Relative entropy and effective field theory

The 3rd International Workshop on BSM Frontiers: Where to Next?
August 22, 2024

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Based on **arXiv:2201.00931** with **Qing-Hong Cao (Peking University)**

arXiv:2211.08065 with **Qing-Hong Cao (Peking University)**, and **Naoto Kan (Osaka University)**

on-going works with **Pietro Conzinu (CERN)**

an on-going work with **Kazuhiro Tatsumi (Kobe University)**

Introduction

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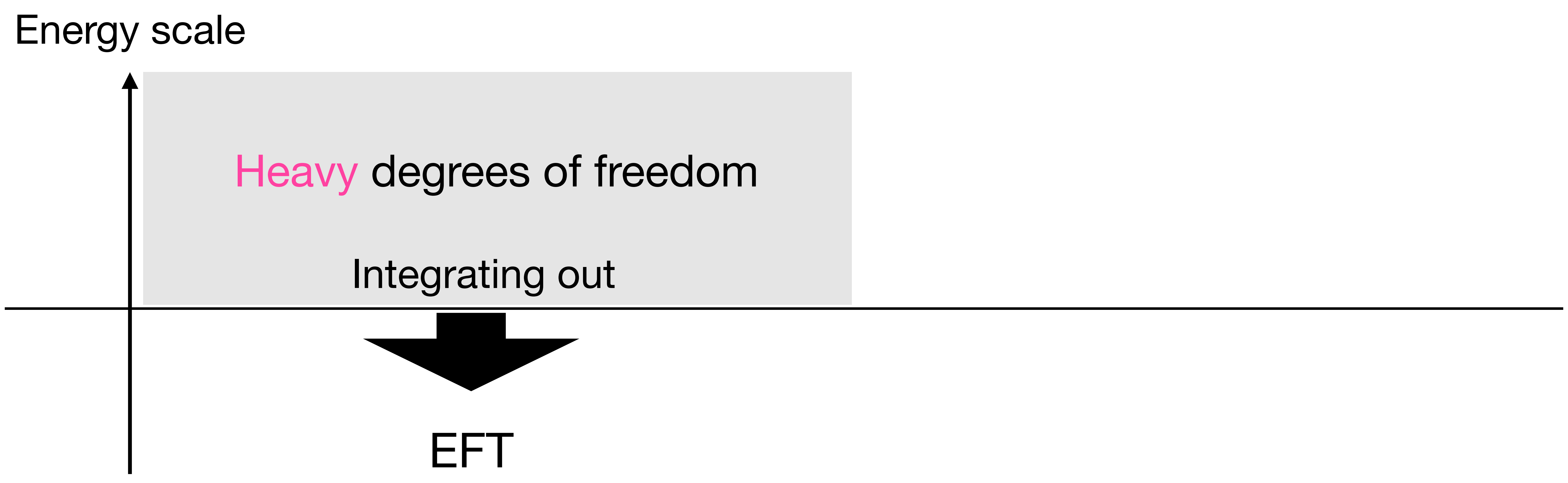
  - EFT is generated by integrating out dynamical degrees of freedom



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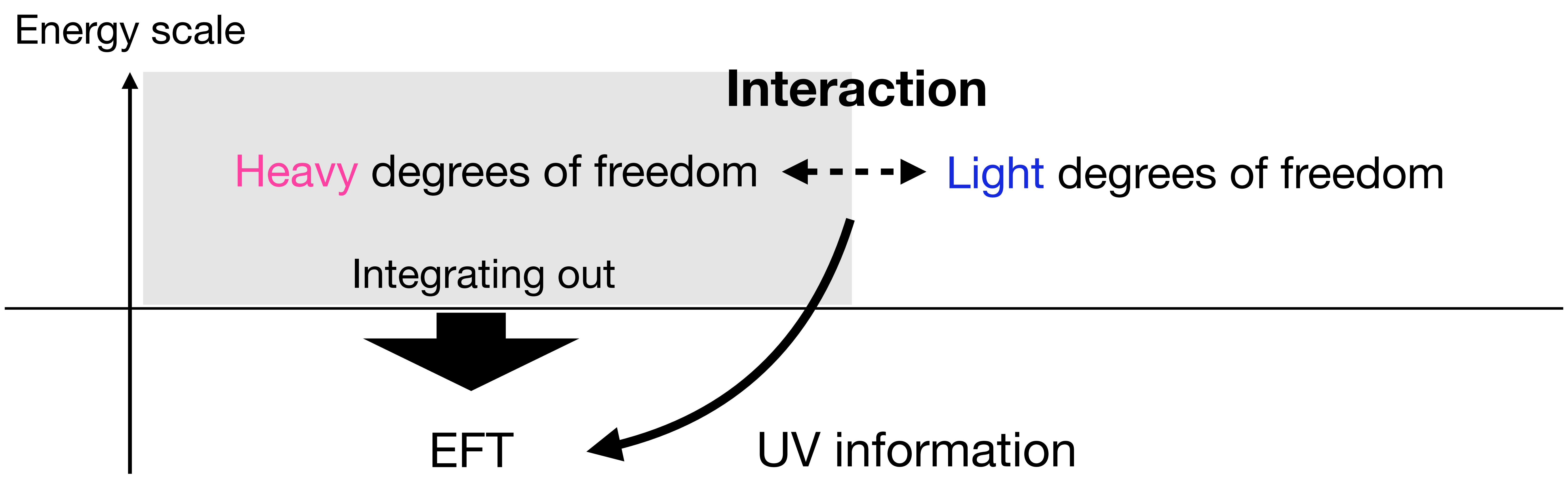

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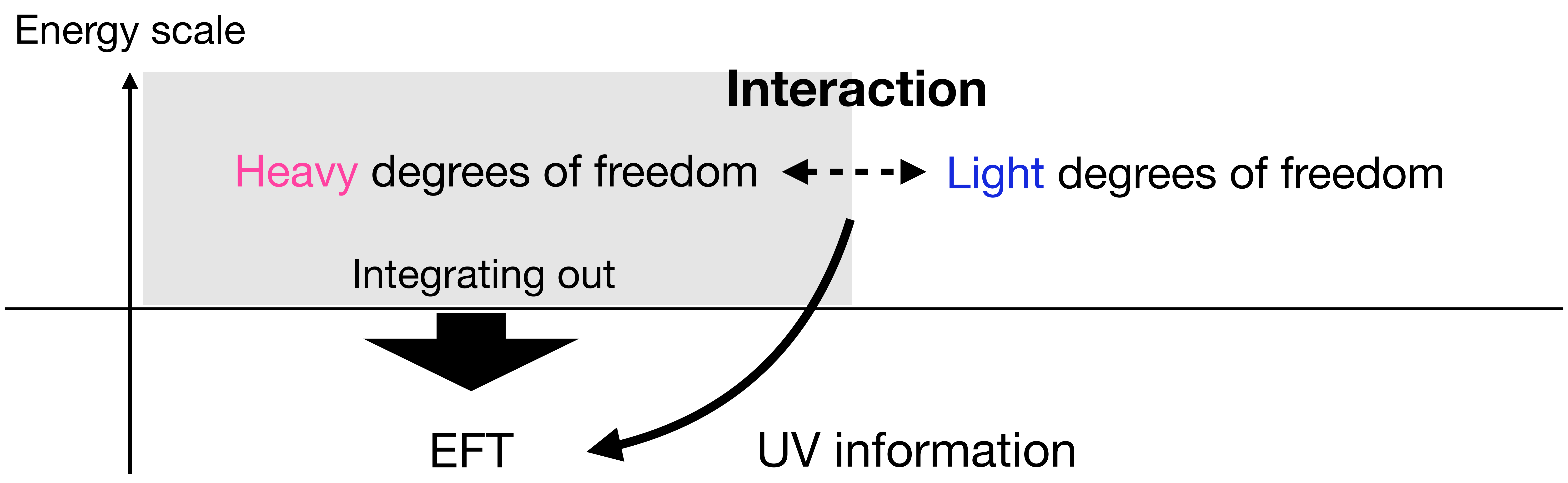

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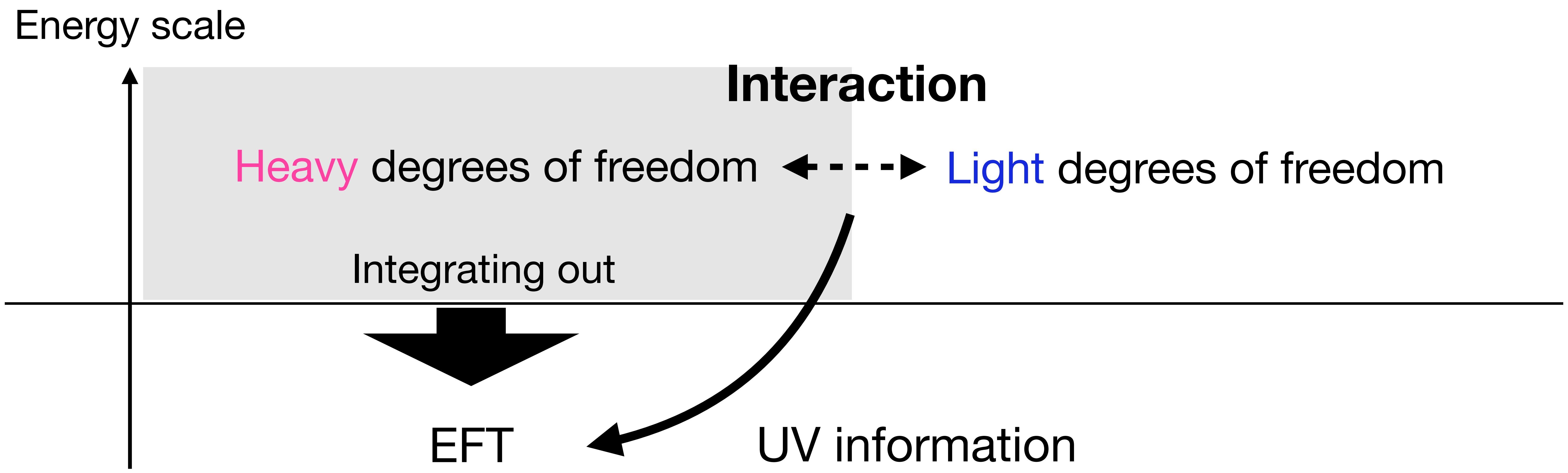

