GKZ hypergeometric systems of four-loop vacuum Feynman integrals

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第二届粒子物理、天体物理和宇宙学前沿课题研讨会

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河北大学 HEBEI UNIVERSITY

Contents

I. Introduction

1. Background

- Higher-order corrections are more important, with the increasing precision at the future colliders: CLIC, ILC, CEPC, FCC, HL-LHC, STCF, SKEKB, $\cdots$
- Vacuum integrals are the important subsets of Feynman integrals, which constitute a main building block in asymptotic expansions of Feynman integrals. The calculation of multi-loop vacuum integrals is a good breakthrough window in the calculation of multi-loop Feynman integrals.
- Considering Feynman integrals as the generalized hypergeometric functions, one finds that the $D-$module of a Feynman diagram is isomorphic to Gel'fand-Kapranov-Zelevinsky (GKZ) $D-$module.

I. Introduction

2. Relevant research

- Hypergeometric functions of some Feynman integrals are obtained from Mellin-Barnes representations.

  Feng, Chang, Chen, Gu, Zhang, *NPB* 927(2018)516 [arXiv:1706.08201]
  Feng, Chang, Chen, Zhang, *NPB* 940(2019)130 [arXiv:1809.00295]
  Gu, Zhang, *CPC* 43(2019)083102 [arXiv:1811.10429]
  Gu, Zhang, Feng, *IJMPA* 35(2020)2050089.

- Using GKZ hypergeometric system, we can obtain the fundamental solution systems of Feynman integrals.

  Feng, Chang, Chen, Zhang, *NPB* 953(2020)114952, [arXiv:1912.01726]
  Feng, Zhang, Chang, *PRD* 106(2022)116025 [arXiv: 2206.04224]
  Feng, Zhang, Dong, Zhou, *EPJC* 83(2023)314 [arXiv:2209.15194].
  Zhang, Feng, *JHEP* 05(2023)075 [arXiv: 2303.02795].
  Zhang, Feng, [arXiv: 2403.13025].

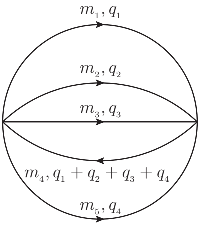
I. Introduction

3. Generally strategy

- We can derive GKZ hypergeometric systems of Feynman integrals, basing on Mellin-Barnes representations and Miller's transformation. We can formulate Feynman integrals as hypergeometric functions through GKZ hypergeometric systems.
- Steps: (1) we write out the GKZ hypergeometric systems satisfied by the Feynman integrals. (2) fundamental solution systems are constructed in neighborhoods of regular singularities of the GKZ hypergeometric systems. The combination coefficients can be determined from Feynman integrals with some special kinematic parameters.

II. 4-loop vacuum: general case

1. 4-loop vacuum with 5 propagates



- Feynman integral of 4-loop vacuum with 5 propagates:

$$U_5 = \left(\Lambda^2_{\rm RE}\right)^{8-2D} \int \frac{d^D\mathbf{q}}{(2\pi)^D} \frac{1}{(q_1^2 - m_1^2)(q_2^2 - m_2^2)(q_3^2 - m_3^2)} \times \frac{1}{[(q_1+q_2+q_3+q_4)^2 - m_4^2](q_4^2 - m_5^2)} \,, \tag{2.1}$$

II. 4-loop vacuum: general case

- Through Mellin-Barnes transformation

$$U_5 = \frac{\left(\Lambda^2_{\rm RE}\right)^{8-2D}}{(2\pi i)^4} \int_{-i\infty}^{+i\infty} d\mathbf{s} \Big[\prod_{i=1}^{4} (-m_i^2)^{s_i} \Gamma(-s_i) \Gamma(1+s_i) \Big] I_q \;, \qquad (2.2)$$

where

$$I_q \equiv \int \frac{d^D\mathbf{q}}{(2\pi)^D} \frac{1}{(q_1^2)^{1+s_1}(q_2^2)^{1+s_2}(q_3^2)^{1+s_3}[(q_1+q_2+q_3+q_4)^2]^{1+s_4}(q_4^2-m_5^2)} \,. \qquad (2.3)$$

II. 4-loop vacuum: general case

- Mellin-Barnes representation of the Feynman integral:

$$U_5 = \frac{-m_5^6}{(2\pi i)^4(4\pi)^8}\left(\frac{4\pi\Lambda_{\rm RE}^2}{m_5^2}\right)^{8-2D}\int_{-i\infty}^{+i\infty} d\mathbf{s}\Big[\prod_{i=1}^{4}\left(\frac{m_i^2}{m_5^2}\right)^{s_i}\Gamma(-s_i)\Big]$$
$$\times\Big[\prod_{i=1}^{4}\Gamma(\frac{D}{2}-1-s_i)\Big]\Gamma(4-\frac{3D}{2}+\sum_{i=1}^{4}s_i)\Gamma(5-2D+\sum_{i=1}^{4}s_i) \quad (2.4)$$

- It is well known that negative integers and zero are simple poles of the function $\Gamma(z)$. As all s_i contours are closed to the right in corresponding complex planes, one finds that the analytic expression of the the four-loop vacuum integral can be written as the linear combination of generalized hypergeometric functions.

