

Wilson Loop

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1. Wilson Loop

What is Wilson Loop

Physical interpretation of the Wilson loop

The static quark potential

2. data analysis

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Wilson Loop

- After we have produced the pure gauge configurations, we can make some physical measurements on these configurations.
- A Wilson loop $W_{\mathcal{L}}$ is made from four pieces, two so-called Wilson lines $S(\mathbf{m}, \mathbf{n}, n_t)$, $S(\mathbf{m}, \mathbf{n}, 0)$, and two temporal transporters $T(\mathbf{n}, n_t)$, $T(\mathbf{m}, n_t)$.

Wilson Loop

- The Wilson line $S(\mathbf{m}, \mathbf{n}, n_t)$ connects the two spatial points \mathbf{m} and \mathbf{n} along some path $\mathcal{C}_{\mathbf{m}, \mathbf{n}}$ with all link variables restricted to time argument n_t ,

$$S(\mathbf{m}, \mathbf{n}, n_t) = \prod_{(\mathbf{k}, j) \in \mathcal{C}_{\mathbf{m}, \mathbf{n}}} U_j(\mathbf{k}, n_t). \quad (1)$$

- The temporal transporter $T(\mathbf{n}, n_t)$ is a straight line of n_t link variables in time direction, all situated at spatial position \mathbf{n} ,

$$T(\mathbf{n}, n_t) = \prod_{j=0}^{n_t-1} U_4(\mathbf{n}, j). \quad (2)$$

Wilson Loop

- Attaching the four pieces to each other gives a closed loop \mathcal{L} ,

$$\mathcal{L} : (m, n_t) \xrightarrow{S} (n, n_t) \xrightarrow{T^\dagger} (n, 0) \xrightarrow{S^\dagger} (m, 0) \xrightarrow{T} (m, n_t). \quad (3)$$

- The Wilson loop $W_{\mathcal{L}}$ is obtained by taking the trace,

$$W_{\mathcal{L}}[U] = \text{tr} \left[S(m, n, n_t) T(n, n_t)^\dagger S(m, n, 0)^\dagger T(m, n_t) \right] \quad (4)$$

$$= \text{tr} \left[\prod_{(k, \mu) \in \mathcal{L}} U_\mu(k) \right] \quad (5)$$

- If the piece of loop $\mathcal{C}_{m,n}$ used in $S(m, n, n_t)$ is a straight line we speak of a planar Wilson loop. Otherwise the Wilson loop is called nonplanar.

Wilson Loop

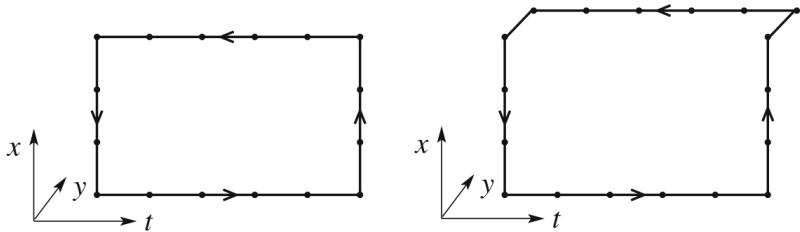


Figure 1: Examples for a planar (left-hand side plot) and a nonplanar (right-hand side) Wilson loop. The horizontal direction is time

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Physical interpretation of the Wilson loop

- Due to the gauge symmetry, We can choice a gauge for gauge field $A_\mu(x)$

$$A_4(x) = 0, \quad (6)$$

- i.e., temporal gauge. So the temporal transporters become trivial,

$$T(\mathbf{n}, n_t) = \prod_{j=0}^{n_t-1} U_4(\mathbf{n}, j) = \mathbb{1}, \quad (7)$$

- we obtain the following chain of identities

$$\langle W_{\mathcal{L}} \rangle = \langle W_{\mathcal{L}} \rangle_{\text{temp}} = \langle \text{tr} [S(\mathbf{m}, \mathbf{n}, n_t) S(\mathbf{m}, \mathbf{n}, 0)^\dagger] \rangle_{\text{temp}}, \quad (8)$$

- The temporal gauge used in (8) makes explicit that the Wilson loop is the correlator of two Wilson lines $S(\mathbf{m}, \mathbf{n}, n_t)$ and $S(\mathbf{m}, \mathbf{n}, 0)$ situated at time slices n_t and 0. Thus we can interpret this correlator using the equation

$$\lim_{T \rightarrow \infty} \langle O_2(t) O_1(0) \rangle_T = \sum_n \langle 0 | \hat{O}_2 | n \rangle \langle n | \hat{O}_1 | 0 \rangle e^{-tE_n}$$

- where we insert the unit operator of the vectors of a complete orthonormal basis as

$$\mathbb{1} = \sum_n |e_n\rangle \langle e_n|$$

in the left-hand side of the equation.

