Majorana Fermions

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Outline

- **Lecture 1 - What is Majorana Fermion?**
  - What is Spin?
  - Two-Component Theory & Parity Violation
  - Dirac vs Majorana Fermions
- **Lecture 2 - Majorana Neutrinos**
  - Neutrino Mixing & Oscillation
  - Neutrinoless Double Beta Decay
  - Leptogenesis
  - Cosmic Neutrino Background

Rotation & Spin

Quantum Mechanics

- **Quantization**

$$[\hat{x}, \hat{p}] = i\hbar$$

$$E \to i\hbar \frac{\partial}{\partial t}, \qquad p \to -i\hbar \frac{\partial}{\partial x}$$

- **Hamiltonian**

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$$

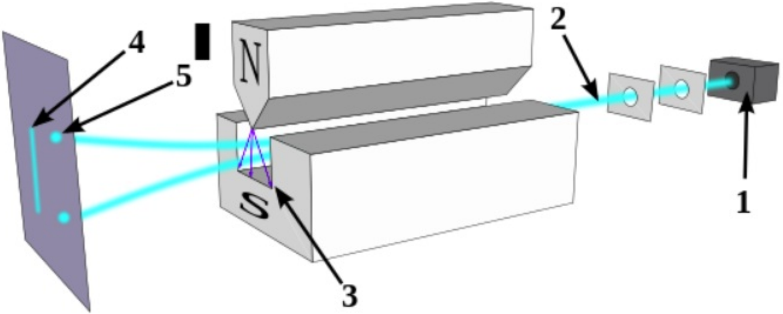
- **Schrodinger Equation**

$$i\hbar \partial_t \psi = \hat{H} \psi$$

- **Evolution**

$$\psi(t) = e^{-i\hat{H}t} \psi(t = 0)$$

Spin



Stern-Gerlach Experiment

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(x) + \mathbf{B} \cdot \mathbf{S}$$

Spin in Quantum Mechanics

- **Spin is introduced by hand**

$$\psi \to \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \psi_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \psi_- \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \sum_{\pm} \psi_\pm \chi_\pm$$

- Spin Operators $\mathbf{S} = \frac{\hbar}{2}\boldsymbol{\sigma}$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Define $S_\pm \equiv S_x \pm i S_y$

$$S_z \chi_\pm = \pm \frac{\hbar}{2} \chi_\pm, \quad S_\pm \chi_\pm = 0, \quad S_\pm \chi_\mp = \chi_\pm$$

The spin operators $\mathbf{S}$ generate $SU(2)$ symmetry. $[S_i, S_j] = i\epsilon_{ijk}\hbar S_k$.
A spin state is a doublet of the $SU(2)$ group.

Rotation & Angular Momentum

- Rotation in 3-D space

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv e^{i\theta J_y} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

- The rotations can also be generated by 3 operators:

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Same relations $[J_i, J_j] = i\epsilon_{ijk} J_k$, but slightly different symmetry group $SO(3)$.

Space-Time & Boost

- Boost along one axis

$$x' = \frac{x - vt}{\sqrt{1-\beta^2}}, \qquad t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1-\beta^2}}$$

- Boost is also rotation!

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \equiv e^{i\beta K_x} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

- Boost also has 3 generators

$$K_x = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_y = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_z = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

Lorentz Symmetry

- **Four dimensional space-time**

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = e^{i(\theta_i J_i + \beta_i K_i)} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

Einstein's special relativity essentially promote time to be one dimension of space-time.

- **Hidden Lorentz Symmetry**

Lorentz vector

$$X^\mu \equiv (t, x, y, z), \quad X_\mu \equiv (t, -x, -y, -z)$$

Lorentz scalar is invariant under Lorentz symmetry:

$$X \cdot X \equiv X^\mu X_\mu = X' \cdot X'$$

Electrodynamics is covariant.

