Wilson Loop

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Outline

1. Wilson Loop
    - What is Wilson Loop
    - Physical interpretation of the Wilson loop
    - Calculation
    - The static quark potential
    - fit the static potential
    - Autocorrelation
    - Error Estimate
    - exercise

Wilson Loop

- After we have produced the pure gauge configurations, we can make some physical measurements on these configurations.
- A Wilson loop $W_{\mathcal{L}}$ is made from four pieces, two so-called Wilson lines S(**m**,**n**,n_t), S(**m**,**n**,0), and two temporal transporters T(**n**,n_t),T(**m**,n_t).

Wilson Loop

- The Wilson line S(**m**,**n**,n_t) connects the two spatial points m and n along some path $\mathcal{C}_{m,n}$ with all link variables restricted to time argument n_t ,

$$S\left(\boldsymbol{m},\boldsymbol{n},n_t\right) = \prod_{(\boldsymbol{k},j)\in\mathcal{C}_{m,n}} U_j\left(\boldsymbol{k},n_t\right). \tag{1}$$

- The temporal transporter T(**n**,n_t) is a straight line of n_t link variables in time direction, all situated at spatial position **n**,

$$T\left(\boldsymbol{n},n_t\right) = \prod_{j=0}^{n_t-1} U_4(\boldsymbol{n},j). \tag{2}$$

Wilson Loop

- Attaching the four pieces to each other gives a closed loop $\mathcal{L}$,

$$\mathcal{L}: \quad (\boldsymbol{m}, n_t) \xrightarrow{S} (\boldsymbol{n}, n_t) \xrightarrow{T^\dagger} (\boldsymbol{n}, 0) \xrightarrow{S^\dagger} (\boldsymbol{m}, 0) \xrightarrow{T} (\boldsymbol{m}, n_t) \, . \tag{3}$$

- The Wilson loop WL is obtained by taking the trace,

$$W_{\mathcal{L}}[U] = \text{tr}\left[S\left(\boldsymbol{m}, \boldsymbol{n}, n_t\right) T\left(\boldsymbol{n}, n_t\right)^\dagger S(\boldsymbol{m}, \boldsymbol{n}, 0)^\dagger T\left(\boldsymbol{m}, n_t\right)\right] \tag{4}$$

$$= \text{tr}\left[\prod_{(k,\mu)\in\mathcal{L}} U_\mu(k)\right] \tag{5}$$

- If the piece of loop $\mathcal{C}_{m,n}$used in S(**m**,**n**,n_t) is a straight line we speak of a planar Wilson loop.Otherwise the Wilson loop is called nonplanar.

Wilson Loop

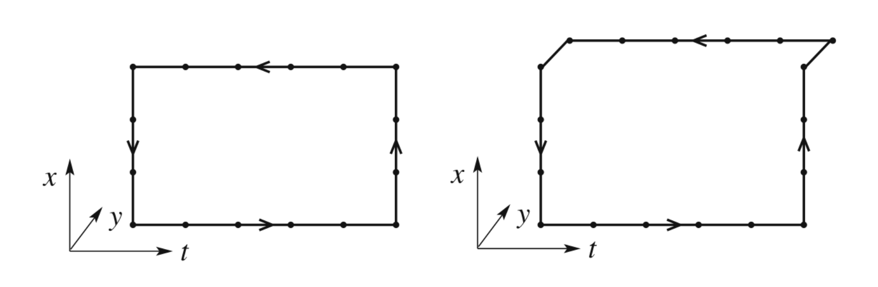

Figure 1: Examples for a planar (left-hand side plot) and a nonplanar (right-hand side) Wilson loop. The horizontal direction is time

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Physical interpretation of the Wilson loop

- Due to the gauge symmetry, We can choice a gauge for gauge field $A_\mu(x)$

$$A_4(x) = 0, \tag{6}$$

- i.e., temporal gauge. So the temporal transporters become trivial,

$$T\left(\boldsymbol{n}, n_t\right) = \prod_{j=0}^{n_t-1} U_4(\boldsymbol{n}, j) = \mathbb{1}, \tag{7}$$