II. 4-loop vacuum: general case

$$U_5 \ni \frac{-m_5^6}{(4\pi)^8}\left(\frac{4\pi\Lambda_{\rm RE}^2}{m_5^2}\right)^{8-2D} \frac{\pi^4}{\sin^4\frac{\pi D}{2}} T_5\left(\mathbf{a},\ \mathbf{b}\ \middle|\ \mathbf{x}\right) , \tag{2.5}$$

$$T_5\left(\mathbf{a},\ \mathbf{b}\ \middle|\ \mathbf{x}\right) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty} A_{n_1n_2n_3n_4} x_1^{n_1}x_2^{n_2}x_3^{n_3}x_4^{n_4} , \tag{2.6}$$

$$A_{n_1n_2n_3n_4} = \frac{\Gamma(a_1+\sum_{i=1}^{4}n_i)\Gamma(a_2+\sum_{i=1}^{4}n_i)}{\prod_{i=1}^{4}n_i!\Gamma(b_i+n_i)} . \tag{2.7}$$

where $x_i = m_i^2/m_5^2$, $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2, b_3, b_4)$ with

$$a_1 = 4 - \frac{3D}{2},\ \ a_2 = 5 - 2D,\ \ b_1 = b_2 = b_3 = b_4 = 2 - \frac{D}{2} , \tag{2.8}$$

II. 4-loop vacuum: general case

- We can define auxiliary function

$$\Phi_5(\mathbf{a},\ \mathbf{b}\ \Big|\ \mathbf{x},\ \mathbf{u},\ \mathbf{v}) = \mathbf{u}^{\mathbf{a}}\mathbf{v}^{\mathbf{b}-\mathbf{e}_4} T_5(\mathbf{a},\ \mathbf{b}\ \Big|\ \mathbf{x})\ , \qquad (2.9)$$

Through Miller's transformation,

$$\vartheta_{u_j}\Phi_5(\mathbf{a},\ \mathbf{b}\ \Big|\ \mathbf{x},\ \mathbf{u},\ \mathbf{v}) = a_j\Phi_5(\mathbf{a},\ \mathbf{b}\ \Big|\ \mathbf{x},\ \mathbf{u},\ \mathbf{v})\ ,\ (j=1,2)\ ,$$
$$\vartheta_{v_k}\Phi_5(\mathbf{a},\ \mathbf{b}\ \Big|\ \mathbf{x},\ \mathbf{u},\ \mathbf{v}) = (b_k-1)\Phi_5(\mathbf{a},\ \mathbf{b}\ \Big|\ \mathbf{x},\ \mathbf{u},\ \mathbf{v})\ ,\ (k=1,\cdots,4)\ , \qquad (2.10)$$

which naturally induces the notion of GKZ hypergeometric system. Euler operators: $\vartheta_{x_k} = x_k\partial_{x_k}$.

II. 4-loop vacuum: general case

- Through the transformation

$$z_j = \frac{1}{u_j},\ (j=1,2),\quad z_{2+k} = v_k,\ z_{6+k} = \frac{x_k}{u_1 u_2 v_k}, \tag{2.11}$$

we have GKZ hypergeometric system for the integral

$$\mathbf{A_5} \cdot \vec{\vartheta}_5 \Phi_5 = \mathbf{B_5} \Phi_5\ , \tag{2.12}$$

$$\mathbf{A_5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}_{6\times 10},$$

$$\vec{\vartheta}_5^{\,T} = (\vartheta_{z_1},\ \cdots,\ \vartheta_{z_{10}})\ ,$$

$$\mathbf{B_5}^T = (-a_1,\ -a_2,\ b_1-1,\ b_2-1,\ b_3-1,\ b_4-1)\ .$$

II. 4-loop vacuum: general case

- Defining the combined variables

$$y_1 = \frac{z_3 z_7}{z_1 z_2} \,, \quad y_2 = \frac{z_4 z_8}{z_1 z_2} \,, \quad y_3 = \frac{z_5 z_9}{z_1 z_2} \,, \quad y_4 = \frac{z_6 z_{10}}{z_1 z_2} \,, \qquad (2.13)$$

we write the solutions as

$$\Phi_5(\mathbf{z}) = \Big(\prod_{i=1}^{10} z_i^{\alpha_i}\Big) \varphi_5(y_1, \ y_2, \ y_3, \ y_4) \,. \qquad (2.14)$$