- The correlator behaves for large total temporal extent T of our Euclidean lattice as (a, b are summed)

$$\langle \text{tr} [S(\mathbf{m}, \mathbf{n}, n_t) S(\mathbf{m}, \mathbf{n}, 0)^\dagger] \rangle_{\text{temp}} = \sum_k \langle 0 | \hat{S}(\mathbf{m}, \mathbf{n})_{ab} | k \rangle \quad (9)$$

$$\langle k | \hat{S}(\mathbf{m}, \mathbf{n})_{ba}^\dagger | 0 \rangle e^{-tE_k} \quad (10)$$

where the Euclidean time argument t is related to n_t via $t = a n_t$ with a being the lattice spacing. The sum in (9) runs over all states $|k\rangle$ that have a non-vanishing overlap with $\hat{S}(\mathbf{m}, \mathbf{n})^\dagger |0\rangle$

- the states $|k\rangle$ with non-vanishing overlap are states describing a static quark-antiquark pair located at spatial positions \mathbf{m} and \mathbf{n} .
- The energy E_1 is thus identified with the energy of the quark-antiquark pair, which is the static potential $V(r)$ at spatial quark separation r ,

$$E_1 = V(r) \quad \text{with} \quad r = a|\mathbf{m} - \mathbf{n}| \quad (11)$$

- Combining (8), (9), and (11) we obtain

$$\langle W_{\mathcal{L}} \rangle \propto e^{-tV(r)} (1 + \mathcal{O}(e^{-t\Delta E})) = e^{-n_t a V(r)} (1 + \mathcal{O}(e^{-n_t a \Delta E})) \quad (12)$$

- Thus we find that we can calculate the static quark-antiquark potential from the large- n_t behavior of the Wilson loop. The corrections in (12) are exponentially suppressed, where ΔE is the difference between $V(r)$ and the first excited energy level of the quark-antiquark pair.

planar and nonplanar

- The Wilson loops we have introduced are not necessarily planar, but also nonplanar.
- Both loops have $n = 5$ (the horizontal direction is time). The planar loop has $r = 3a$, the nonplanar loop has $r = \sqrt{3^2 + 1}a = \sqrt{10}a$.
- Thus with nonplanar Wilson loops we can calculate the potential $V(r)$ not only at distances r that are integer multiples of a , but also at intermediate points.
- Nonplanar Wilson loops also allow one to study whether rotational invariance is eventually restored when approaching the continuum limit.

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The static quark potential

- The wilson loop can be expressed as a path integral:

$$\langle W_C \rangle = \frac{1}{Z} \int \mathcal{D}[U] \exp \left(-\frac{\beta}{3} \sum_P \text{Re tr} [\not{K} - U_P] \right) \text{tr} \left[\prod_{l \in C} U_l \right] \quad (13)$$

- This expression can be rewritten as (the constant factor $\exp(-\beta/3 \sum_P \text{Re tr} [\not{K}])$ canceled with the denominators.)

$$\begin{aligned} \langle W_C \rangle &= \frac{1}{Z'} \int \mathcal{D}[U] \exp \left(\frac{\beta}{3} \sum_P \text{Re tr} [U_P] \right) \text{tr} \left[\prod_{l \in C} U_l \right] \\ &= \frac{1}{Z'} \int \mathcal{D}[U] \exp \left(\frac{\beta}{6} \sum_P (\text{tr} [U_P] + \text{tr} [U_P^\dagger]) \right) \text{tr} \left[\prod_{l \in C} U_l \right] \end{aligned} \quad (14)$$

A few integrals for $SU(3)$

- Some useful integrals over products of entries U_{ab} of group elements U in the fundamental representation:

$$\begin{aligned}\int_{SU(3)} dU U_{ab} &= 0 \\ \int_{SU(3)} dU U_{ab} U_{cd} &= 0 \\ \int_{SU(3)} dU U_{ab} (U^\dagger)_{cd} &= \frac{1}{3} \delta_{ad} \delta_{bc} \\ \int_{SU(3)} dU U_{ab} U_{cd} U_{ef} &= \frac{1}{6} \epsilon_{ace} \epsilon_{bdf}\end{aligned}\tag{15}$$