Symmetry ⇔ Group

- The four-dimensional rotations

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = e^{i(\theta_i J_i + \beta_i K_i)} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

  Can also be treated as symmetry transformation.

$$\begin{aligned} [J_i, J_j] &= +i\epsilon_{ijk} J_k, \\ [K_i, K_j] &= -i\epsilon_{ijk} J_k, \\ [J_i, K_j] &= +i\epsilon_{ijk} K_k \end{aligned}$$

- Rotation mixes with boost.

Decomposition of Lorentz Group

- Lorentz Group Generators

$$\begin{aligned} [J_i, J_j] &= +i\epsilon_{ijk} J_k, \\ [K_i, K_j] &= -i\epsilon_{ijk} J_k, \\ [J_i, K_j] &= +i\epsilon_{ijk} K_k \end{aligned}$$

- Redefinitions

$$A_i \equiv \frac{1}{2}(J_i + iK_i) \qquad J_i = A_i + B_i$$

$$B_i \equiv \frac{1}{2}(J_i - iK_i) \qquad K_i = -i(A_i - B_i)$$

- Disentangled group generators

$$\begin{aligned} [A_i, A_j] &= i\epsilon_{ijk} A_k \\ [B_i, B_j] &= i\epsilon_{ijk} B_k \\ [A_i, B_j] &= 0 \end{aligned}$$

Spinor

- Disentangled group generators

$$[A_i, A_j] = i\epsilon_{ijk}A_k, \quad [B_i, B_j] = i\epsilon_{ijk}B_k, \quad [A_i, B_j] = 0$$

- Two sets of $SU(2)$ group generators:

$$SO(3,1) = SU(2)_L \times SU(2)_R$$

The Lorentz group is equivalent to two $SU(2)$ groups.

- Parity $\Leftrightarrow$ space inversion:

$$\begin{aligned} \mathbf{x} \to -\mathbf{x}, \qquad & \mathbf{x}' \to -\mathbf{x}' \\ \mathbf{J} \to \mathbf{J} \qquad & \mathbf{K} \to -\mathbf{K} \\ \mathbf{A} \leftrightarrow \mathbf{B} \qquad & SU(2)_L \leftrightarrow SU(2)_R \end{aligned}$$

Under parity transformation, $SU(2)_L$ and $SU(2)_R$ interchanges.

Left- and Right-Handed Fermions

- The smallest nontrivial representations,

$$\chi \in (2,1) \qquad \xi \in (1,2)$$

- Dirac spinor

$$\psi = \begin{pmatrix} \chi \\ \xi \end{pmatrix} \in (2,2) = (2,1) \otimes (1,2)$$

- Left- and right-handed fermions

$$\psi_L = P_L \psi, \qquad \psi_R = P_R \psi,$$

where P_L & P_R are projectors

$$P_L = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_R = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I} \end{pmatrix}$$

Parity Violation

PHYSICAL REVIEW VOLUME 104, NUMBER 1 OCTOBER 1, 1956

Question of Parity Conservation in Weak Interactions*

T. D. LEE, *Columbia University, New York, New York*

AND

C. N. YANG,† *Brookhaven National Laboratory, Upton, New York*
(Received June 22, 1956)

The question of parity conservation in β decays and in hyperon and meson decays is examined. Possible experiments are suggested which might test parity conservation in these interactions.

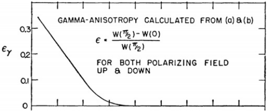
Experimental Test of Parity Conservation in Beta Decay*

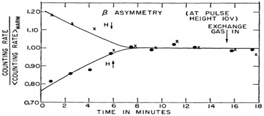
C. S. WU, *Columbia University, New York, New York*