Here $\vec{\alpha}^{\,T} = (\alpha_1, \ \alpha_2, \ \cdots \ , \ \alpha_{10})$ denotes a sequence of complex number such that

$$\mathbf{A_5} \cdot \vec{\alpha} = \mathbf{B_5} \ . \qquad (2.15)$$

II. 4-loop vacuum: general case

- Correspondingly the dual matrix $\tilde{\mathbf{A}}_5$ of $\mathbf{A}_5$ is

$$\tilde{\mathbf{A}}_5 = \begin{pmatrix} -1 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.16)$$

The row vectors of the matrix $\tilde{\mathbf{A}}_5$ induce the integer sublattice $\mathbf{B}$ which can be used to construct the formal solutions in hypergeometric series.

- We denote the submatrix composed of the first, third, fourth and fifth column vectors of the dual matrix as $\tilde{\mathbf{A}}_{1345}$

$$\tilde{\mathbf{A}}_{1345} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix} . \quad (2.17)$$

II. 4-loop vacuum: general case

- Obviously $\det \tilde{\mathbf{A}}_{1345} = 1 \neq 0$, and

$$\mathbf{B}_{1345} = \tilde{\mathbf{A}}_{1345}^{-1} \cdot \tilde{\mathbf{A}}_{5}$$
$$= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 \end{pmatrix} \quad (2.18)$$

Taking 4 row vectors of the matrix $\mathbf{B}_{1345}$ as the basis of integer lattice, one constructs the GKZ hypergeometric series solutions in parameter space through choosing the sets of column indices $I_i \subset [1, 10]$ $(i = 1, \cdots, 16)$ which are consistent with the basis of integer lattice $\mathbf{B}_{1345}$.

II. 4-loop vacuum: general case

- We take the set of column indices $I_1 = [2, 6, \cdots, 10]$, i.e. the implement $J_1 = [1, 10] \setminus I_1 = [1, 3, 4, 5]$. The choice on the set of indices implies the exponent numbers $\alpha_1 = \alpha_3 = \alpha_4 = \alpha_5 = 0$. One can have

$$\alpha_2 = a_1 - a_2,\ \alpha_6 = \sum_{i=1}^{4} b_i - a_1 - 4,\ \alpha_7 = 1 - b_1,$$
$$\alpha_8 = 1 - b_2,\ \alpha_9 = 1 - b_3,\ \alpha_{10} = \sum_{i=1}^{3} b_i - a_1 - 3\ . \quad (2.19)$$

Combined with Eq. (2.8), we can have

$$\alpha_2 = \frac{D}{2} - 1,\ \alpha_6 = -\frac{D}{2},\ \alpha_7 = \alpha_8 = \alpha_9 = \frac{D}{2} - 1,\ \alpha_{10} = -1\ . \quad (2.20)$$

II. 4-loop vacuum: general case

- According the basis of integer lattice $\mathbf{B}_{1345}$, the hypergeometric series solution can be

$$\Phi^{(1)}_{[1345]} = y_1^{\frac{D}{2}-1} y_2^{\frac{D}{2}-1} y_3^{\frac{D}{2}-1} y_4^{-1} \sum_{\mathbf{n}=0}^{\infty} c^{(1)}_{[1345]}(\mathbf{n}) \left(\frac{1}{y_4}\right)^{n_1} \left(\frac{y_1}{y_4}\right)^{n_2} \left(\frac{y_2}{y_4}\right)^{n_3} \left(\frac{y_3}{y_4}\right)^{n_4} ,$$

$$c^{(1)}_{[1345]}(\mathbf{n}) = \frac{\Gamma(\frac{D}{2} + \sum_{i=1}^{4} n_i)\Gamma(1 + \sum_{i=1}^{4} n_i)}{\prod_{i=1}^{4} n_i!\Gamma(\frac{D}{2} + n_i)} . \quad (2.21)$$

Here, the convergent region is

$$\Xi_{[1345]} = \{(y_1, y_2, y_3, y_4) \Big| 1 < |y_4|, |y_1| < |y_4|, |y_2| < |y_4|, |y_3| < |y_4|\} ,$$

which shows that $\Phi^{(1)}_{[1345]}(\alpha, z)$ is in neighborhood of regular singularity ∞.

