\mu(\partial_\mu + igA_\mu)\psi(x) \to \bar{\psi}(x)\Omega^\dagger(x)\gamma^\mu(\partial_\mu + igA'_\mu)\Omega(x)\psi(x)$$

- For it to be invariant, we need

$$A'_\mu(x) = \Omega(x)A_\mu(x)\Omega^\dagger(x) + i(\partial_\mu\Omega(x))\Omega^\dagger(x)$$

- Analogous to QED, we can define the field strength

$$F_{\mu\nu}(x) = D_\mu A_\nu(x) - D_\nu A_\mu(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + ig[A_\mu(x), A_\nu(x)]$$

- And thus the gauge part of QCD Lagrangian

$$L_g = \frac{1}{2}\mathrm{Tr}[F^{\mu\nu}(x)F_{\mu\nu}(x)]$$

Gauge theories: Quantum chromodynamics

- The gauge field can be decomposed in terms of color components (T_i is an appropriate matrix representation)

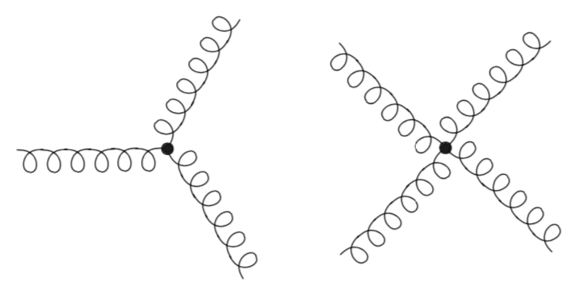
$$A_\mu(x) = \sum_{i=1}^{8} A_\mu^i(x) T_i$$

$$F_{\mu\nu}(x) = \sum_{i=1}^{8} F_{\mu\nu}^i(x) T_i, \quad F_{\mu\nu}^i(x) = \partial_\mu A_\nu^i(x) - \partial_\nu A_\mu^i(x) - f_{ijk} A_\mu^j(x) A_\nu^k(x)$$

- The Lagrangian becomes

$$L_g = \frac{1}{4} \sum_{i=1}^{8} F^{\mu\nu,i}(x) F_{\mu\nu}^i(x)$$

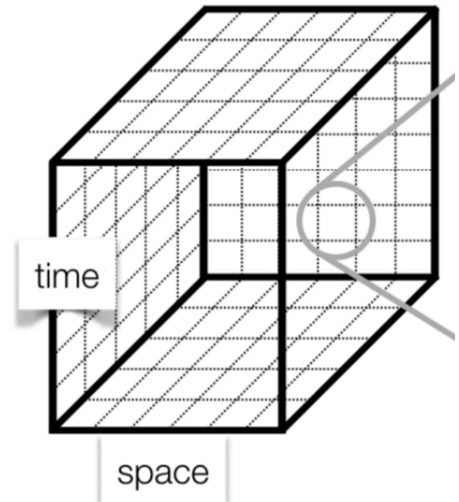
- It appears as 8 copies of QED gauge Lagrangian, but there is a crucial difference coming from $f_{ijk} A_\mu^j(x) A_\nu^k(x)$, it leads to cubic and quadratic gluon self-interactions



Gauge theories on the lattice

Gauge theories on the lattice

- Euclidean formulation of QFTs can be conveniently realized on a discrete lattice



A spacetime point is characterized by

$na = (n_1, n_2, n_3, n_4)a$

- We need to discretize the Lagrangian
- Discretized derivative

$$\partial_\mu \psi(x) \to \frac{\psi(n+\hat{\mu}) - \psi(n-\hat{\mu})}{2a}$$

$$\bar{\psi}(x)\partial_\mu \psi(x) \to \bar{\psi}(n) \sum_{\mu=1}^{4} \frac{\psi(n+\hat{\mu}) - \psi(n-\hat{\mu})}{2a}$$

- Again, local gauge invariance dictates the existence of gauge fields and their transformation properties

Gauge theories on the lattice

- Under discrete gauge transformation

$$\psi(n) \to \psi'(n) = \Omega(n)\psi(n), \quad \bar{\psi}(n) \to \bar{\psi}'(n) = \bar{\psi}(n)\Omega^\dagger(n)$$