Differences between theories with and without interaction characterize UV information

⇒ Relative entropy characterizes their difference

Relative entropy

Relative entropy

- Definition of relative entropy b/w two probability distribution functions ρ_A and ρ_B

$$S(\rho_A || \rho_B) \equiv \mathrm{Tr}\left[\rho_A \ln \rho_A - \rho_A \ln \rho_B\right]$$

Relative entropy

※ $\mathrm{Tr}[\rho_A] = \mathrm{Tr}[\rho_B] = 1, \ \ \rho_A = \rho_A^\dagger, \ \ \rho_B = \rho_B^\dagger$

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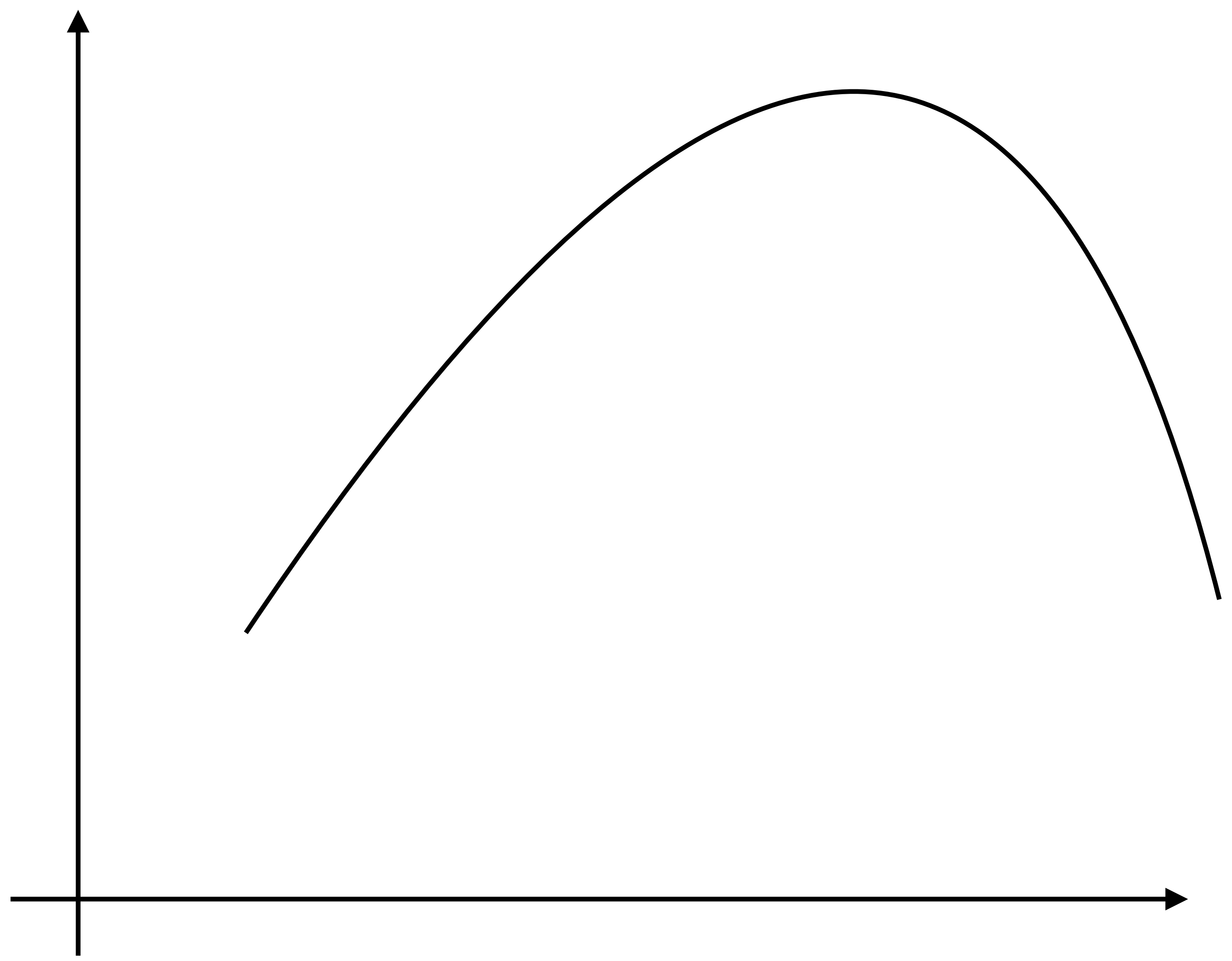

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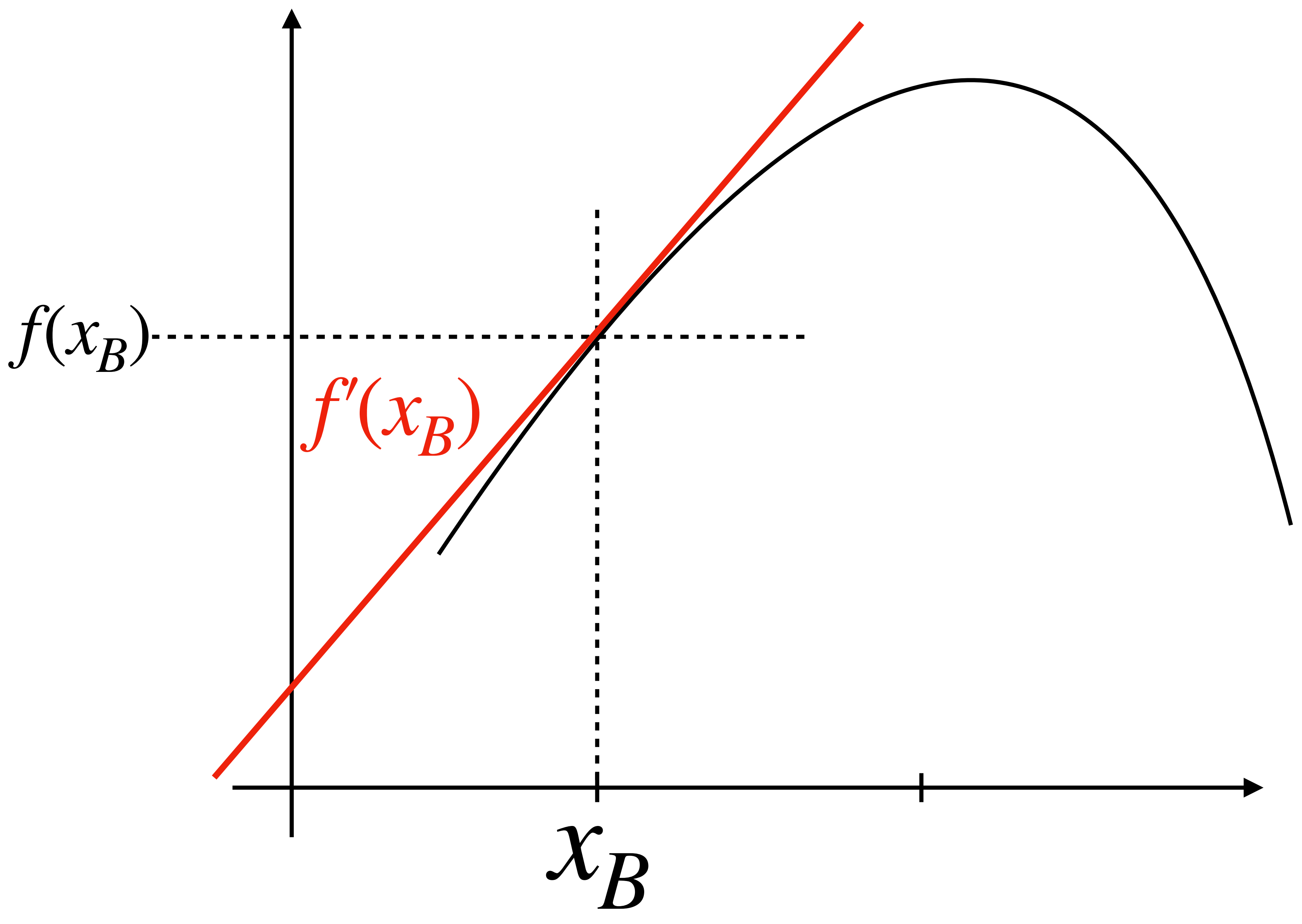