AND

E. AMBLER, R. W. HAYWARD, D. D. HOPPES, AND R. P. HUDSON, *National Bureau of Standards, Washington, D. C.*
(Received January 15, 1957)

$$^{60}_{27}Co \to {}^{60}_{28}Ni + e + \bar{\nu}_e + 2\gamma$$





Parity Violation in Standard Model

- **Electroweak symmetries $SU(2)_L \times U(1)_Y$**

$$\begin{pmatrix} \nu_L \\ \ell_L \end{pmatrix} \in (\mathbf{2}, -\frac{\mathbf{1}}{\mathbf{2}}) \qquad \ell_R \in (\mathbf{1}, -\mathbf{1})$$

Electric charge $\mathbf{Q} = \mathbf{T_3} + \mathbf{Y}$ determined by anomaly cancellation

$$Q(\ell_L) = -\frac{\mathbf{1}}{\mathbf{2}} - \frac{\mathbf{1}}{\mathbf{2}} = -\mathbf{1}, \quad Q(\ell_R) = \mathbf{0} - \mathbf{1} = -\mathbf{1}, \quad Q(\nu_L) = \frac{\mathbf{1}}{\mathbf{2}} - \frac{\mathbf{1}}{\mathbf{2}} = \mathbf{0}$$

There is no right-handed neutrino! **Charge Quantization**

- **Charged Current Interaction**

$$g W^+_\mu \frac{1}{\sqrt{2}} (\bar{\nu}_L \gamma^\mu e_L) + h.c.$$

only involves left-handed fermions $\Rightarrow$ **Parity Violation ???**

Explaining Parity Violation

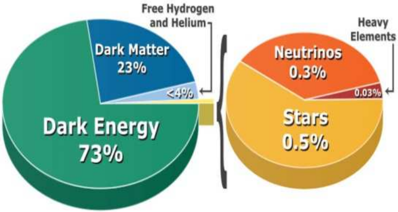
- **Left-Right Symmetric Model**

$$\begin{array}{ccc} SU(2)_L \times SU(2)_R & \times & U(1)_{B-L} \\ & \downarrow & \\ SU(2)_L & \times & U(1)_Y \end{array}$$

Introducing $W_R^{\pm}$, Z_R, and ν_R

- **Mirror Worlds**

6 copies $\left\{ \begin{array}{ll} \textbf{SM} & \textbf{Mirror 1} \\ \textbf{Mirror 2} & \textbf{Mirror 3} \\ \textbf{Mirror 4} & \textbf{Mirror 5} \end{array} \right.$



Dirac Equation

- **Hamiltonian vs Dispersion Relation**

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(x) \quad \Rightarrow \quad E = \frac{p^2}{2m} + V$$

Not consistent with special relativity, $E^2 = \mathbf{p}^2 + m^2$.

- **Generalization**

$$i\hbar\partial_t\psi = \hat{H}\psi$$

to accommodate $(E + \mathbf{p})(E - \mathbf{p}) = m^2$.

$$i\partial_t\psi = \hat{H}\psi \equiv (\alpha_i \hat{\mathbf{p}}_i + \beta m)\,\psi$$

Quadratic form

$$-\partial_t^2\psi = \left[\alpha_i\alpha_j\hat{\mathbf{p}}_i\hat{\mathbf{p}}_j + \{\alpha_i, \beta\}\hat{\mathbf{p}}_i m + m^2\beta^2\right]\psi$$

Dirac Equation

- Quadratic Form:

$$
\begin{aligned}
-\partial_t^2 \psi &= \left[\alpha_i \alpha_j \hat{\mathbf{p}}_i \hat{\mathbf{p}}_j + \{\alpha_i, \beta\} \hat{\mathbf{p}}_i m + m^2 \beta^2\right] \psi \\
&= \left[\hat{\mathbf{p}} \cdot \hat{\mathbf{p}} + m^2\right] \psi
\end{aligned}
$$

The coefficients should satisfy

$$
\{\alpha_i, \alpha_j\} = 2\delta_{ij}, \qquad \{\alpha_i, \beta\} = 0, \quad \beta^2 = 1
$$

$$
\boldsymbol{\alpha} = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}
$$