II. 4-loop vacuum: general case

- In a similar way, we can obtain other fifteen hypergeometric solutions which are consistent with the basis of integer lattice $\mathbf{B}_{1345}$, and the convergent region is also $\Xi_{[1345]}$.
- The above sixteen hypergeometric series solutions $\Phi^{(i)}_{[1345]}(\alpha, z)$ whose convergent region is $\Xi_{[1345]}$ can constitute a fundamental solution system.
- Multiplying one of the row vectors of the matrix $\mathbf{B}_{1345}$ by -1, the induced integer matrix can also be chosen as a basis of the integer lattice space of certain hypergeometric series.

II. 4-loop vacuum: general case

- Taking 4 row vectors of the following matrix as the basis of integer lattice,

$$\mathbf{B}_{\tilde{1}345} = \text{diag}(-1,1,1,1)\cdot\mathbf{B}_{1345} = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 \end{pmatrix} \quad (2.22)$$

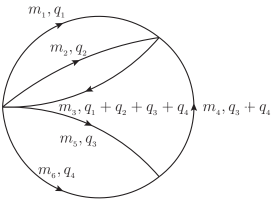
one obtains sixteen hypergeometric series solutions $\Phi^{(i)}_{[\tilde{1}345]}(\alpha, z)$ similarly. The convergent region is

$$\Xi_{[\tilde{1}345]} = \{(y_1,\ y_2,\ y_3,\ y_4)\Big|\,|y_1|<1,\ |y_2|<1,\ |y_3|<1,\ |y_4|<1\}\ ,$$

which shows that $\Phi^{(i)}_{[\tilde{1}345]}(\alpha, z)$ are in neighborhood of regular singularity 0 and can constitute a fundamental solution system.

II. 4-loop vacuum: general case

2. 4-loop vacuum with 6 propagates for type A



- Feynman integral of 4-loop vacuum with 6 propagates A:

$$U_{6A} = \left(\Lambda^2_{\rm RE}\right)^{8-2D} \int \frac{d^D\mathbf{q}}{(2\pi)^D} \frac{1}{(q_1^2 - m_1^2)(q_2^2 - m_2^2)[(q_1 + q_2 + q_3 + q_4)^2 - m_3^2]} \times \frac{1}{[(q_3 + q_4)^2 - m_4^2](q_3^2 - m_5^2)(q_4^2 - m_6^2)} . \tag{2.23}$$

II. 4-loop vacuum: general case

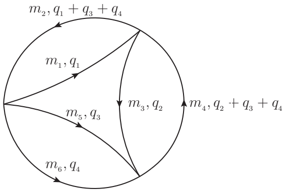
GKZ hypergeometric system for the 4-loop vacuum integral A

$$\mathbf{A_{6A}} \cdot \vec{\vartheta}_{6A} \Phi_{6A} = \mathbf{B_{6A}} \Phi_{6A} \,, \tag{2.24}$$

$$\begin{aligned}
\mathbf{A_{6A}} &= \begin{pmatrix} \mathbf{I_{11\times 11}} & \mathbf{A_{X6A}} \end{pmatrix}_{11\times 16} \,, \\
\mathbf{A_{X6A}}^T &= \begin{pmatrix}
1 & 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & -1 \\
1 & 1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & -1 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0
\end{pmatrix} \,, \\
\vec{\vartheta}_{6A}^{\,T} &= (\vartheta_{z_1}, \cdots, \vartheta_{z_{16}}) \,, \\
\mathbf{B_{6A}}^T &= (-a_1, \cdots, -a_5, b_1 - 1, \cdots, b_6 - 1) \,.
\end{aligned} \tag{2.25}$$

II. 4-loop vacuum: general case

3. 4-loop vacuum with 6 propagates for type B



- Feynman integral of 4-loop vacuum with 6 propagates B:

$$U_{6B} = \left(\Lambda^2_{\rm RE}\right)^{8-2D} \int \frac{d^D\mathbf{q}}{(2\pi)^D} \frac{1}{(q_1^2 - m_1^2)[(q_1 + q_3 + q_4)^2 - m_2^2]} \times \frac{1}{(q_2^2 - m_3^2)[(q_2 + q_3 + q_4)^2 - m_4^2](q_3^2 - m_5^2)(q_4^2 - m_6^2)} . \quad (2.26)$$