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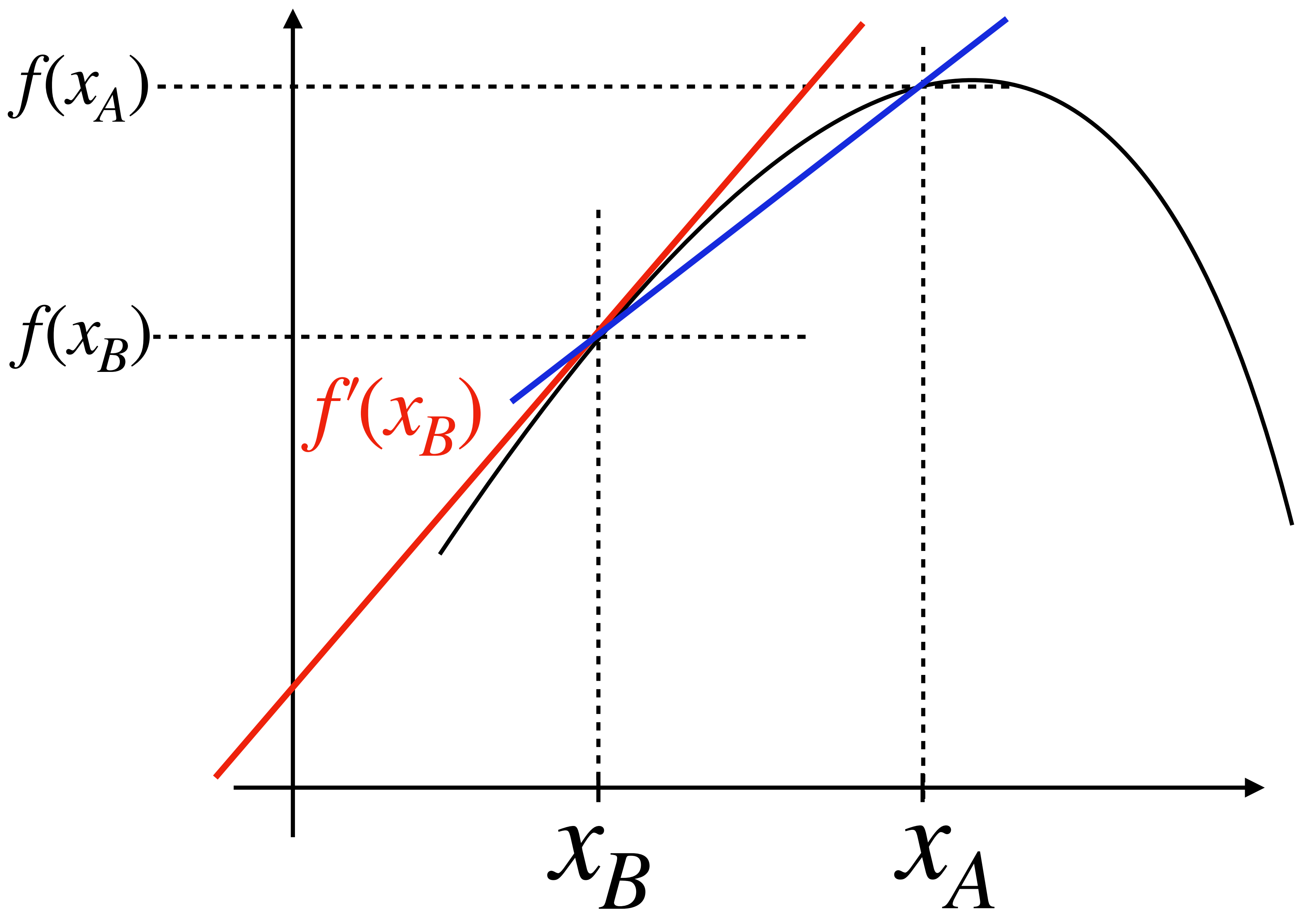

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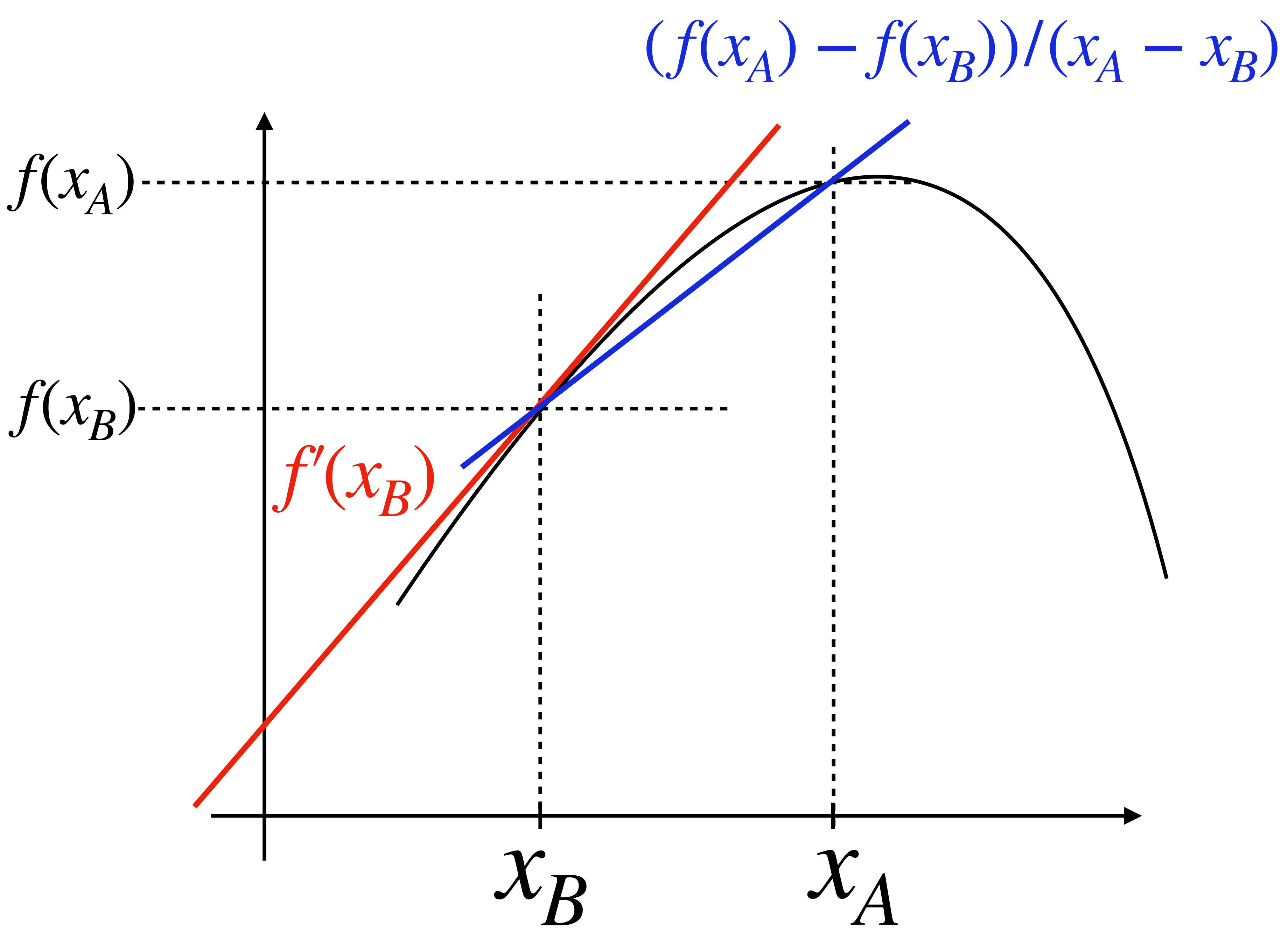