- Lorentz Invariant Form

$$
\left[i \partial_t \gamma^0 - i \partial_i \gamma^i - m\right] \psi = [i \partial_\mu \gamma^\mu - m] \psi = 0
$$

Plane-Wave Solution

- **Plane-wave with definite energy & momentum**

$$\psi(t, \mathbf{x}) = \psi e^{-i(tE - \mathbf{x}\cdot\mathbf{p})} \quad \Rightarrow \quad (\not{p} - m)\psi = 0$$

A free particle behaves as plane wave

- **Equation of Motion**

$$\begin{pmatrix} -m & p \cdot \sigma \\ p \cdot \bar{\sigma} & -m \end{pmatrix} \begin{pmatrix} \chi \\ \xi \end{pmatrix} = 0$$

The left- and right-handed spinors are correlated:

$$\chi = \frac{p \cdot \sigma}{m} \xi, \quad \xi = \frac{p \cdot \bar{\sigma}}{m} \chi$$

The EOM reduces 4 dof to 2 dof.

Plane Wave Solution from Boost & Rotation

- In the rest frame: $(m\gamma_0 - m)\psi_0 = 0$

$$m \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \psi_0 = 0, \quad \Rightarrow \quad \psi_0^s = \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix}$$

  with two spin polarizations $s = \pm 1$.

- Rest frame $\rightarrow$ general case:

$$\psi(\vec{p}) = R_z(\phi) R_y(\theta) B_z(\beta) \psi_0$$

  - $B_z(\beta)$: Boost along z-axis
  - $R_y(\theta)$: Rotate angle θ from z-axis
  - $R_z(\phi)$: Rotate angle ϕ around z-axis

Two Component Theory

- Explicit parametrization

$$u_h(p) = \begin{pmatrix} \omega_{-h}\chi_h \\ \omega_{+h}\chi_h \end{pmatrix} \quad \chi_+ = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) e^{i\phi} \end{pmatrix} \quad \chi_- = \begin{pmatrix} -\sin\left(\frac{\theta}{2}\right) e^{-i\phi} \\ \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

  where $\omega_\pm \equiv \sqrt{E \pm |\mathbf{p}|}$ & $\mathbf{p} = |\mathbf{p}|(\sin\theta\cos\phi, \sin\theta\cos\phi, \cos\theta)$

- Massless limit

$$\omega_- = \sqrt{E - |\mathbf{p}|} \to 0$$

  Only two components of the 4-spinor can survive:

$$u_+(p) = \sqrt{2E} \begin{pmatrix} 0 \\ \chi_+ \end{pmatrix} \quad u_-(p) = \sqrt{2E} \begin{pmatrix} \chi_- \\ 0 \end{pmatrix}$$

  which is exactly the case for neutrinos.

Two-Component Theory & Parity Violation

Parity Nonconservation and a Two-component Theory of the Neutrino†

T. D. LEE
Columbia University, New York, New York
and
C. N. YANG
Institute for Advanced Study, Princeton, New Jersey

(Received January 10, 1957; revised manuscript received January 17, 1957)

Summary

A two-component theory of the neutrino is discussed. The theory is possible only if parity is not conserved in interactions involving the neutrino. Various experimental implications are analyzed. Some general remarks concerning nonconservation are made.

Physical Review, **105**, 5, 1671-5 (1957)

Two-Component Theory & Weyl Spinor

- Only two components of the 4-spinor can survive:

$$u_+(p) = \sqrt{2E}\begin{pmatrix} 0 \\ \chi_+ \end{pmatrix} \qquad u_-(p) = \sqrt{2E}\begin{pmatrix} \chi_- \\ 0 \end{pmatrix}$$

Either left or right handed components $\Rightarrow$ **Parity Violation**.