II. 4-loop vacuum: general case

GKZ hypergeometric system for the 4-loop vacuum integral B

$$\mathbf{A_{6B}}\cdot\vec{\vartheta}_{6B}\Phi_{6B}=\mathbf{B_{6B}}\Phi_{6B}\;, \tag{2.27}$$

$$\mathbf{A_{6B}}=\left(\begin{array}{cc}\mathbf{I_{13\times13}} & \mathbf{A_{X6B}}\end{array}\right)_{13\times18}\;,$$

$$\mathbf{A_{X6B}}^T=$$

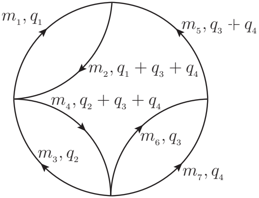
$$\left(\begin{array}{ccccccccccccc}
1 & 1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1\\
1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1\\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & -1\\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & -1\\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0
\end{array}\right),$$

$$\vec{\vartheta}_{6B}^{\;T}=(\vartheta_{z_1},\cdots,\;\vartheta_{z_{18}})\;,$$

$$\mathbf{B_{6B}}^T=(-a_1,\;\cdots,\;-a_6,\;b_1-1,\;\cdots,\;b_7-1)\;. \tag{2.28}$$

II. 4-loop vacuum: general case

4. 4-loop vacuum with 7 propagates for type A



- Feynman integral of 4-loop vacuum with 7 propagates A:

$$U_{7A} = \left(\Lambda^2_{\rm RE}\right)^{8-2D} \int \frac{d^D\mathbf{q}}{(2\pi)^D} \frac{1}{(q_1^2 - m_1^2)[(q_1 + q_3 + q_4)^2 - m_2^2](q_2^2 - m_3^2)}$$
$$\times \frac{1}{[(q_2 + q_3 + q_4)^2 - m_4^2][(q_3 + q_4)^2 - m_5^2](q_3^2 - m_6^2)(q_4^2 - m_7^2)}. \quad (2.29)$$

II. 4-loop vacuum: general case

GKZ hypergeometric system for the 4-loop vacuum integral

$$\mathbf{A_{7A}} \cdot \vec{\vartheta}_{7A} \Phi_{7A} = \mathbf{B_{7A}} \Phi_{7A} \, , \tag{2.30}$$

$$\mathbf{A_{7A}} = \left(\begin{array}{cc} \mathbf{I_{14\times 14}} & \mathbf{A_{X7A}} \end{array} \right)_{14\times 20} \, ,$$

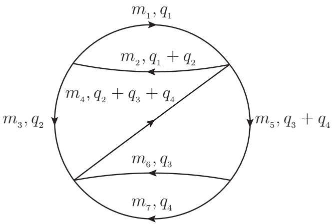
$$\mathbf{A_{X7A}}^T = \left(\begin{array}{cccccccccccccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{array} \right),$$

$$\vec{\vartheta}_{7A}^{\,T} = (\vartheta_{z_1}, \cdots, \vartheta_{z_{20}}) \, ,$$

$$\mathbf{B_{7A}}^T = (-a_1, \cdots, -a_7, b_1 - 1, \cdots, b_7 - 1) \, . \tag{2.31}$$

II. 4-loop vacuum: general case

5. 4-loop vacuum with 7 propagates for type B



- Feynman integral of 4-loop vacuum with 7 propagates B:

$$U_{7B} = \left(\Lambda^2_{\rm RE}\right)^{8-2D} \int \frac{d^D\mathbf{q}}{(2\pi)^D} \frac{1}{(q_1^2 - m_1^2)[(q_1 + q_2)^2 - m_2^2](q_2^2 - m_3^2)} \times \frac{1}{[(q_2 + q_3 + q_4)^2 - m_4^2][(q_3 + q_4)^2 - m_5^2](q_3^2 - m_6^2)(q_4^2 - m_7^2)}. \quad (2.32)$$

II. 4-loop vacuum: general case

GKZ hypergeometric system for the 4-loop vacuum integral

$$\mathbf{A_{7B}} \cdot \vec{\vartheta}_{7B} \Phi_{7B} = \mathbf{B_{7B}} \Phi_{7B} \;, \tag{2.33}$$

$$\mathbf{A_{7B}} = \begin{pmatrix} \mathbf{I}_{16\times 16} & \mathbf{A_{X7B}} \end{pmatrix}_{16\times 22} \;,$$

$$\mathbf{A_{X7B}}^{T} =$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\vec{\vartheta}_{7B}^{\,T} = (\vartheta_{z_1}, \cdots, \vartheta_{z_{22}}) \;,$$

$$\mathbf{B_{7B}}^{T} = (-a_1, \cdots, -a_8, b_1 - 1, \cdots, b_8 - 1) \;. \tag{2.34}$$