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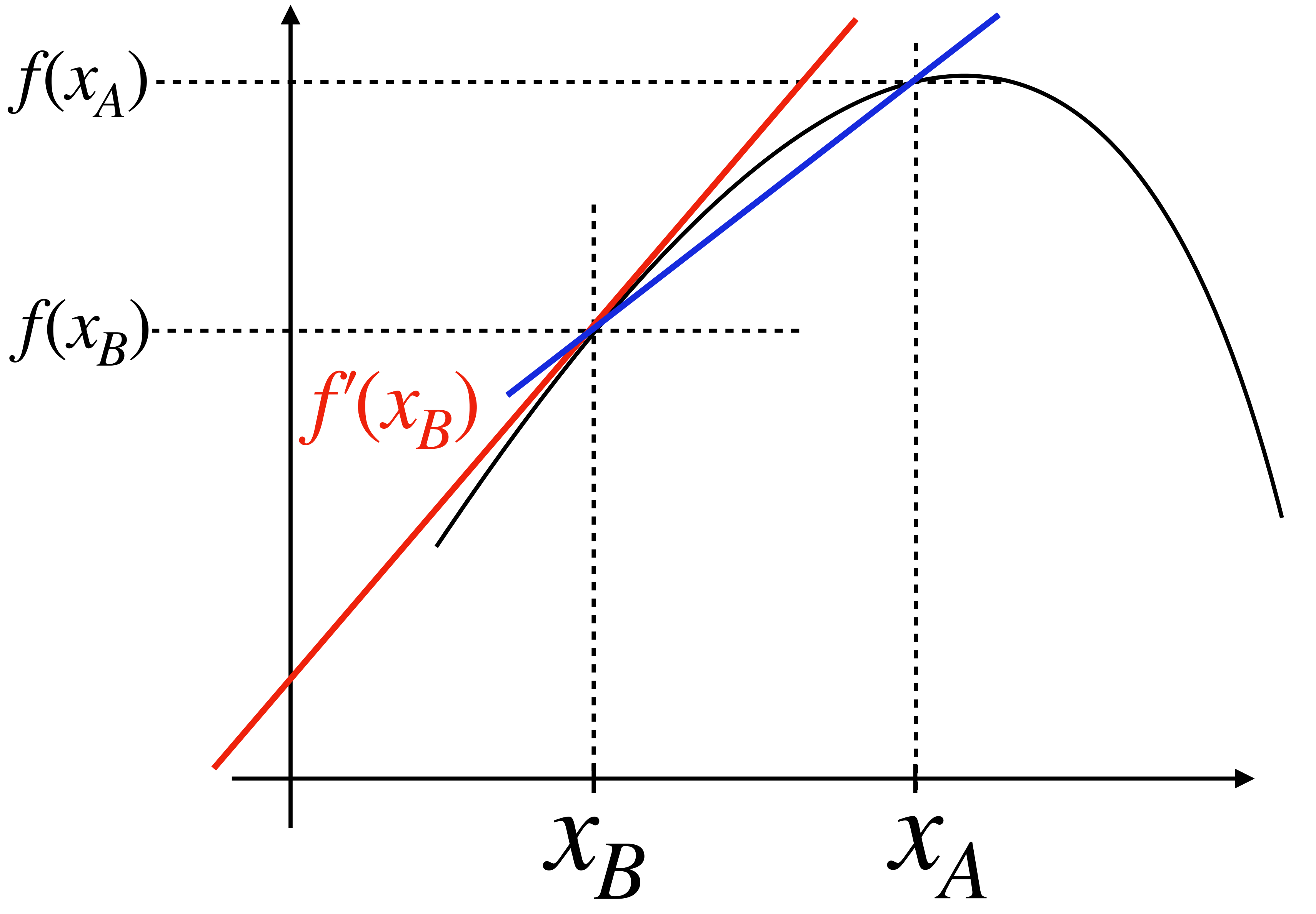

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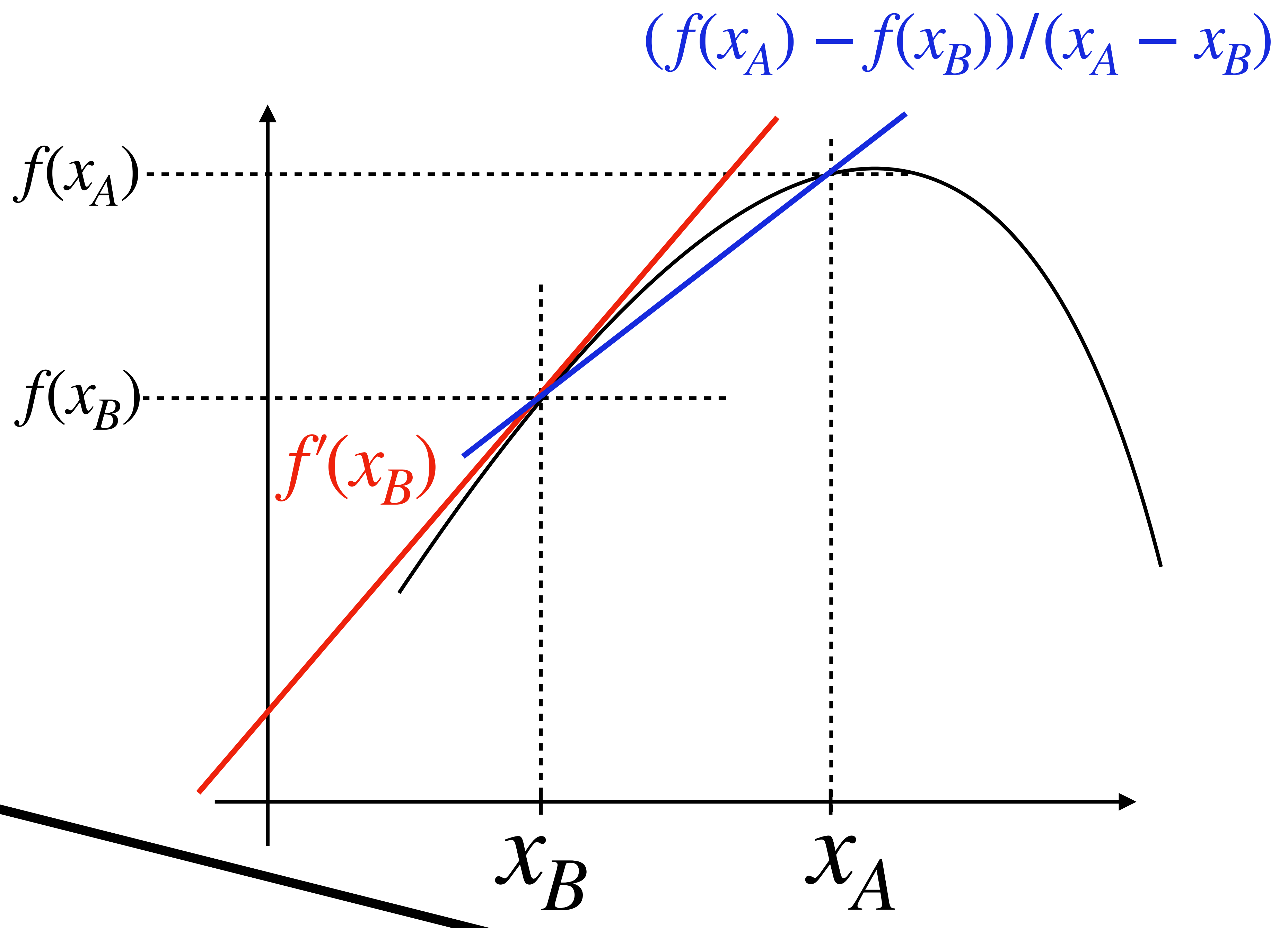

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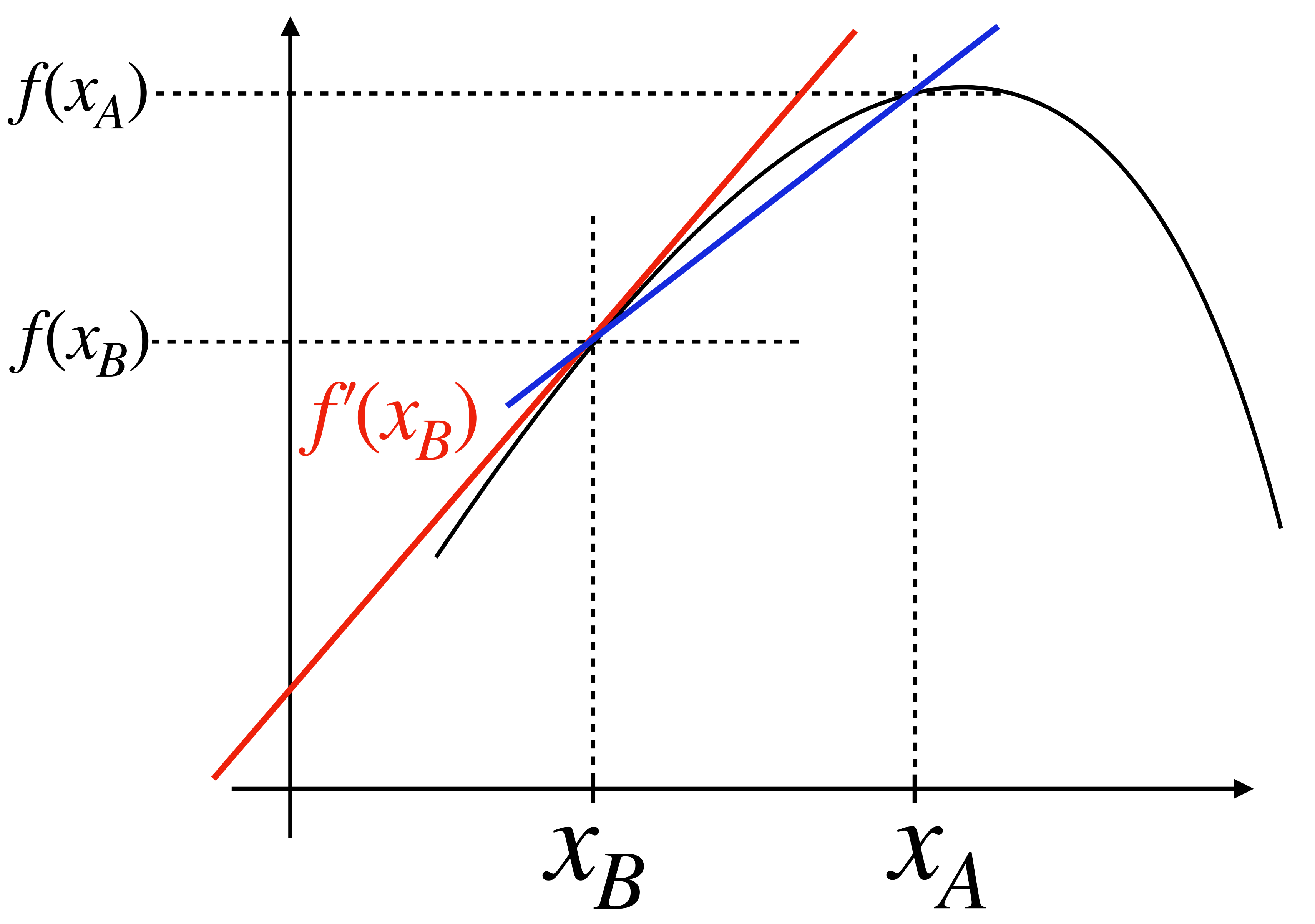