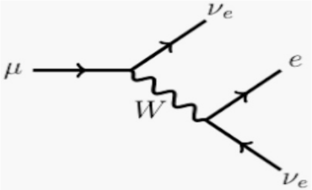
- Spin eigenstates along momentum

$$\begin{aligned} \chi_+ &= \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ +\sin\left(\frac{\theta}{2}\right) e^{i\phi} \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \chi_- &= \begin{pmatrix} -\sin\left(\frac{\theta}{2}\right) e^{-i\phi} \\ \cos\left(\frac{\theta}{2}\right) \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

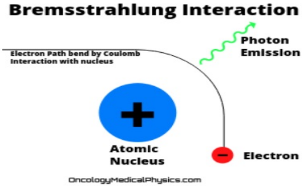
Left-handed spinor has only spin-down polarization & the opposite for right-handed spinor.

Living Particles

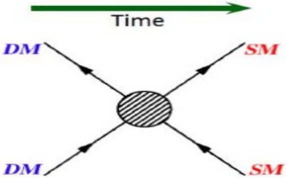
- Decay



- Radiation



- Annihilation



Harmonic Oscillator

- **Hamiltonian**

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$

Dimensionless operators

$$\hat{Q} \equiv \sqrt{\frac{m\omega}{\hbar}}\hat{x}, \quad \hat{P} \equiv \frac{1}{\sqrt{m\omega\hbar}}\hat{p}$$

$$\hat{H} = \frac{1}{2}\hat{\omega}\left[\hat{P}^2 + \hat{Q}^2\right]$$

- **Ladder Operators**

$$a \equiv \frac{1}{\sqrt{2}}(\hat{Q} + i\hat{P}), \qquad a^\dagger \equiv \frac{1}{\sqrt{2}}(\hat{Q} - i\hat{P})$$

$$\hat{H} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) \qquad \hat{H}|n\rangle = \hat{\omega}\left(n + \frac{1}{2}\right)|n\rangle$$

$$a|n\rangle = \sqrt{n}|n-1\rangle \;\&\; a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \Rightarrow a^\dagger a|n\rangle = n|n\rangle.$$

Second Quantization - Scalar as An Example

- **Scalar field**

$$\phi(\mathbf{x},t) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\left(a_\mathbf{p} e^{-ip\cdot x} + a^\dagger_\mathbf{p} e^{+ip\cdot x}\right)$$
$$= \int \frac{d^4 p}{(2\pi)^3}\delta(p^2 - m^2)\left(a_\mathbf{p} e^{-ip\cdot x} + a^\dagger_\mathbf{p} e^{+ip\cdot x}\right)$$

Containing both $a_\mathbf{p}$ & $a^\dagger_\mathbf{p}$ for a scalar particle with $\mathbf{p}$.

- **Commutation relation**

$$\left[a_\mathbf{p}, a^\dagger_{\mathbf{p}'}\right] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}'), \quad \left[a_\mathbf{p}, a_{\mathbf{p}'}\right] = \left[a^\dagger_\mathbf{p}, a^\dagger_{\mathbf{p}'}\right] = 0$$

- **Identical particles**

$$|\phi(\mathbf{p})\rangle = a^\dagger_\mathbf{p}|0\rangle, \quad |\phi(\mathbf{p})\phi(\mathbf{q})\rangle = a^\dagger_\mathbf{p} a^\dagger_\mathbf{q}|0\rangle, \quad \cdots$$

Spin-$1/2$ Fermions

- **Field**

$$\psi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \sum_s \left[a^s_{\mathbf{p}} u^s(p) e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}} v^s(p) e^{ip\cdot x} \right]$$

with $s = \pm 1$ for the two spins

- **Two sets of operators**

$$\begin{cases} \text{Particle:} & a^s_{\mathbf{p}}, a^{s\dagger}_{\mathbf{p}}, \quad |f^s\rangle = a^{s\dagger}|0\rangle, \quad \textbf{positive frequency} \\ \text{Anti-Particle:} & b^s_{\mathbf{p}}, b^{s\dagger}_{\mathbf{p}}, \quad |\bar{f}^s\rangle = b^{s\dagger}|0\rangle, \quad \textbf{negative frequency} \end{cases}$$

satisfying anti-commutation relations:

$$\{a^r_{\mathbf{p}}, a^{s\dagger}_{\mathbf{q}}\} = \{b^r_{\mathbf{p}}, b^{s\dagger}_{\mathbf{q}}\} = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \delta^{rs}$$

In total, there are four degrees of freedom.