III. 4-loop vacuum: special case

1. 4-loop vacuum with 5 propagates: 2 massive $m_1, m_5 \neq 0$

- GKZ hypergeometric system can be simplified as

$$\mathbf{A_{51}} \cdot \vec{\vartheta}_{51} \Phi_{51} = \mathbf{B_5} \Phi_{51} \ , \tag{3.1}$$

$$\mathbf{A}_{51} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} . \tag{3.2}$$

$$\vec{\vartheta}_{51}^{\,T} = (\vartheta_{z_1}, \cdots, \vartheta_{z_7}) \ , \tag{3.3}$$

III. 4-loop vacuum: special case

1. 4-loop vacuum with 5 propagates：2 massive $m_1, m_5 \neq 0$

- The dual matrix $\tilde{\mathbf{A}}_{51}$ of $\mathbf{A}_{51}$ is

$$\tilde{\mathbf{A}}_{51} = \begin{pmatrix} -1 & -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{3.4}$$

- For integer sublattice $\mathbf{B}_{51} = \tilde{\mathbf{A}}_{51}$, one of corresponding hypergeometric series solution can be written as

$$\Phi^{(1)}_{[51]}(y_1) = {}_2F_1\left(\begin{array}{cc} 4-\frac{3D}{2}, & 5-2D \\ & 2-\frac{D}{2} \end{array} \middle| y_1 \right) , \tag{3.5}$$

with $y_1 = x_1 = m_1^2/m_5^2$, and ${}_2F_1$ is Gauss function:

$${}_2F_1\left(\begin{array}{cc} a, & b \\ & c \end{array} \middle| x \right) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(c)_n} x^n , \tag{3.6}$$

with $(a)_n = \Gamma(a+n)/\Gamma(a)$.

III. 4-loop vacuum: special case

1. 4-loop vacuum with 5 propagates：2 massive $m_1, m_5 \neq 0$

- For integer sublattice $\mathbf{B}_{51} = \tilde{\mathbf{A}}_{51}$, the another hypergeometric series solution can be written as

$$\Phi^{(2)}_{[51]}(y_1) = (y_1)^{D/2-1} \, {}_2F_1 \left(\begin{array}{c} 3-D, \quad 4-\frac{3D}{2} \\ \frac{D}{2} \end{array} \middle| \, y_1 \right) . \quad (3.7)$$

Here, the convergent region of $\Phi^{1,2}_{[51]}(y_1)$ is $|y_1| < 1$.

- In the region $|y_1| < 1$, the scalar integral is a linear combination of two fundamental solutions:

$$\Phi_{51}(y_1) = C^{(1)}_{[51]} \Phi^{(1)}_{[51]}(y_1) + C^{(2)}_{[51]} \Phi^{(2)}_{[51]}(y_1) . \quad (3.8)$$

III. 4-loop vacuum: special case

1. 4-loop vacuum with 5 propagates： 2 massive $m_1, m_5 \neq 0$

- Multiplying one of the row vectors of the integer matrix $\mathbf{B}_{51}$ by -1, the corresponding system of fundamental solutions for is similarly composed by two Gauss functions:

$$\Phi^{(3)}_{[51]}(y_1) = (y_1)^{\frac{3D}{2}-4}\; {}_2F_1\left(\begin{array}{c} 4-\frac{3D}{2},\;\; 3-D \\ \frac{D}{2} \end{array}\middle|\, \frac{1}{y_1}\right)\,,$$

$$\Phi^{(4)}_{[51]}(y_1) = (y_1)^{2D-5}\; {}_2F_1\left(\begin{array}{c} 5-2D,\;\; 4-\frac{3D}{2} \\ 2-\frac{D}{2} \end{array}\middle|\, \frac{1}{y_1}\right)\,, \quad (3.9)$$

  which the convergent region is $|y_1| > 1$.