- we expand the Boltzmann factor of (14) in β using the Taylor expansion for the exponential function,

$$\exp\left(\frac{\beta}{6} \sum_P (\text{tr}[U_P] + \text{tr}[U_P^\dagger])\right) = \sum_{i,j=0}^{\infty} \frac{1}{i!j!} \left(\frac{\beta}{6}\right)^{i+j} \quad (16)$$

$$\times \left(\sum_P \text{tr}[U_P]\right)^i \left(\sum_P \text{tr}[U_P^\dagger]\right)^j \quad (17)$$

- Note that in this expansion we have separated the contributions from clockwise oriented plaquettes U_P^\dagger and counter-clockwise oriented plaquettes U_P .
- This is important since for the leading term in the expansion only those plaquettes oriented oppositely to the Wilson loop contribute.

- For the normalization factor Z' it is straightforward to determine the leading contribution in the small- β expansion. Already the first term with $i = j = 0$ in (14) gives a nonvanishing contribution to the integral and we obtain

$$Z' = \int \mathcal{D}[U] \exp \left(\frac{\beta}{6} \sum_P \left(\text{tr} [U_P] + \text{tr} [U_P^\dagger] \right) \right) \quad (18)$$

$$= \int \mathcal{D}[U] (1 + \mathcal{O}(\beta)) = 1 + \mathcal{O}(\beta^2) \quad (19)$$

- The expansion of the numerator of (14) is less straightforward. The leading term in the expansion of the Boltzmann factor are vanished due to $SU(3)$ group integrals as in (15).
- We have to expand the Boltzmann factor in small β . This brings down additional link variables from the exponent and in this way we can saturate the integrals over the links to obtain nonvanishing contributions.

- If we consider the contour \mathcal{C} of the Wilson loop to be a $n_r \times n_t$ rectangle of links, then the minimal area $\mathcal{A}_{\mathcal{C}}$ spanned by this contour contains $n_A = n_r n_t$ plaquettes.
- The physical area $\mathcal{A}_{\mathcal{C}}$ is related to the extension of the Wilson loop in physical units an_r, an_t by
$$\mathcal{A}_{\mathcal{C}} = a^2 n_A = an_r an_t.$$
- we find nonvanishing contributions only when each link variable $U_{\mu}(n)$ in the loop is paired with its conjugate partner $U_{\mu}(n)^{\dagger}$. since we have plaquettes in our action, this must continue until we have filled the contour \mathcal{C} with n_A plaquettes obtained from the expansion of the Boltzmann factor.

- we need at least $n_A = n_r n_t$ plaquettes from the exponent, the necessary term in the expansion (16) of the exponential is of order n_A . Explicitly this leading term reads

$$\begin{aligned}
 & \int \mathcal{D}[U] \frac{1}{n_A!} \left(\frac{\beta}{6}\right)^{n_A} \left(\sum_P \text{tr} [U_P^\dagger]\right)^{n_A} \text{tr} \left[\prod_{l \in \mathcal{C}} U_l\right] \\
 &= \left(\frac{\beta}{6}\right)^{n_A} \int \mathcal{D}[U] \prod_{P \in \mathcal{A}_\mathcal{C}} \text{tr} [U_P^\dagger] \text{tr} \left[\prod_{l \in \mathcal{C}} U_l\right] \\
 &= \text{tr}[\mathbb{K}] \left(\frac{\beta}{6}\right)^{n_A} \left(\frac{1}{3}\right)^{n_A} = 3 \exp\left(n_A \ln\left(\frac{\beta}{18}\right)\right)
 \end{aligned} \tag{20}$$

- Combining (3.67) and (3.68) we find

$$\langle W_C \rangle = 3 \exp \left(n_A \ln \left(\frac{\beta}{18} \right) \right) (1 + \mathcal{O}(\beta)) \quad (21)$$

$$= 3 \exp \left(n_r n_t \ln \left(\frac{\beta}{18} \right) \right) (1 + \mathcal{O}(\beta)) \quad (22)$$

- According to (3.56) this expression has to be compared to the asymptotic form, i.e., for large $t = an_t$ we have

$$\langle W_C \rangle \propto \exp(-an_t V(r)) \quad (23)$$

- Thus, we conclude that in the strong coupling limit (note that $r = an_r$)

$$V(r) = \sigma r \quad (24)$$

- Where the string tension σ is given by the leading order expression

$$\sigma = -\frac{1}{a^2} \ln \left(\frac{\beta}{18} \right) (1 + \mathcal{O}(\beta)) \quad (25)$$

- Such a term in the potential gives rise to the important feature of confinement.