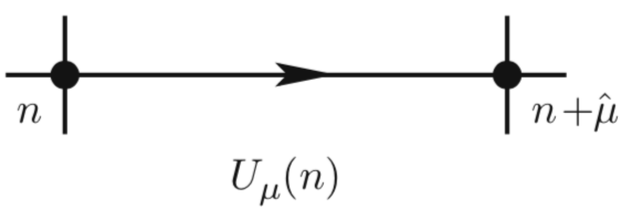
- We have

$$\bar{\psi}(n)\psi(n+\hat{\mu}) \to \bar{\psi}'(n)\psi'(n+\hat{\mu}) = \bar{\psi}(n)\Omega^\dagger(n)\Omega(n+\hat{\mu})\psi(n+\hat{\mu}) \neq \bar{\psi}(n)\psi(n+\hat{\mu})$$

- Gauge non-invariance can be compensated if we introduce a field $U_\mu(n)$ to form a combination $\bar{\psi}(n)U_\mu(n)\psi(n+\hat{\mu})$ and let it transform as

$$U_\mu(n) \to U'_\mu(n) = \Omega(n)U_\mu(n)\Omega^\dagger(n+\hat{\mu})$$

- $U_\mu(n)$ links the fermion fields at different spacetime points, and thus is called a link variable



$$U_{-\mu}(n+\hat{\mu}) = U_\mu^\dagger(n)$$

Gauge theories on the lattice

- Now we have a gauge-invariant expression

$$\bar{\psi}(x)\partial_\mu\psi(x) \rightarrow \bar{\psi}(n)\sum_{\mu=1}^{4}\frac{U_\mu(n)\psi(n+\hat{\mu}) - U_{-\mu}(n)\psi(n-\hat{\mu})}{2a}$$

- $U_\mu(n)$ plays the same role on the lattice as that the gauge field plays in the continuum, its continuum counterpart is the so-called gauge transporter

$$U(x,y) = \mathcal{P}\exp\left[i\int_{C_{x,y}} ds^\mu A_\mu(s)\right]$$

  which connects fermions at different spacetime points x, y to form a gauge-invariant combination

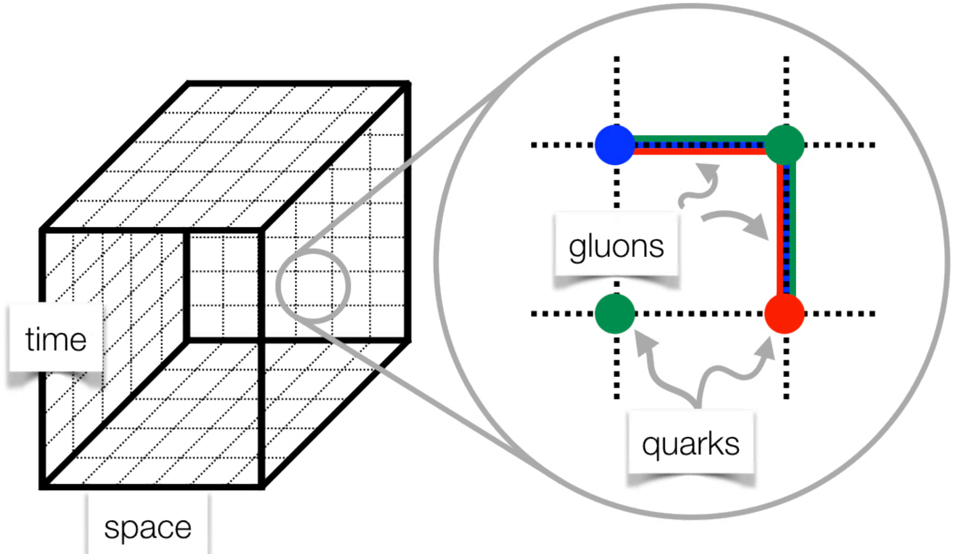
- Its discrete version is (accurate to O(a))