- In the region $|y_1| > 1$, the scalar integral is a linear combination of two fundamental solutions:

$$\Phi_{51}(y_1) = C^{(3)}_{[51]}\Phi^{(3)}_{[51]}(y_1) + C^{(4)}_{[51]}\Phi^{(4)}_{[51]}(y_1)\,. \quad (3.10)$$

III. 4-loop vacuum: special case

1. 4-loop vacuum with 5 propagates：2 massive $m_1, m_5 \neq 0$

- As $m_1^2 \ll m_5^2$, $m_2 = m_3 = m_4 = 0$, $I_1 = I_{1,0} + \cdots$, where

$$I_{1,0} = \left(\Lambda_{\rm RE}^2\right)^{8-2D} \int \frac{d^D\mathbf{q}}{(2\pi)^D} \frac{1}{q_1^2 q_2^2 q_3^2 (q_1 + q_2 + q_3 + q_4)^2 (q_3^2 - m_5^2)}$$
$$= \frac{-m_5^6}{(4\pi)^8} \left(\frac{4\pi\Lambda_{\rm RE}^2}{m_5^2}\right)^{8-2D} \Gamma(4 - \frac{3D}{2})\Gamma(5 - 2D) \; . \qquad (3.11)$$

This result indicates

$$C_{[51]}^{(1)} = \frac{-m_5^6}{(4\pi)^8} \left(\frac{4\pi\Lambda_{\rm RE}^2}{m_5^2}\right)^{8-2D} \Gamma(4 - \frac{3D}{2})\Gamma(5 - 2D) \; . \qquad (3.12)$$

III. 4-loop vacuum: special case

1. 4-loop vacuum with 5 propagates： 2 massive $m_1, m_5 \neq 0$

- As $m_1^2 \gg m_5^2$ and $m_2 = m_3 = m_4 = 0$, $I_1 = I_{1,\infty} + \cdots$, where

$$I_{1,\infty} = \left(\Lambda_{\rm RE}^2\right)^{8-2D} \int \frac{d^D\mathbf{q}}{(2\pi)^D} \frac{1}{(q_1^2 - m_1^2)q_2^2 q_3^2 (q_1 + q_2 + q_3 + q_4)^2 q_3^2}$$

$$= \frac{-m_1^6}{(4\pi)^8} \left(\frac{4\pi\Lambda_{\rm RE}^2}{m_1^2}\right)^{8-2D} \Gamma(4 - \frac{3D}{2})\Gamma(5 - 2D) \ . \qquad (3.13)$$

This result indicates

$$C_{[51]}^{(4)} = \frac{-m_5^6}{(4\pi)^8} \left(\frac{4\pi\Lambda_{\rm RE}^2}{m_5^2}\right)^{8-2D} \Gamma(4 - \frac{3D}{2})\Gamma(5 - 2D) \ . \qquad (3.14)$$

III. 4-loop vacuum: special case

1. 4-loop vacuum with 5 propagates： 2 massive $m_1, m_5 \neq 0$

- Mellin-Barnes representation

$$U_5 = \frac{-m_5^6}{(4\pi)^8}\Big(\frac{4\pi\Lambda_{\rm RE}^2}{m_5^2}\Big)^{8-2D}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}ds_1\Big(\frac{m_1^2}{m_5^2}\Big)^{s_1}\Gamma(-s_1)$$
$$\times\Gamma(\frac{D}{2}-1-s_1)\Gamma(4-\frac{3D}{2}+s_1)\Gamma(5-2D+s_1)\,. \qquad (3.15)$$

The residue of simple pole of $\Gamma(-s_1)$ provides $C_{[51]}^{(1)}\Phi_{[51]}^{(1)}(y_1)$, that of simple pole of $\Gamma(D/2-1-s_1)$ provides $C_{[51]}^{(2)}\Phi_{[51]}^{(2)}(y_1)$, that of simple pole of $\Gamma(4-\frac{3D}{2}+s_1)$ provides $C_{[51]}^{(3)}\Phi_{[51]}^{(3)}(y_1)$, that of simple pole of $\Gamma(5-2D+s_1)$ provides $C_{[51]}^{(4)}\Phi_{[51]}^{(4)}(y_1)$.

$$C_{[51]}^{(2)} = C_{[51]}^{(3)} = \frac{-m_5^6}{(4\pi)^8}\Big(\frac{4\pi\Lambda_{\rm RE}^2}{m_5^2}\Big)^{8-2D}\Gamma(4-\frac{3D}{2})\Gamma(3-D)\Gamma(1-\frac{D}{2})\,. \qquad (3.16)$$