Equations of Motions $(i\partial\!\!\!/ - m)\psi = 0$

$$\psi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_\mathbf{p}} \sum_s \left[a^s_\mathbf{p} u^s(p) e^{-ip\cdot x} + b^{s\dagger}_\mathbf{p} v^s(p) e^{ip\cdot x} \right]$$

- **Particle**

$$(p\!\!\!/ - m)u = \begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m \end{pmatrix} u = 0$$

Particle has 2 dof

$$u_h(p) = \begin{pmatrix} \omega_{-h}\chi_h \\ \omega_{+h}\chi_h \end{pmatrix}$$

- **Anti-Particle**

$$(p\!\!\!/ + m)v = \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix} v = 0$$

Anti-particle also has 2 dof

$$v_h(p) = \begin{pmatrix} -h\omega_{+h}\chi_{-h} \\ +h\omega_{-h}\chi_{-h} \end{pmatrix}$$

Creation & Annihilation in Pair

Since particle & anti-particle are described by different operators, why they can be created & annihilated in pair?

- **Interaction term should be Lorentz invariant**

$$\overline{\psi}\mathcal{O}\psi \in \mathcal{L}$$

Spin $\frac{1}{2} \times \frac{1}{2} = 0 + 1$ could be invariant (0).

- **Creation**

$$\langle f \bar{f} | \overline{\psi}\mathcal{O}\psi | 0 \rangle = \langle 0 | ab(\bar{u}a^\dagger + \bar{v}b)\mathcal{O}(ua + vb^\dagger)|0\rangle = \bar{u}\mathcal{O}v$$

- **Annihilation**

$$\langle 0 | \overline{\psi}\mathcal{O}\psi | f \bar{f} \rangle = \langle 0 | (\bar{u}a^\dagger + \bar{v}b)\mathcal{O}(ua + vb^\dagger)a^\dagger b^\dagger |0\rangle = \bar{v}\mathcal{O}u$$

For others

$$\langle ff, \bar{f}\bar{f} | \overline{\psi}\mathcal{O}\psi | 0 \rangle = \langle 0 | \overline{\psi}\mathcal{O}\psi | ff, \bar{f}\bar{f} \rangle = 0$$

Charge Conjugation

- **If there is charge**, $\psi \to e^{i\alpha(x)Q}\psi$:

$$a^s_{\mathbf{p}} \to e^{i\alpha(x)Q} a^s_{\mathbf{p}}, \qquad b^s_{\mathbf{p}} \to e^{-i\alpha(x)Q} b^s_{\mathbf{p}},$$

  Particle & anti-particle have opposite charges.

- **Use charge conjugation to relate particle with anti-particle?**

$$\mathbb{C} a^s_{\mathbf{p}} \mathbb{C} = b^s_{\mathbf{p}}, \qquad \mathbb{C} b^s_{\mathbf{p}} \mathbb{C} = a^s_{\mathbf{p}}$$

  Spinor transforms as

$$\begin{aligned} \psi(\mathbf{p}) &= \sum_s \left[a^s_{\mathbf{p}} u^s(p) e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}} v^s(p) e^{ip\cdot x} \right] \\ \mathbb{C}\psi(\mathbf{p})\mathbb{C} &= \sum_s \left[b^s_{\mathbf{p}} u^s(p) e^{-ip\cdot x} + a^{s\dagger}_{\mathbf{p}} v^s(p) e^{ip\cdot x} \right] \end{aligned}$$