- we obtain the following chain of identities

$$\langle W_{\mathcal{L}} \rangle = \langle W_{\mathcal{L}} \rangle_{\text{temp}} = \left\langle \text{tr}\left[S\left(\boldsymbol{m}, \boldsymbol{n}, n_t\right) S(\boldsymbol{m}, \boldsymbol{n}, 0)^{\dagger}\right]\right\rangle_{\text{temp}}, \tag{8}$$

- The temporal gauge used in (8) makes explicit that the Wilson loop is the correlator of two Wilson lines S(**m**,**n**,n_t) and S(**m**,**n**,0) situated at time slices n_t and 0. Thus we can interpret this correlator using the equation

$$\lim_{T\to\infty} \langle O_2(t) O_1(0) \rangle_T = \sum_n \left\langle 0 \left| \widehat{O}_2 \right| n \right\rangle \left\langle n \left| \widehat{O}_1 \right| 0 \right\rangle \mathrm{e}^{-tE_n}$$

- where we inserting the unit operator of the vectors of a complete orthonormal basis as

$$\not{1} = \sum_n |e_n\rangle \langle e_n|$$

in the left-hand side of the equation.

- The correlator behaves for large total temporal extent T of our Euclidean lattice as (a, b are summed)

$$\left\langle \text{tr}\left[S\left(\boldsymbol{m},\boldsymbol{n},n_t\right)S(\boldsymbol{m},\boldsymbol{n},0)^\dagger\right]\right\rangle_{\text{temp}} = \sum_k \left\langle 0 \left| \widehat{S}(\boldsymbol{m},\boldsymbol{n})_{ab} \right| k \right\rangle \tag{9}$$

$$\left\langle k \left| \widehat{S}(\boldsymbol{m},\boldsymbol{n})^\dagger_{ba} \right| 0 \right\rangle \mathrm{e}^{-tE_k} \tag{10}$$

where the Euclidean time argument t is related to n_t via $t = a\, n_t$ with a being the lattice spacing. The sum in (9) runs over all states $|k\rangle$ that have a non-vanishing overlap with $\hat{S}(\mathbf{m},\mathbf{n})^\dagger|0\rangle$

- the states $|k\rangle$ with non-vanishing overlap are states describing a static quark–antiquark pair located at spatial positions **m** and **n**.
- The energy E_1 is thus identified with the energy of the quark–antiquark pair, which is the static potential $V(r)$ at spatial quark separation r,

$$E_1 = V(r) \quad \text{with} \quad r = a|\boldsymbol{m} - \boldsymbol{n}| \tag{11}$$

- Combining (8), (9), and (11) we obtain

$$\langle W_{\mathcal{L}} \rangle \propto \mathrm{e}^{-tV(r)} \left(1 + \mathcal{O}\left(\mathrm{e}^{-t\Delta E}\right)\right) = \mathrm{e}^{-n_t a V(r)} \left(1 + \mathcal{O}\left(\mathrm{e}^{-n_t a \Delta E}\right)\right) \tag{12}$$

- Thus we find that we can calculate the static quark–antiquark potential from the large-n_t behavior of the Wilson loop. The corrections in (12) are exponentially suppressed, where ΔE is the difference between $V(r)$ and the first excited energy level of the quark–antiquark pair.

planar and nonplanar

- The Wilson loops we have introduced are not necessarily planar, but also nonplanar.
- Both loops have $n = 5$ (the horizontal direction is time). The planar loop has $r = 3a$, the nonplanar loop has $r = \sqrt{3^2 + 1}a = \sqrt{10}a$.
- Thus with nonplanar Wilson loops we can calculate the potential $V(r)$ not only at distances r that are integer multiples of a, but also at intermediate points.
- Nonplanar Wilson loops also allow one to study whether rotational invariance is eventually restored when approaching the continuum limit.