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Autocorrelation

- The statistical analysis of the measured observables is the important final step of a Monte Carlo simulation.
- Since in our case the data sample is the result of a (computer-)time series in our Monte Carlo simulation there is high chance that the observables are in fact correlated. This so-called autocorrelation leads to a nonvanishing AUTOCORRELATION FUNCTION, which we define as

$$\begin{aligned} C_X (X_i, X_{i+t}) &= \langle (X_i - \langle X_i \rangle) (X_{i+t} - \langle X_{i+t} \rangle) \rangle \\ &= \langle X_i X_{i+t} \rangle - \langle X_i \rangle \langle X_{i+t} \rangle \end{aligned}$$

- For a Markov chain in equilibrium the autocorrelation function depends only on the (computer time) separation t and we write

$$C_X(t) = C_X(X_i, X_{i+t})$$

- Note that $C_X(0) = \sigma_X^2$. In a typical situation the normalized correlation function Γ_X exhibits exponential behavior asymptotically for large t :

$$\Gamma_X(t) \equiv \frac{C_X(t)}{C_X(0)} \sim \exp\left(-\frac{t}{\tau_{X,\text{exp}}}\right)$$

- one calls $\tau_{X,\text{exp}}$ the exponential autocorrelation time for X .

- For uncorrelated data, the variance of estimator is

$$\begin{aligned}\sigma_{\hat{X}}^2 &= \left\langle (\hat{X} - \langle X \rangle)^2 \right\rangle = \left\langle \left(\frac{1}{N} \sum_{i=1}^N (X_i - \langle X \rangle) \right)^2 \right\rangle \\ &= \frac{1}{N^2} \left\langle \sum_{i,j=1}^N (X_i - \langle X \rangle) (X_j - \langle X \rangle) \right\rangle \quad . \quad (26) \\ &= \frac{1}{N} \langle X^2 \rangle - \langle X \rangle^2 + \frac{1}{N^2} \sum_{i \neq j} \langle X_i X_j \rangle\end{aligned}$$

- For correlated random variables X_i the terms with $i \neq j$ in the second line of (26) do not vanish and one can continue this equation to obtain for the correlated case

$$\begin{aligned}
 \sigma_{\hat{X}}^2 &= \frac{1}{N^2} \sum_{i,j=1}^N C_X(|i-j|) = \frac{1}{N^2} \sum_{t=-(N-1)}^{N-1} \sum_{k=1}^{N-|t|} C_X(|t|) \\
 &= \sum_{t=-N}^N \frac{N-|t|}{N^2} C_X(|t|) = \frac{C_X(0)}{N} \sum_{t=-N}^N \Gamma_X(|t|) \left(1 - \frac{|t|}{N}\right) \\
 &\approx \frac{\sigma_X^2}{N} 2 \left(\frac{1}{2} + \sum_{t=1}^N \Gamma_X(|t|) \right) \equiv \frac{\sigma_X^2}{N} 2\tau_{X, \text{int}}
 \end{aligned}$$

- We introduced the integrated autocorrelation time

$$\tau_{X, \text{int}} = \frac{1}{2} + \sum_{t=1}^N \Gamma_X(t).$$

- This definition is motivated by the observation that for exponential behavior

$$\tau_{X, \text{int}} = \frac{1}{2} + \sum_{t=1}^N \Gamma_X(|t|) \approx \int_0^{\infty} dt e^{-t/\tau} = \tau \text{ (for large } \tau \text{)}.$$

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- If it is too expensive to compute the autocorrelation time, there are simpler statistical methods for obtaining at least some estimate for the correlation of the data.
- Data blocking methods
- Statistical bootstrap
- Jackknife

Data blocking methods

- One divides the data into sub-blocks of data of size K , computes the block mean values, and considers them as new variables X_i .
- The variance of these blocked X_i then should decrease like $1/K$ if the original data were independent.
- One repeats this for a sequence of different values for K . As soon as the $1/K$ behavior is observed for large enough K one may consider these block variables as statistically independent.
- Once the data (or the block results) can be considered independent, one may determine the expectation values of the observables of interest and their errors.