III. 4-loop vacuum: special case

2. 4-loop vacuum with 6 propagates： 3 massive

- Type A: i9-13th, 64GB: FeynGKZ $\sim$ 0.1 s, FIESTA$\sim$ 1500 s

```
SumLim = 15;
ParameterSub = {De → 4 - 2 × 0.001, ϵ → 0.001, a1 → 1, a2 → 1, a3 → 1,
   a4 → 1, a5 → 1, a6 → 1, m1 → 0.01, m2 → 0.1 , m6 → 10};
NumericalSum[SeriesSolution, ParameterSub, SumLim];

Numerical result = 1.2624 × 10^11
Time Taken 0.059701 seconds
```

```
SumLim = 15;
ParameterSub = {De → 4 - 2 × 0.001, ϵ → 0.001, a1 → 1, a2 → 1, a3 → 1,
   a4 → 1, a5 → 1, a6 → 1, m1 → 0.01, m2 → 0.1 , m6 → 10 };
FIESTAEvaluate[MomentumRep, LoopMomenta, InvariantList, ParameterSub];

FIESTA Value = 1.26241 × 10^11
Time Taken 1525.73 seconds
```

- Type B: i9-13th, 64GB: FeynGKZ $\sim$ 0.1 s, FIESTA$\sim$ 500 s

```
SumLim = 15;
ParameterSub = {De → 4 - 2 × 0.001, ϵ → 0.001, a1 → 1, a2 → 1, a3 → 1,
   a4 → 1, a5 → 1, a6 → 1, m1 → 0.1, m2 → 0.2 , m6 → 10};
NumericalSum[SeriesSolution, ParameterSub, SumLim];

Numerical result = -4.07562 × 10^11
Time Taken 0.092314 seconds
```

```
SumLim = 15;
ParameterSub = {De → 4 - 2 × 0.001, ϵ → 0.001, a1 → 1, a2 → 1, a3 → 1,
   a4 → 1, a5 → 1, a6 → 1, m1 → 0.1, m2 → 0.2 , m6 → 10 };
FIESTAEvaluate[MomentumRep, LoopMomenta, InvariantList, ParameterSub];

FIESTA Value = -4.07552 × 10^11
Time Taken 570.975 seconds
```

III. 4-loop vacuum: special case

3. 4-loop vacuum with 7 propagates： 3 massive

- Type A: i9-13th, 64GB: FeynGKZ $\sim$ 0.1 s, FIESTA$\sim$ 500 s

```
SumLim = 15;
ParameterSub = {De → 4 - 2 × 0.001, ϵ → 0.001, a1 → 1, a2 → 1, a3 → 1,
   a4 → 1, a5 → 1, a6 → 1, a7 → 1, m1 → 0.01, m2 → 0.1 , m7 → 10 };
NumericalSum[SeriesSolution, ParameterSub, SumLim];
Numerical result = -8.23731 × 10^12
Time Taken 0.082945 seconds

SumLim = 15;
ParameterSub = {De → 4 - 2 × 0.001, ϵ → 0.001, a1 → 1, a2 → 1, a3 → 1,
   a4 → 1, a5 → 1, a6 → 1, a7 → 1, m1 → 0.01, m2 → 0.1 , m7 → 10 };
FIESTAEvaluate[MomentumRep, LoopMomenta, InvariantList, ParameterSub];
FIESTA Value = -8.23727 × 10^12
Time Taken 587.565 seconds
```

- Type B: i9-13th, 64GB: FeynGKZ $\sim$ 0.1 s, FIESTA$\sim$ 6000 s

```
SumLim = 15;
ParameterSub = {De → 4 - 2 × 0.001, ϵ → 0.001, a1 → 1, a2 → 1, a3 → 9 / 10,
   a4 → 1, a5 → 1, a6 → 1, a7 → 1, m1 → 0.01, m2 → 0.1, m7 → 10 };
NumericalSum[SeriesSolution, ParameterSub, SumLim];
Numerical result = 1.43756 × 10^8
Time Taken 0.094408 seconds

SumLim = 15;
ParameterSub = {De → 4 - 2 × 0.001, ϵ → 0.001, a1 → 1, a2 → 1, a3 → 9 / 10,
   a4 → 1, a5 → 1, a6 → 1, a7 → 1, m1 → 0.01, m2 → 0.1, m7 → 10 };
FIESTAEvaluate[MomentumRep, LoopMomenta, InvariantList, ParameterSub];
FIESTA Value = 1.43754 × 10^8
Time Taken 6370.08 seconds
```

IV. Summary

- **Using Mellin-Barnes representation and Miller's transformation, we derive GKZ hypergeometric systems of 4-loop vacuum Feynman integrals.**
- **In the neighborhoods of origin 0 including infinity ∞, we can obtain analytical hypergeometric series solutions through GKZ hypergeometric systems.**
- **One can see that the computing time using the GKZ hypergeometric series solutions is less than that using numerical program FIESTA.**
- **In order to derive the fundamental solution system in neighborhoods of all possible regular singularities, next we will embed the vacuum integrals in Grassmannian manifold.**

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THANKS!