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input file

```xml
<chroma>
    <Param>
        <InlineMeasurements>
            <elem>
                <Name>WILSLP</Name>
                <Frequency>2</Frequency>
                <Param>
                    <version>3</version>
                    <kind>7</kind>
                    <j_decay>3</j_decay>
                    <t_dir>3</t_dir>
                    <GaugeState>
                        <Name>SIMPLE_GAUGE_STATE</Name>
                        <GaugeBC>
                            <Name>PERIODIC_GAUGEBC</Name>
                        </GaugeBC>
                    </GaugeState>
                </Param>
                <NamedObject>
                    <gauge_id>default_gauge_field</gauge_id>
                </NamedObject>
            </elem>
        </InlineMeasurements>
        <nrow>24 24 24 32</nrow>
    </Param>
```

Figure 2: The input file for computing wilson loop

parameters

- Parameter "kind" is used to control which kinds of wilson loop are calculated.
- There are three kinds of wilson loop in chroma.
  - "space-like"
  - "time-like"
  - off-axis "time-like"
- Set kind 7 mean calculation all three cases.

xml

- The results are stored in output.xml. We can use the “elementtree” which is a python package to read the results in .xml files.
- The result of wilson loop can be stored in a three-dimensional array, such as (conf, tlength, rlength).
- We can apply bootstrap or jackknife methods to estimate errors.

xml


```xml
<wils_loop1>
  <lengthr>24</lengthr>
  <wloop1>
    <elem>
      <r>0</r>
      <loop>0.593639134045096 0.383657559147797 0.252756190741583 0.16706592846726 0.110677066702199 0.07323100943
    </elem>
    <elem>
      <r>1</r>
      <loop>0.383593083331063 0.19012362611784 0.101552140925667 0.0553684481699792 0.0303952806627127 0.016783064
    </elem>
    <elem>
      <r>2</r>
      <loop>0.252683752292945 0.101340327968154 0.0468444105583397 0.0229603029351851 0.0114138013470634 0.0054538
    </elem>
    <elem>
      <r>3</r>
      <loop>0.167030569303193 0.0550769625693254 0.0225417793236289 0.0101988316367707 0.00466471543446183 0.00172
    </elem>
    <elem>
      <r>4</r>
      <loop>0.110554988875505 0.0302910141581682 0.0112564003889854 0.00491093893725804 0.00201728593561667 0.0004
    </elem>
    <elem>
      <r>5</r>
      <loop>0.0733118269589373 0.0166671969163673 0.0056917660787625 0.00197730908860176 0.000366102289530654 0.00
    </elem>
    <elem>
      <r>6</r>
      <loop>0.0485978833789361 0.00894917624710141 0.00283515087342205 0.000844505082801832 0.000242980251629579 -
    </elem>
    <elem>
      <r>7</r>
      <loop>0.0322650521274582 0.00497864585325959 0.0013256229346642 0.000648393534869111 0.000144230832238614 9.
    </elem>
    <elem>
      <r>8</r>
      <loop>0.0212682536075674 0.00291870420933602 0.000734577819832468 0.000272637871877147 1.96716717611362e-05
    </elem>
    <elem>
      <r>9</r>
      <loop>0.0140434032749072 0.00131171465875631 0.000170602776785603 0.000341874507002539 0.000106433833385308
    </elem>
```


Figure 3: xml output file for wilson loop

results

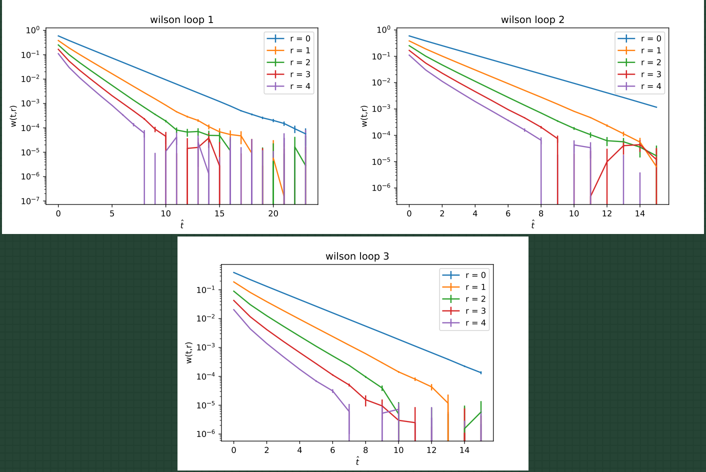

Figure 4: The wilson loops vary with t dimensions. Each line represents a different distance r.