Statistical bootstrap

- Let us call the value of the observable obtained from the original data set $\hat{\theta}$.
- One recreates from the sample repeatedly other samples by choosing randomly N data out of the original set. Let us assume we have done this K times and thus have K sets of N data values each.

- For each of these sets one computes the observable θ resulting in values θ_k with $k = 1, \dots, K$. Then one determines

$$\tilde{\theta} \equiv \frac{1}{K} \sum_{k=1}^K \theta_k, \quad \sigma_{\tilde{\theta}}^2 \equiv \frac{1}{K} \sum_{k=1}^K (\theta_k - \tilde{\theta})^2$$

- These are estimators for $\langle \theta \rangle$ and σ_{θ}^2 . They are not unbiased and therefore $\tilde{\theta} \neq \hat{\theta}$ for finite K . The difference is called bias and gives an idea on how far away the result may be from the true $\langle \theta \rangle$. As final result for the observable one quotes $\langle \theta \rangle = \tilde{\theta} \pm \sigma_{\tilde{\theta}}$

Jackknife

- We start with a data set of size N and an observable θ like for the statistical bootstrap. The value of the observable computed for the original set is again called $\hat{\theta}$.
- One now constructs N subsets by removing the n th entry of the original set ($n = 1, \dots, N$) and determines the value θ_n for each set.
- Then

$$\sigma_{\hat{\theta}}^2 \equiv \frac{N-1}{N} \sum_{n=1}^N (\theta_n - \hat{\theta})^2$$

The square root of the variance gives an estimate for the standard deviation of $\hat{\theta}$.

- For the final result one quotes either $\langle \theta \rangle = \hat{\theta} \pm \sigma_{\hat{\theta}}$ or replaces $\hat{\theta}$ by the unbiased estimator. The bias may be determined from

$$\tilde{\theta} \equiv \frac{1}{N} \sum_{n=1}^N \theta_n$$

leading to $\hat{\theta} - (N - 1)(\tilde{\theta} - \hat{\theta})$ as the unbiased estimator for $\langle \theta \rangle$.

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input file

```
<chroma>
  <Param>
    <InlineMeasurements>
      <elem>
        <Name>WILSLP</Name>
        <Frequency>2</Frequency>
        <Param>
          <version>3</version>
          <kind>7</kind>
          <j_decay>3</j_decay>
          <t_dir>3</t_dir>
          <GaugeState>
            <Name>SIMPLE_GAUGE_STATE</Name>
            <GaugeBC>
              <Name>PERIODIC_GAUGEBC</Name>
            </GaugeBC>
          </GaugeState>
        </Param>
        <NamedObject>
          <gauge_id>default_gauge_field</gauge_id>
        </NamedObject>
      </elem>
    </InlineMeasurements>
    <nrow>24 24 24 32</nrow>
  </Param>
```

Figure 2: The input file for computing wilson loop

- Parameter “kind” is used to control which kinds of wilson loop are calculated.
- There are three kinds of wilson loop in chroma.
 - ▶ “space-like”
 - ▶ “time-like”
 - ▶ off-axis “time-like”
- Set kind 7 mean calculation all three cases.