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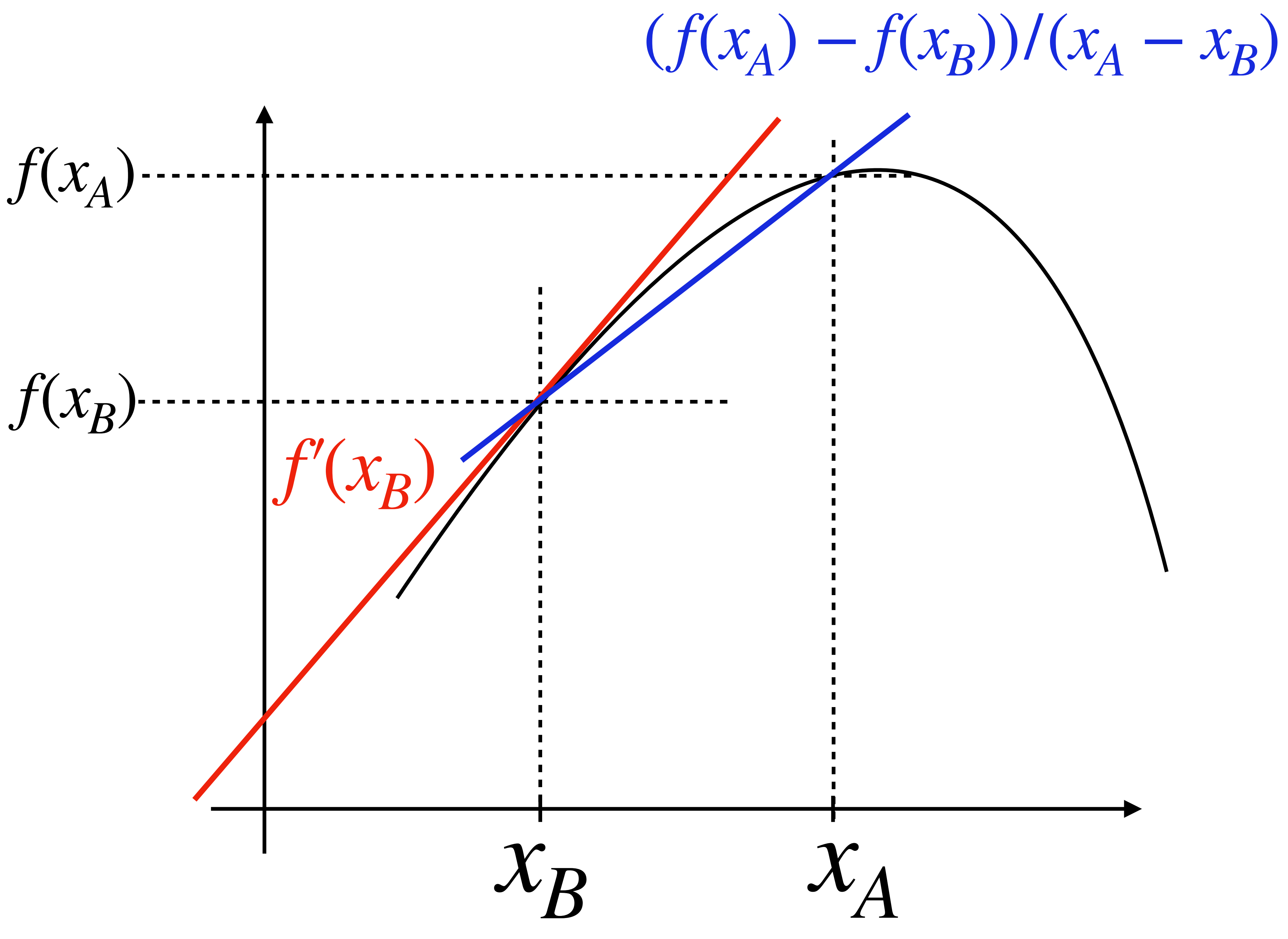

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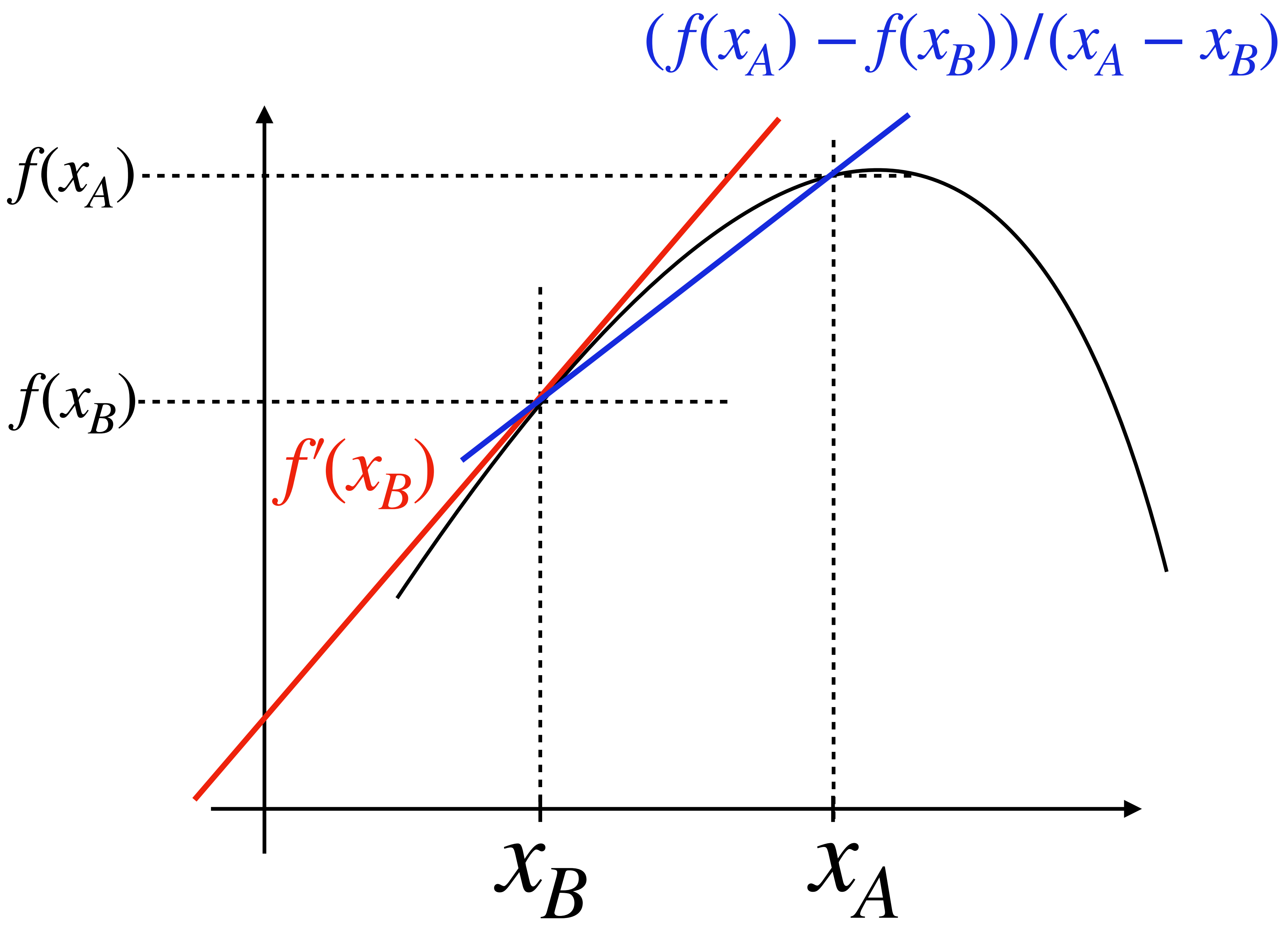