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A few integrals for SU(3)

- Some useful integrals over products of entries U_{ab} of group elements U in the fundamental representation:

$$\begin{aligned} &\int_{\mathrm{SU}(3)} \mathrm{d}U U_{ab} = 0 \\ &\int_{\mathrm{SU}(3)} \mathrm{d}U U_{ab} U_{cd} = 0 \\ &\int_{\mathrm{SU}(3)} \mathrm{d}U U_{ab} \left(U^{\dagger}\right)_{cd} = \frac{1}{3}\delta_{ad}\delta_{bc} \\ &\int_{\mathrm{SU}(3)} \mathrm{d}U U_{ab} U_{cd} U_{ef} = \frac{1}{6}\epsilon_{ace}\epsilon_{bdf} \end{aligned} \tag{13}$$

- A relation that will be useful later:

$$\int dU \, \mathrm{tr}[VU] \, \mathrm{tr}\left[U^\dagger W\right] = \frac{1}{3} \mathrm{tr}[VW]$$

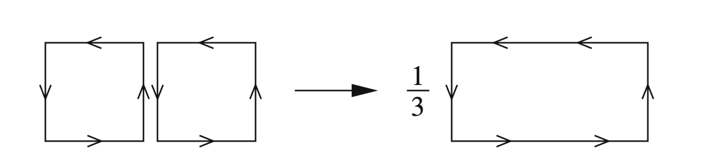

Figure 5: Integrating out the common link of a product of two plaquettes

The static quark potential

- The wilson loop can be expressed as a path integral:

$$\langle W_{\mathcal{C}} \rangle = \frac{1}{Z} \int \mathcal{D}[U] \exp\left(-\frac{\beta}{3} \sum_P \mathrm{Re}\,\mathrm{tr}\,[\mathbb{1} - U_P] \right) \mathrm{tr}\left[\prod_{l \in \mathcal{C}} U_l \right] \tag{14}$$

- This expression can be rewritten as(the constant factor $\exp\left(-\beta/3 \sum_P \mathrm{Re}\,\mathrm{tr}[\mathbb{1}]\right)$canceled with the denominators.)

$$\begin{aligned}\langle W_{\mathcal{C}} \rangle &= \frac{1}{Z'} \int \mathcal{D}[U] \exp\left(\frac{\beta}{3} \sum_P \mathrm{Re}\,\mathrm{tr}\,[U_P] \right) \mathrm{tr}\left[\prod_{l \in \mathcal{C}} U_l \right] \\ &= \frac{1}{Z'} \int \mathcal{D}[U] \exp\left(\frac{\beta}{6} \sum_P \left(\mathrm{tr}\,[U_P] + \mathrm{tr}\left[U_P^\dagger\right] \right) \right) \mathrm{tr}\left[\prod_{l \in \mathcal{C}} U_l \right]\end{aligned} \tag{15}$$

- we expand the Boltzmann factor of (15) in β using the Taylor expansion for the exponential function,

$$\exp\left(\frac{\beta}{6}\sum_P\left(\mathrm{tr}\left[U_P\right]+\mathrm{tr}\left[U_P^\dagger\right]\right)\right)=\sum_{i,j=0}^{\infty}\frac{1}{i!j!}\left(\frac{\beta}{6}\right)^{i+j} \tag{16}$$

$$\times\left(\sum_P\mathrm{tr}\left[U_P\right]\right)^i\left(\sum_P\mathrm{tr}\left[U_P^\dagger\right]\right)^j \tag{17}$$

- Note that in this expansion we have separated the contributions from clockwise oriented plaquettes $U_P^\dagger$ and counter-clockwise oriented plaquettes U_P.
- This is important since for the leading term in the expansion only those plaquettes oriented oppositely to the Wilson loop contribute.