xml

- The results are stored in output.xml. We can use the elementtree which is a python package to read the results in .xml files.
- The result of wilson loop can be stored in a three-dimensional array, such as (conf, tlength, rlength).
- We can apply bootstrap or jackknife methods to estimate errors.

```

<wils_loop1>
  <lengthr>24</lengthr>
  <wloop1>
    <elem>
      <r>0</r>
      <loop>0.593639134045096 0.383657559147797 0.252756190741583 0.16706592846726 0.110677066702199 0.07323100943
    </elem>
    <elem>
      <r>1</r>
      <loop>0.383593083331063 0.19012362611784 0.101552140925667 0.0553684481699792 0.0303952806627127 0.016783064
    </elem>
    <elem>
      <r>2</r>
      <loop>0.252683752292945 0.101340327968154 0.0468444105583397 0.0229603029351851 0.0114138013470634 0.0054536
    </elem>
    <elem>
      <r>3</r>
      <loop>0.1670305693803193 0.0550769625693254 0.0225417793236289 0.0101988316367707 0.00466471543446183 0.00172
    </elem>
    <elem>
      <r>4</r>
      <loop>0.110554988875505 0.0302910141581682 0.0112564003889854 0.00491093893725804 0.00201728593561667 0.0004
    </elem>
    <elem>
      <r>5</r>
      <loop>0.0733118269589373 0.0166671969163673 0.0056917660787625 0.00197730908860176 0.000366102289530654 0.00
    </elem>
    <elem>
      <r>6</r>
      <loop>0.0485978833789361 0.00894917624710141 0.00283515087342205 0.000844505082801832 0.000242980251629579
    </elem>
    <elem>
      <r>7</r>
      <loop>0.0322650521274582 0.00497864585325959 0.0013256229346642 0.000648393534869111 0.000144230832238614 9.
    </elem>
    <elem>
      <r>8</r>
      <loop>0.0212682536075674 0.00291870420933602 0.000734577819832468 0.000272637871877147 1.96716717611362e-05
    </elem>
    <elem>
      <r>9</r>
      <loop>0.0140434032749072 0.00131171465875631 0.000170602776785603 0.000341874507002539 0.000106433833385308
    </elem>
  </wloop1>
</wils_loop1>

```

Figure 3: xml output file for wilson loop

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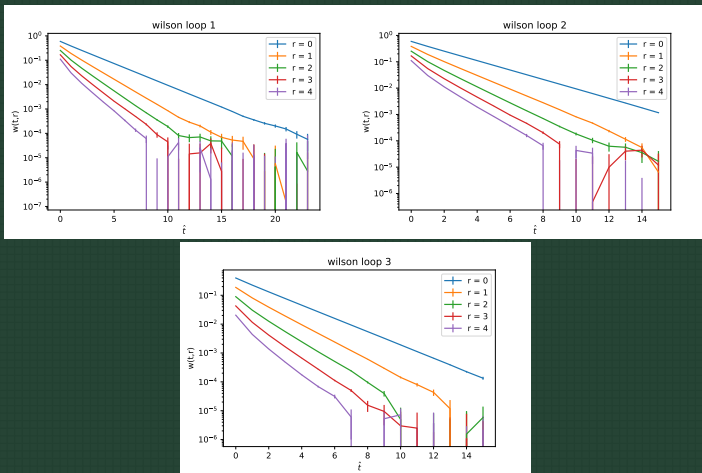


Figure 4: The wilson loops vary with t dimentions. Each line represents a different distance r .

static potential

- According to the Eq. 12, when t is large, Wilson loop decays exponentially with t .
- We can use the following formula to obtain the potential with different r which is similar in meson mass calculation:

$$V(t, r) = \log\left(\frac{W(t, r)}{W(t+1, r)}\right).$$

- When t is large enough, $V(t, r)$ is close to the static potential of quark-antiquark.

results

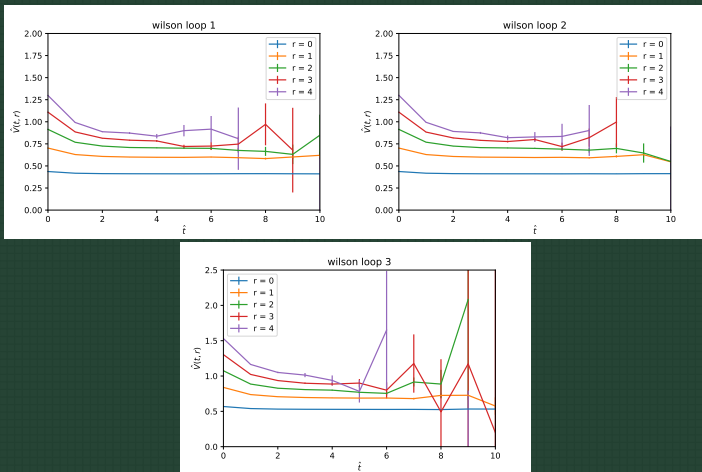


Figure 5: The potential $V(t, r)$ obtained by wilson loops.

results

- Finally, we obtain the vary of static potential with distance r

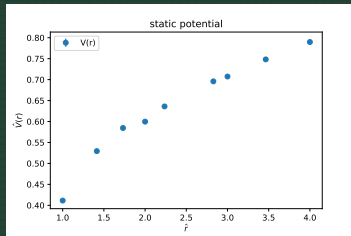


Figure 6: The static quark potential $V(r)$ vary with distance r .