Charge Conjugation of Fields

- Relation among spinors

$$u^s(p) = -i\gamma^2 (v^s(p))^*, \quad v^s(p) = -i\gamma^2 (u^s(p))^*$$

- Spinor transformations

$$\begin{aligned}
\psi(\mathbf{p}) &= \sum_s \left[a^s_{\mathbf{p}} u^s(p) e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}} v^s(p) e^{ip\cdot x} \right] \\
\mathbb{C}\psi(\mathbf{p})\mathbb{C} &= -i\gamma^2 \sum_s \left[b^s_{\mathbf{p}} (v^s(p))^* e^{-ip\cdot x} + a^{s\dagger}_{\mathbf{p}} (u^s(p))^* e^{ip\cdot x} \right] \\
&= -i\gamma^2 \psi^*(\mathbf{p}) \\
&\equiv \psi^c(\mathbf{p})
\end{aligned}$$

Dirac vs Majorana Fields

- Definition: **Majorana field is its own charge conjugation!**

$$\psi(\mathbf{p}) = \psi^c(\mathbf{p})$$

- Reconstruction in Dirac field

$$\psi^D(\mathbf{p}) = \sum_s \left[a^s_{\mathbf{p}} u^s(p) e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}} v^s(p) e^{ip\cdot x} \right]$$

Decomposition: $\psi^{\pm}_M(\mathbf{p}) \equiv \psi(\mathbf{p}) \pm \psi^c(\mathbf{p})$, each with 2 dof

$$\psi^M_{\pm}(\mathbf{p}) = \sum_s \left[(a^s_{\mathbf{p}} \pm b^s_{\mathbf{p}}) u^s(p) e^{-ip\cdot x} + (a^s_{\mathbf{p}} \pm b^s_{\mathbf{p}})^\dagger v^s(p) e^{ip\cdot x} \right]$$

Basically the difference appears in the operator part.

Massless limit

- Dirac vs Majorana

$$\psi^D(\mathbf{p}) = \sum_s \left[a^s_{\mathbf{p}} u^s(p) e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}} v^s(p) e^{ip\cdot x}\right]$$

$$\psi^M(\mathbf{p}) = \sum_s \left[a^s_{\mathbf{p}} u^s(p) e^{-ip\cdot x} + a^{s\dagger}_{\mathbf{p}} v^s(p) e^{ip\cdot x}\right]$$

- Massless limit

$$u_+(p) = \sqrt{2E}\begin{pmatrix} 0 \\ \chi_+ \end{pmatrix} \qquad u_-(p) = \sqrt{2E}\begin{pmatrix} \chi_- \\ 0 \end{pmatrix}$$

$$v_+(p) = -\sqrt{2E}\begin{pmatrix} \chi_- \\ 0 \end{pmatrix} \qquad v_-(p) = -\sqrt{2E}\begin{pmatrix} 0 \\ \chi_+ \end{pmatrix}$$

Is Majorana fermion its own antiparticle?

- Left vs Right

$$\psi_L^M(\mathbf{p}) = \sqrt{2E}\sum_s \left[a_{\mathbf{p}}^{-} e^{-ip\cdot x} - a_{\mathbf{p}}^{+\dagger} e^{ip\cdot x}\right] \begin{pmatrix} \chi_- \\ 0 \end{pmatrix}$$

$$\psi_R^M(\mathbf{p}) = \sqrt{2E}\sum_s \left[a_{\mathbf{p}}^{+} e^{-ip\cdot x} - a_{\mathbf{p}}^{-\dagger} e^{ip\cdot x}\right] \begin{pmatrix} 0 \\ \chi_+ \end{pmatrix}$$

Although the spinor seems the same, the particle and anti-particle are associated with different spin/helicity.

Majorana particle is NOT its own anti-particle!

Thank You!