- For the normalization factor Z' it is straightforward to determine the leading contribution in the small- β expansion. Already the first term with $i = j = 0$ in (15) gives a nonvanishing contribution to the integral and we obtain

$$Z' = \int \mathcal{D}[U] \,\mathrm{exp}\left(\frac{\beta}{6}\sum_P \left(\mathrm{tr}\,[U_P] + \mathrm{tr}\left[U_P^\dagger\right]\right)\right) \tag{18}$$

$$= \int \mathcal{D}[U](1 + \mathcal{O}(\beta)) = 1 + \mathcal{O}\left(\beta^2\right) \tag{19}$$

- The expansion of the numerator of (15) is less straightforward. The leading term in the expansion of the Boltzmann factor are vanished due to $SU(3)$ group integratals as in (13).
- We have to expand the Boltzmann factor in small β. This brings down additional link variables from the exponent and in this way we can saturate the integrals over the links to obtain nonvanishing contributions.

- If we consider the contour $\mathcal{C}$ of the Wilson loop to be a $n_r \times n_t$ rectangle of links, then the minimal area $\mathcal{A}_\mathcal{C}$ spanned by this contour contains $n_A = n_r n_t$ plaquettes.
- The physical area $\mathcal{A}_\mathcal{C}$ is related to the extension of the Wilson loop in physical units an_r, an_t by $\mathcal{A}_\mathcal{C} = a^2 n_A = an_r an_t$.
- we find nonvanishing contributions only when each link variable $U_\mu(n)$ in the loop is paired with its conjugate partner $U_\mu(n)^\dagger$. since we have plaquettes in our action, this must continue until we have filled the contour $\mathcal{C}$ with n_A plaquettes obtained from the expansion of the Boltzmann factor.

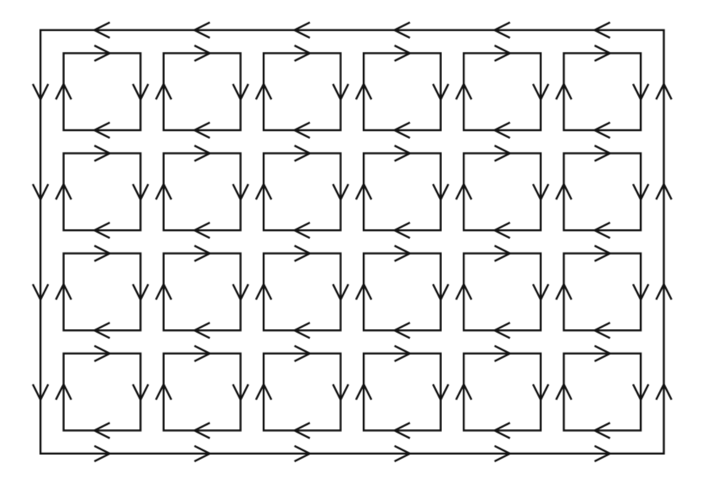

Figure 6: Leading contribution in the strong coupling (small β) expansion of the Wilson loop.

- we need at least $n_A = n_r n_t$ plaquettes from the exponent, the necessary term in the expansion (16) of the exponential is of order n_A. Explicitly this leading term reads

$$
\begin{aligned}
&\int \mathcal{D}[U] \frac{1}{n_A!} \left(\frac{\beta}{6}\right)^{n_A} \left(\sum_P \text{tr}\left[U_P^\dagger\right]\right)^{n_A} \text{tr}\left[\prod_{l\in\mathcal{C}} U_l\right] \\
&= \left(\frac{\beta}{6}\right)^{n_A} \int \mathcal{D}[U] \prod_{P\in\mathcal{A}_C} \text{tr}\left[U_P^\dagger\right] \text{tr}\left[\prod_{l\in\mathcal{C}} U_l\right] \\
&= \text{tr}[\mathbb{1}] \left(\frac{\beta}{6}\right)^{n_A} \left(\frac{1}{3}\right)^{n_A} = 3 \exp\left(n_A \ln\left(\frac{\beta}{18}\right)\right)
\end{aligned}
\tag{20}
$$