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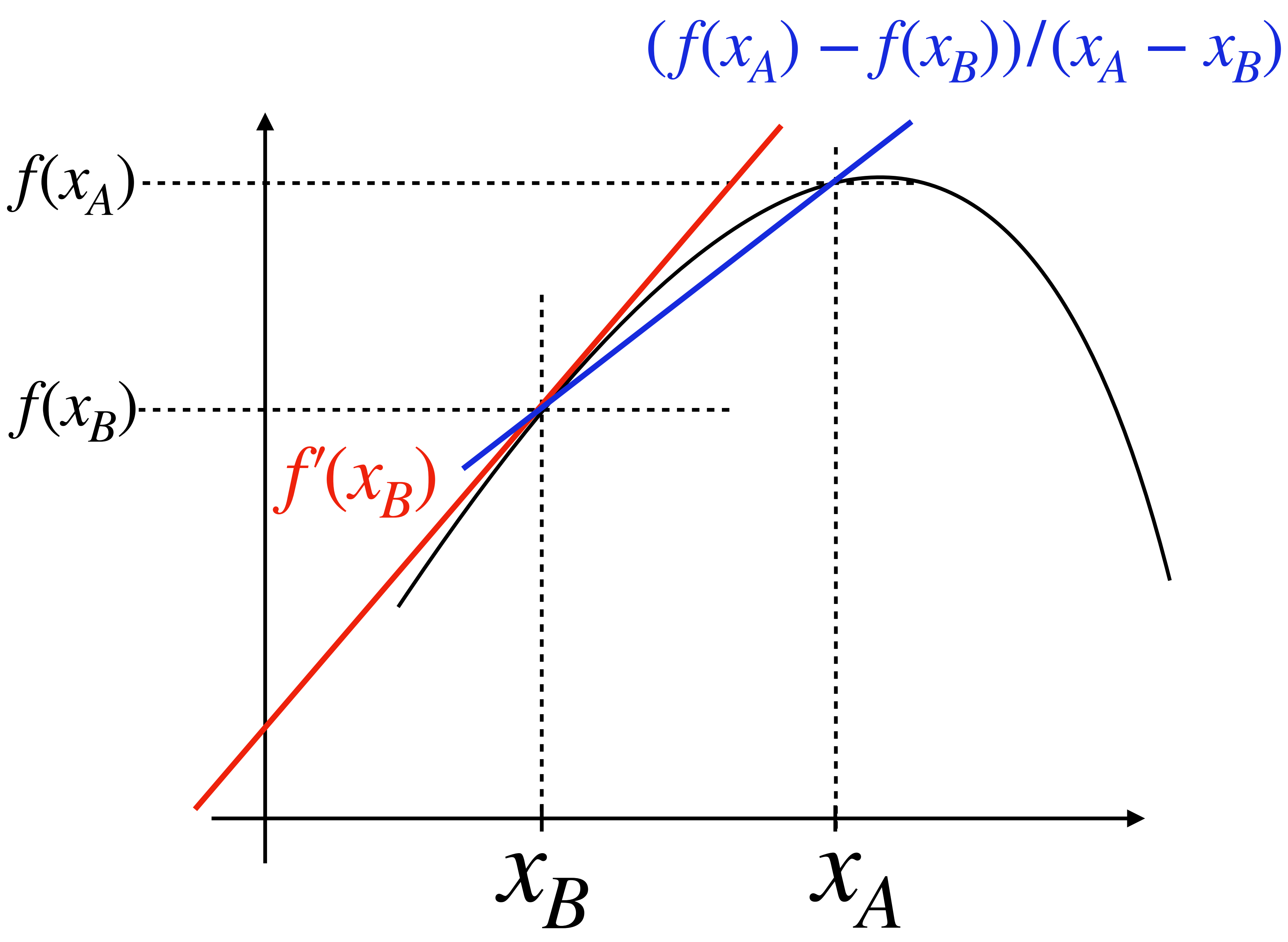

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Relative entropy characterizes difference between two probability distributions

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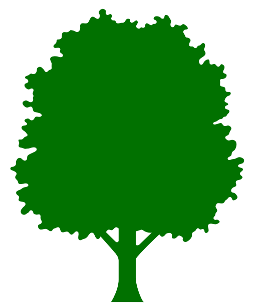
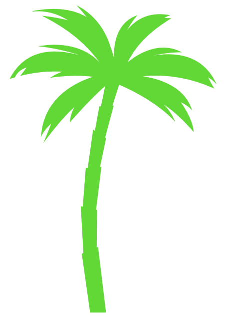
 $\mapsto \rho_A$  $\mapsto \rho_B$

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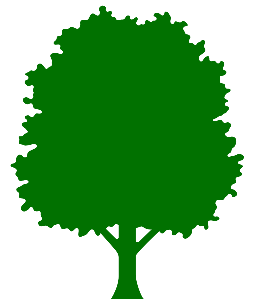
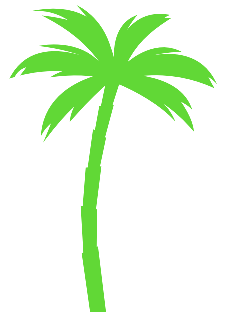
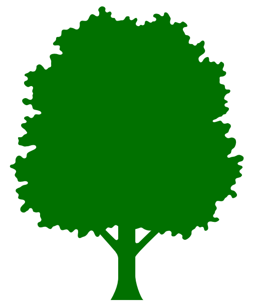
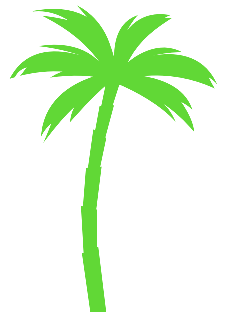
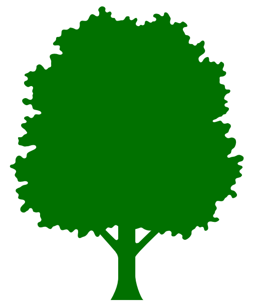
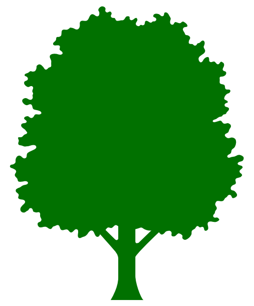
 $\mapsto \rho_A$  $\mapsto \rho_B$

$S($ $||$ $) > 0$ $S($ $||$ $) = 0$

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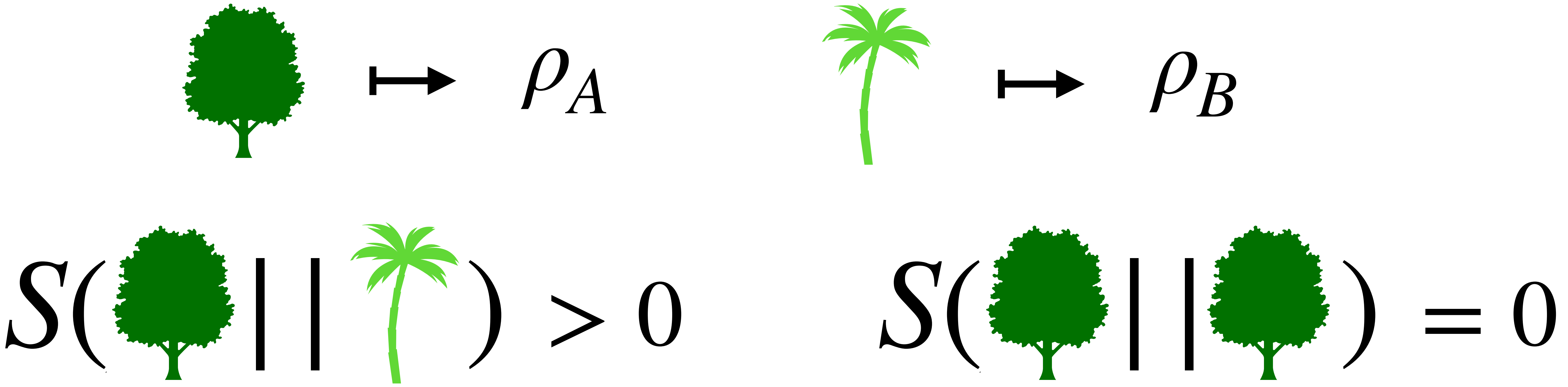

What about relative entropy b/w theories with and without interaction?

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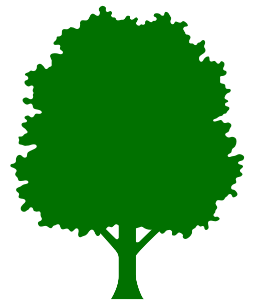
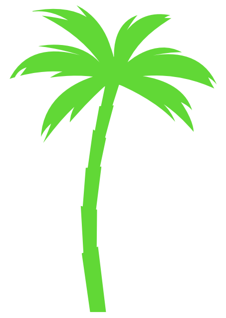
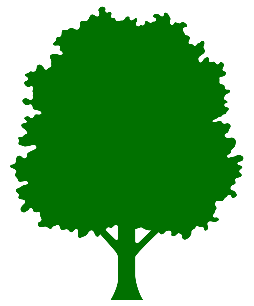
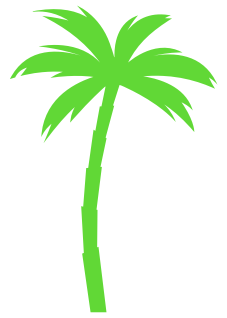
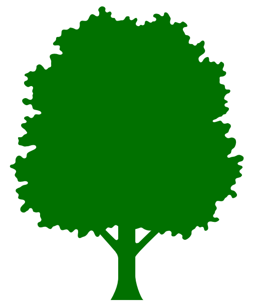
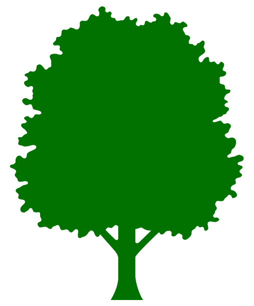
 $\mapsto \rho_A$  $\mapsto \rho_B$

$S($  $||$  $) > 0$ $S($  $||$  $) = 0$

What about relative entropy b/w theories with and without interaction?

⇒ We have to define probability distribution for each theory.