$$U_\mu(n) = e^{iaA_\mu(n)}$$

  and gives the continuum action in $a \rightarrow 0$ limit

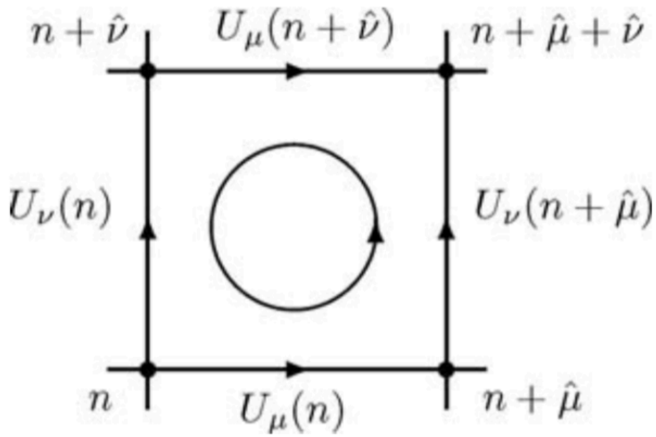
Gauge theories on the lattice

- Euclidean formulation of QFTs can be conveniently realized on a discrete lattice



Gauge theories on the lattice

- Gauge part of the action can be constructed from a closed loop formed by link variables, called a plaquette



$$U_{\mu\nu}(n) = U_\mu(n)U_\nu(n+\hat{\mu})U_{-\mu}(n+\hat{\mu}+\hat{\nu})U_{-\nu}(n+\hat{\nu}) = U_\mu(n)U_\nu(n+\hat{\mu})U_\mu^\dagger(n+\hat{\nu})U_\nu^\dagger(n)$$

- Then the gauge action can be constructed as (Wilson)

$$S_g = \frac{2}{g^2}\sum_n \sum_{\mu<\nu} \text{Re Tr}[1 - U_{\mu\nu}(n)]$$

- This is the first lattice formulation of QCD gauge action

Numerical Simulations

The path integral are evaluated approximately by N sample configurations $\{U_n\}$ with the distribution probability $\exp\{-S_g[U_n]\}$. An observable O is estimated as the average over the N configurations:

$$\langle O \rangle = \frac{1}{N}\sum_n O[U_n] + \mathcal{O}(1/\sqrt{N})$$

How to generate the configurations?

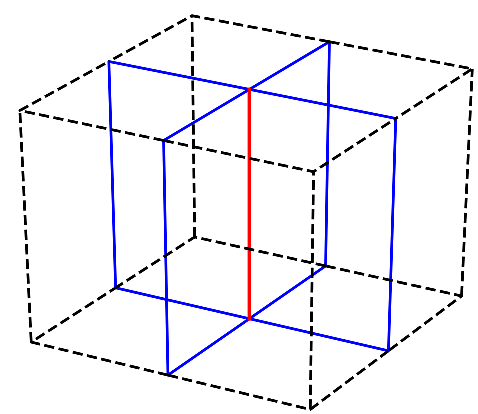
Numerical Simulations

Metropolis algorithm:

- Start from some configuration, choose a site n and a direction μ, change this link variable $U_\mu(n) \to U'_\mu(n)$.
- Calculate the change of the action

$$\Delta S = S(U'_\mu(n)) - S(U_\mu(n)) = -\frac{\beta}{3}\mathrm{Re}\,\mathrm{tr}[(\mathrm{U}'_\mu(\mathrm{n}) - \mathrm{U}_\mu(\mathrm{n}))\mathrm{A}]\,.$$

- Accept the new variable $U'_\mu(n)$ if $r < \exp(-\Delta S)$, where r is a random number uniformly distributed in [0,1).
- Repeat these step from the beginning.



Numerical Simulations

- **Heatbath**
  - The candidate link $U'_{\mu}(n)$ is chosen according to its local probability:

$$dP(U) = dU exp(\frac{\beta}{3} \mathrm{Re}\,\mathrm{tr[UA]})$$

  - More efficient than metropolis, but suffers critical slowing down.
- **Overrelaxation**
  - The candidate link is chosen such that the action is preserved. Such a change is always accepted.
  - Not ergodic, must be used in combination with an ergodic algorithm such as Heatbath.

Numerical Simulations

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Start from a slightly perturbed unit gauge

Random seeds.

Number of updates before equilibrium

Autocorrelation length

Numerical Simulations

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Gauge action type

β value

Boundary condition

Number of Overrelaxation sweeps

Number of Heatbath sweeps

Lattice size

Numerical Simulations