- Combining (3.67) and (3.68) we find

$$\langle W_{\mathcal{C}} \rangle = 3 \exp\left(n_A \ln\left(\frac{\beta}{18} \right) \right) (1 + \mathcal{O}(\beta)) \quad (21)$$

$$= 3 \exp\left(n_r n_t \ln\left(\frac{\beta}{18} \right) \right) (1 + \mathcal{O}(\beta)) \quad (22)$$

- According to (3.56) this expression has to be compared to the asymptotic form, i.e., for large $t = a n_t$ we have

$$\langle W_{\mathcal{C}} \rangle \propto \exp\left(-a n_t V(r) \right) \quad (23)$$

- Thus, we conclude that in the strong coupling limit (note that $r = an_r$)

$$V(r) = \sigma r \tag{24}$$

- Where the string tension σ is given by the leading order expression

$$\sigma = -\frac{1}{a^2} \ln\left(\frac{\beta}{18}\right)(1 + \mathcal{O}(\beta)) \tag{25}$$

- Such a term in the potential gives rise to the important feature of confinement.
- In QED the static potential is of the Coulomb-type and is the same like in QCD when $\alpha_s \to 0$。 so the potential also have the Coulomb-type term.
- the static QCD potential can be parameterized by

$$V(r) = A + \frac{B}{r} + \sigma r \tag{26}$$

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static potential

- According to the Eq. 12, when t is large, Wilson loop decays exponentially with t.
- We can use the following formula to obtain the potential with different r which is similar to meson mass calculation:

$$V(t,r) = \log(\frac{W(t,r)}{W(t+1,r)}).$$

- When t is large enough, $V(t,r)$ is close to the static potential of quark-antiquark.

results

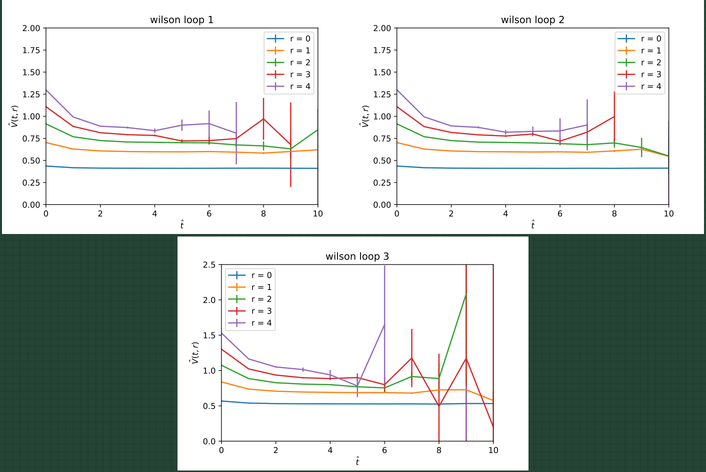

Figure 7: The potential $V(t,r)$ obtained by wilson loops.

results

- Finally, we obtain the vary of static potential with distance r

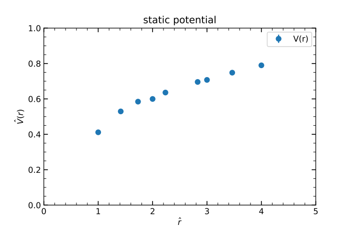

Figure 8: The static quark potential $V(r)$ vary with distance r.

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Autocorrelation

- The statistical analysis of the measured observables is the important final step of a Monte Carlo simulation.
- Since in our case the data sample is the result of a (computer-)time series in our Monte Carlo simulation there is high chance that the observables are in fact correlated. This so-called autocorrelation leads to a nonvanishing AUTOCORRELATION FUNCTION, which we define as

$$
\begin{aligned}
C_X\left(X_i, X_{i+t}\right) &= \left\langle\left(X_i - \left\langle X_i\right\rangle\right)\left(X_{i+t} - \left\langle X_{i+t}\right\rangle\right)\right\rangle \\
&= \left\langle X_i X_{i+t}\right\rangle - \left\langle X_i\right\rangle\left\langle X_{i+t}\right\rangle
\end{aligned}
$$

- For a Markov chain in equilibrium the autocorrelation function depends only on the (computer time) separation t and we write

$$C_X(t) = C_X\left(X_i, X_{i+t}\right)$$

- Note that $C_X(0) = \sigma_X^2$. In a typical situation the normalized correlation function Γ_X exhibits exponential behavior asymptotically for large t :

$$\Gamma_X(t) \equiv \frac{C_X(t)}{C_X(0)} \sim \exp\left(-\frac{t}{\tau_{X,\mathrm{exp}}}\right)$$

- one calls $\tau_{X,\,\mathrm{exp}}$ the exponential autocorrelation time for X.