Probability distributions of theories

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- We define probability distributions of theory described by Euclidean action I as follows:

Probability distribution function: $P[\phi, \Phi] = e^{-I[\phi,\Phi]}/Z$

Partition function: $Z = \int d[\phi]d[\Phi]e^{-I[\phi,\Phi]}$

where I: Euclidean action, ϕ: light fields, Φ: heavy fields

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- Relative entropy between two theories

$$S(P_A || P_B) \equiv \int d[\phi]d[\Phi]\left(P_A \ln P_A - P_A \ln P_B\right) \geq 0$$

where $P_A = e^{-I_A}/Z_A, \quad P_B = e^{-I_B}/Z_B$

Definition of two theories

Interaction b/w ϕ and Φ

- We consider theories described by $$I_0[\phi, \Phi] + I_I[\phi, \Phi]$$

※ Φ: heavy fields, ϕ: light fields

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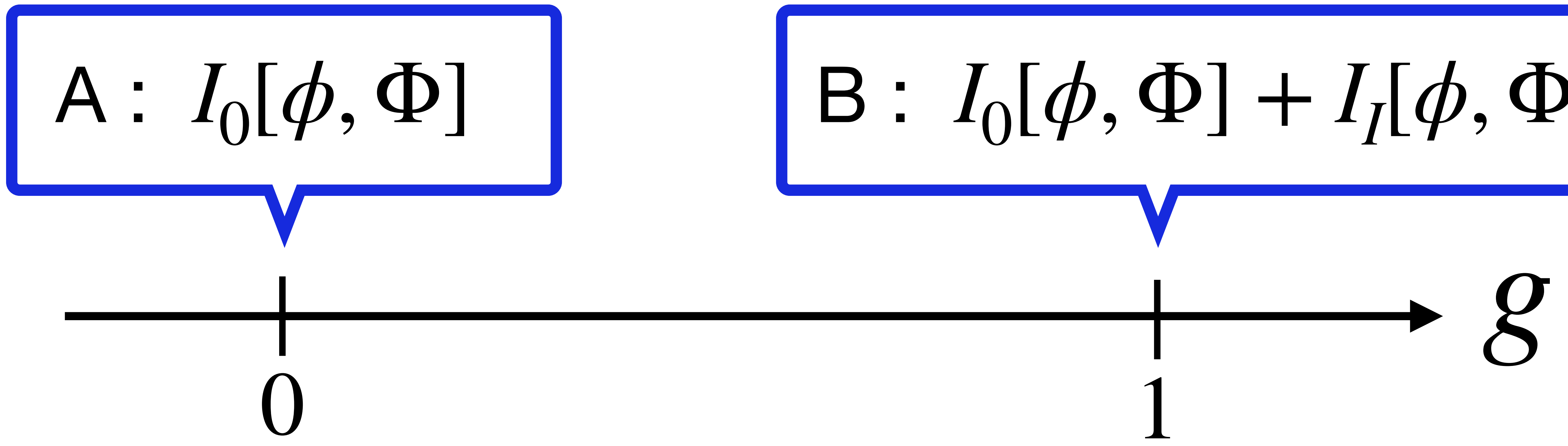

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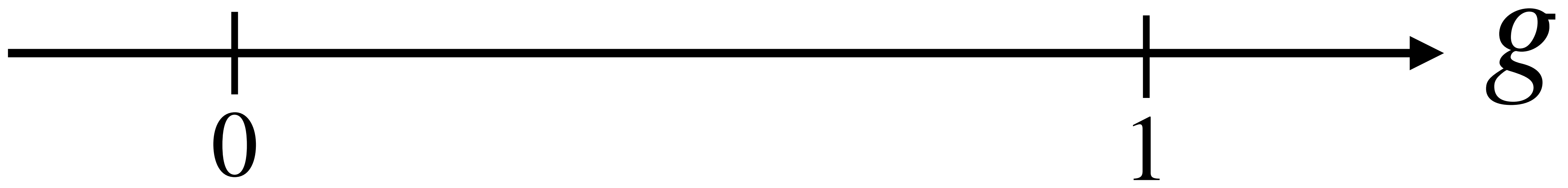
A : $I_0[\phi, \Phi]$

B : $I_0[\phi, \Phi] + I_I[\phi, \Phi]$



We consider relative entropy $S(P_A || P_B)$

※ (Φ, ϕ) of A is the same as that of B

Relative entropy between two theories

$$S(P_A||P_B) = \int d[\phi]d[\Phi]\left[P_A \ln P_A - P_A \ln P_B\right]$$

Relative entropy between two theories

$$S(P_A || P_B) = \int d[\phi]d[\Phi]\left[P_A \ln P_A - P_A \ln P_B\right] \qquad P_A = e^{-I_0[\phi,\Phi]}/Z_0,\ P_B = e^{-(I_0[\phi,\Phi]+g\cdot I_I[\phi,\Phi])}/Z_g$$

Relative entropy between two theories

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$$= -\ln Z_0 + \ln Z_g + g\cdot\int d[\phi]d[\Phi]P_A I_I$$

$$\int d[\phi]d[\Phi]P_A = 1$$

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$g \to 0$

$$\lim_{g\to 0} P_B = P_A$$

$$= W_0 - W_g + g\cdot\int d[\phi]d[\Phi]P_A I_I$$

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Relative entropy between two theories

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$S(P_A||P_B)$ yields constraints on the Euclidean effective actions

Relative entropy between two theories

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The same inequality arises from quantum mechanical approach

Relative entropy between two theories

$$S(P_A||P_B) = \int d[\phi]d[\Phi]\left[P_A \ln P_A - P_A \ln P_B\right]$$

$P_A = e^{-I_0[\phi,\Phi]}/Z_0 \quad P_B = e^{-(I_0[\phi,\Phi]+gI_I[\phi,\Phi])}/Z_g$

$$= W_0 - W_g + g\left(\frac{\partial W_g}{\partial g}\right)_{g=0} \geq 0$$

Effective actions: $W_g = -\ln Z_g, \quad W_0 = -\ln Z_0$

$S(P_A||P_B)$ yields constraints on the Euclidean effective actions

even in quantum mechanical system

$$S(P_A||P_B) \to \mathrm{tr}\left[P_A \ln P_A - P_A \ln P_B\right]$$

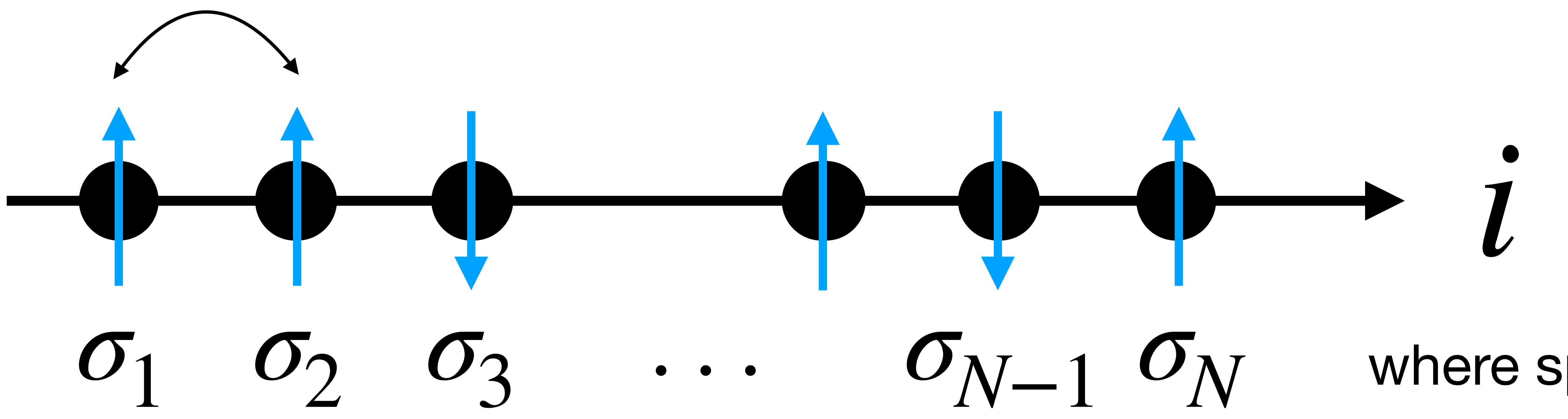
$P_A \to e^{-H_0}/Z_0 \qquad P_B \to e^{-(H_0+gH_I)}/Z_g$