- For uncorrelated data, the variance of estimator is

$$\sigma_{\hat{X}}^2 = \left\langle (\hat{X} - \langle X \rangle)^2 \right\rangle = \left\langle \left(\frac{1}{N} \sum_{i=1}^{N} (X_i - \langle X \rangle) \right)^2 \right\rangle$$
$$= \frac{1}{N^2} \left\langle \sum_{i,j=1}^{N} (X_i - \langle X \rangle)(X_j - \langle X \rangle) \right\rangle \quad . \qquad (27)$$
$$= \frac{1}{N} \langle X^2 \rangle - \langle X \rangle^2 + \frac{1}{N^2} \sum_{i \neq j} \langle X_i X_j \rangle$$

- For correlated random variables X_i the terms with $i \neq j$ in the second line of (27) do not vanish and one can continue this equation to obtain for the correlated case

$$
\begin{aligned}
\sigma^2_{\hat{X}} &= \frac{1}{N^2} \sum_{i,j=1}^{N} C_X(|i-j|) = \frac{1}{N^2} \sum_{t=-(N-1)}^{N-1} \sum_{k=1}^{N-|t|} C_X(|t|) \\
&= \sum_{t=-N}^{N} \frac{N-|t|}{N^2} C_X(|t|) = \frac{C_X(0)}{N} \sum_{t=-N}^{N} \Gamma_X(|t|) \left(1 - \frac{|t|}{N}\right) \\
&\approx \frac{\sigma^2_X}{N} 2 \left(\frac{1}{2} + \sum_{t=1}^{N} \Gamma_X(|t|)\right) \equiv \frac{\sigma^2_X}{N} 2\tau_{X,\,\text{int}}
\end{aligned}
$$

- We introduced the integrated autocorrelation time

$$\tau_{X,\,\text{int}} = \frac{1}{2} + \sum_{t=1}^{N} \Gamma_X(t).$$

- This definition is motivated by the observation that for exponential behavior

$$\tau_{X,\,\text{int}} = \frac{1}{2} + \sum_{t=1}^{N} \Gamma_X(|t|) \approx \int_0^\infty \text{d}t e^{-t/\tau} = \tau(\text{ for large } \tau).$$

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- If it is too expensive to compute the autocorrelation time, there are simpler statistical methods for obtaining at least some estimate for the correlation of the data.
- Data blocking methods
- Statistical bootstrap
- Jackknife

Data blocking methods

- One divides the data into sub-blocks of data of size K, computes the block mean values, and considers them as new variables X_i.
- The variance of these blocked X_i then should decrease like $1/K$ if the original data were independent.
- One repeats this for a sequence of different values for K. As soon as the $1/K$ behavior is observed for large enough K one may consider these block variables as statistically independent.
- Once the data (or the block results) can be considered independent, one may determine the expectation values of the observables of interest and their errors.

Statistical bootstrap

- Let us call the value of the observable obtained from the original data set $\widehat{\theta}$.
- One recreates from the sample repeatedly other samples by choosing randomly N data out of the original set. Let us assume we have done this K times and thus have K sets of N data values each.

- For each of these sets one computes the observable θ resulting in values θ_k with $k = 1, \ldots, K$. Then one determines

$$\widetilde{\theta} \equiv \frac{1}{K}\sum_{k=1}^{K}\theta_k, \quad \sigma^2_{\tilde{\theta}} \equiv \frac{1}{K}\sum_{k=1}^{K}\left(\theta_k - \widetilde{\theta}\right)^2$$

- These are estimators for $\langle\theta\rangle$ and σ^2_{θ}. They are not unbiased and therefore $\widetilde{\theta} \neq \widehat{\theta}$ for finite K. The difference is called bias and gives an idea on how far away the result may be from the true $\langle\theta\rangle$. As final result for the observable one quotes $\langle\theta\rangle = \widetilde{\theta} \pm \sigma_{\tilde{\theta}}$

Jackknife

- We start with a data set of size N and an observable θ like for the statistical bootstrap. The value of the observable computed for the original set is again called $\hat{\theta}$.
- One now constructs N subsets by removing the n th entry of the original set $(n = 1, \ldots, N)$ and determines the value θ_n for each set.
- Then

$$\sigma^2_{\widehat{\theta}} \equiv \frac{N-1}{N} \sum_{n=1}^{N} \left(\theta_n - \widehat{\theta} \right)^2$$

The square root of the variance gives an estimate for the standard deviation of $\hat{\theta}$.

- For the final result one quotes either $\langle\theta\rangle = \widehat{\theta} \pm \sigma_{\widehat{\theta}}$ or replaces $\widehat{\theta}$ by the unbiased estimator. The bias may be determined from

$$\widetilde{\theta} \equiv \frac{1}{N}\sum_{n=1}^{N}\theta_n$$

leading to $\widehat{\theta} - (N-1)(\widetilde{\theta} - \widehat{\theta})$ as the unbiased estimator for $\langle\theta\rangle$.

why Jackknife

- When we want to estimate some function of the average of x, i.e. $f(X)$.A poor way to estimate this would be from

$$\bar{f} \equiv \overline{f(x)} = \frac{1}{N}\sum_{i=1}^{N} f_i \quad \text{where } f_i = f\left(x_i\right)$$

- This is bad because it is biased, i.e.

$$\langle \bar{f} \rangle \neq f(X)$$

- The difference is given by

$$
\begin{aligned}
\langle \bar{f} \rangle - f(X) &= \int P(x)(f(x) - f(X))dx \\
&= f'(X) \int P(x)(x - X)dx + \frac{1}{2} f''(X) \int P(x)(x -
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2} f''(X) \left[\langle x^2 \rangle - \langle x \rangle^2 \right] + \cdots \\
&= \frac{1}{2} f''(X) \sigma^2 + \cdots
\end{aligned}
$$

- A better, i.e. less biased, estimate for $f(X)$ is clearly $f(\bar{x})$.

$$\langle f(\bar{x})\rangle - f(X) = \frac{1}{2} f''(X)\left[\langle \bar{x}^2 \rangle - \langle \bar{x} \rangle^2\right] + \cdots$$
$$= \frac{1}{2N} f''(X)\sigma^2 + \cdots$$

- When we want to estimate for $f(X)$ and its error, we can use jackknife or bootstrap resampling methods.

Outline

1. Wilson Loop
    - What is Wilson Loop
    - Physical interpretation of the Wilson loop
    - Calculation
    - The static quark potential
    - fit the static potential
    - Autocorrelation
    - Error Estimate
    - exercise

exercise

1. According to Eq. 35, plot the static potential $V(t, r)$.
2. Choice $V(r, t = 3)$, plot V(r) .

exercise

- 3. the vacuum expectation value of the Wilson loop is connected to the static potential via

$$\langle W_{\mathcal{C}} \rangle = C\,\mathsf{exp}(-tV(r)) = C\,\mathsf{exp}\left(-n_t a V(na)\right)$$

- As V(r) can parameterized as in Eq. 26, So

$$\begin{aligned} \langle W_{\mathcal{C}} \rangle &= C\,\mathsf{exp}\left(-n_t a(A + \frac{B}{a n_r} + \sigma n_r a)\right) \\ &= C\,\mathsf{exp}\left(-n_t a A - \frac{Bnt}{n_r} - \sigma n_r n_t a^2)\right) \end{aligned}$$

- Using this formula to fit the Wilson loop and get the fit value of B and σ
- Finally, the lattice spacing can get from these two parameters

$$a = \frac{1}{2}\sqrt{\frac{\sigma a^2}{1.65 + B}}$$

谢谢