$$= W_0 - W_g + g\left(\frac{\partial W_g}{\partial g}\right)_{g=0} \geq 0$$

$W_g = -\ln Z_g, \ W_0 = -\ln Z_0$

Example: Ising model

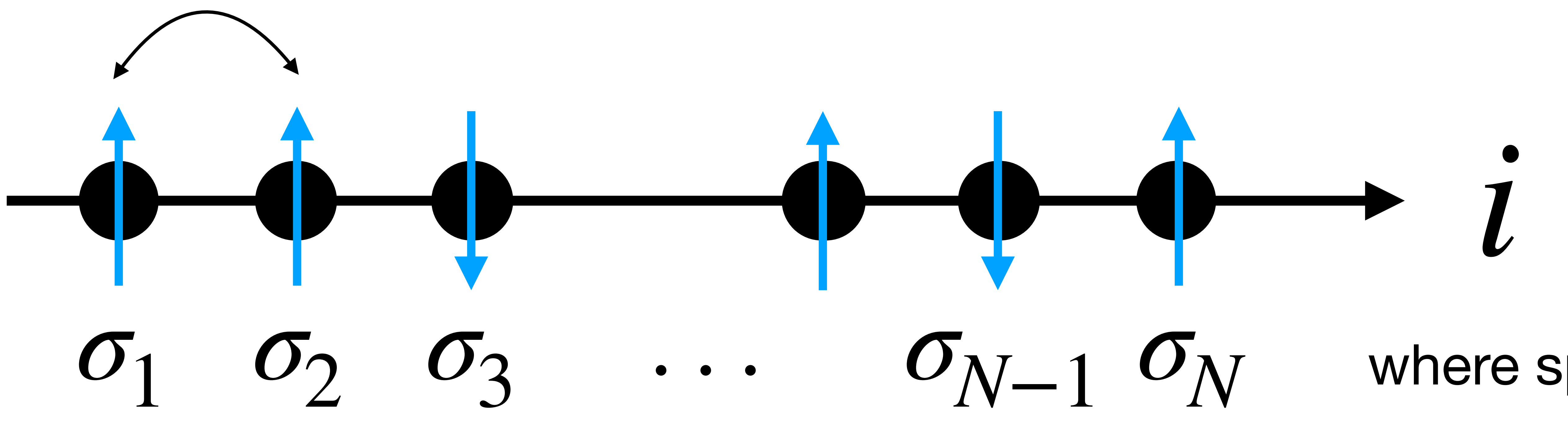
Ex. One dimension J : interaction strength



where spin of i: $\sigma_i = \pm 1$

Example: Ising model

Ex. One dimension J : interaction strength



where spin of i: $\sigma_i = \pm 1$

Theory	Hamiltonian	Probability	Partition function
A magnetic field $= 0$	$H_0 = -J \sum_{i=1}^{N} \sigma_i \sigma_{i+1}$		
B			

Example: Ising model

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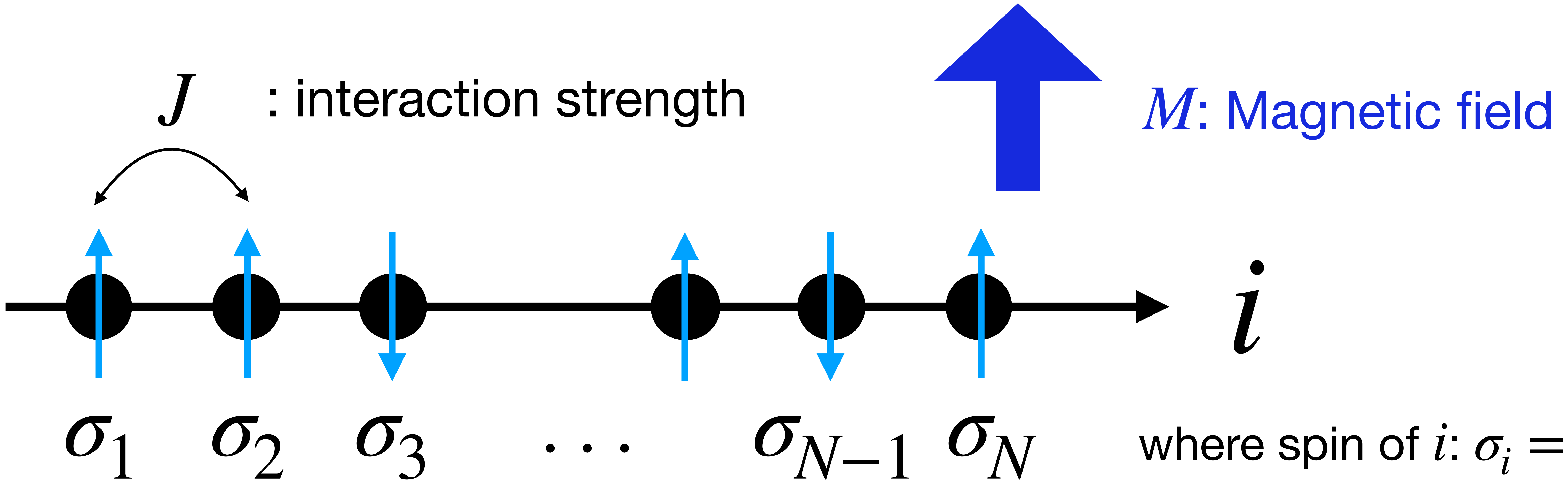

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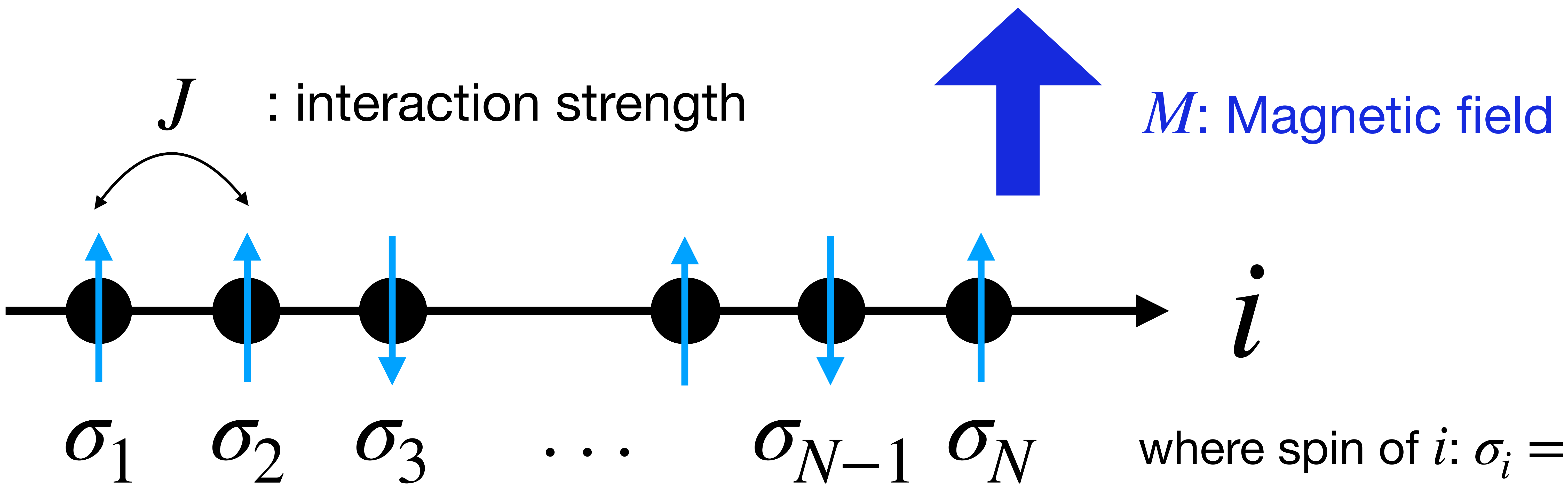

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Interaction b/w dynamical spin and BG magnetic field

Example: Ising model

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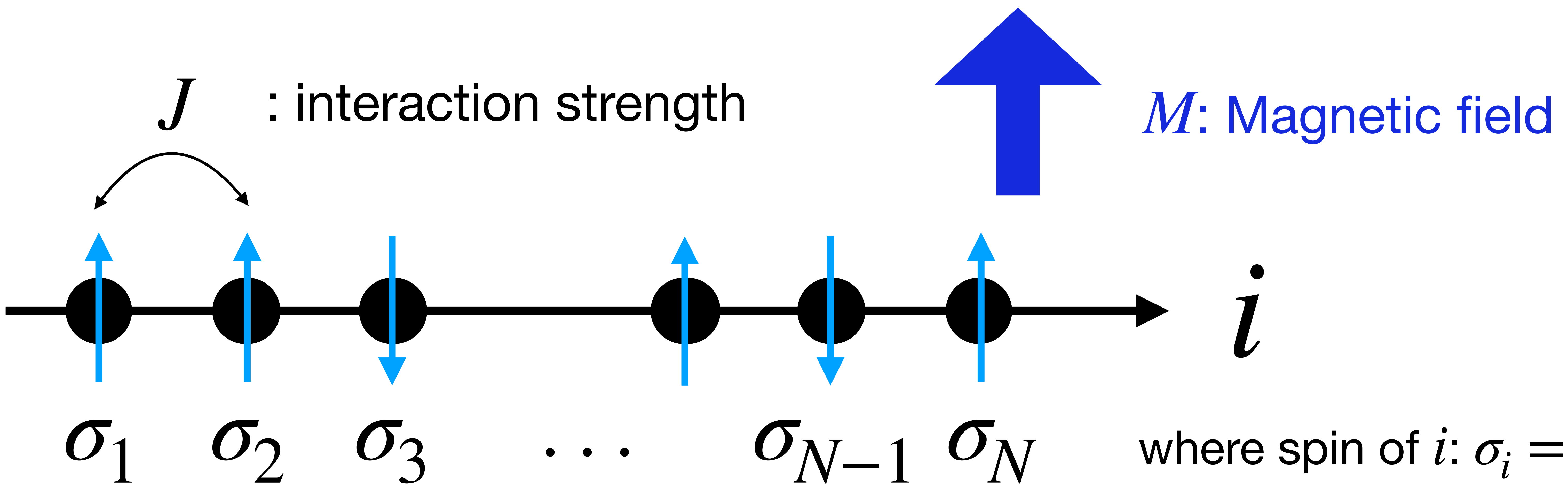

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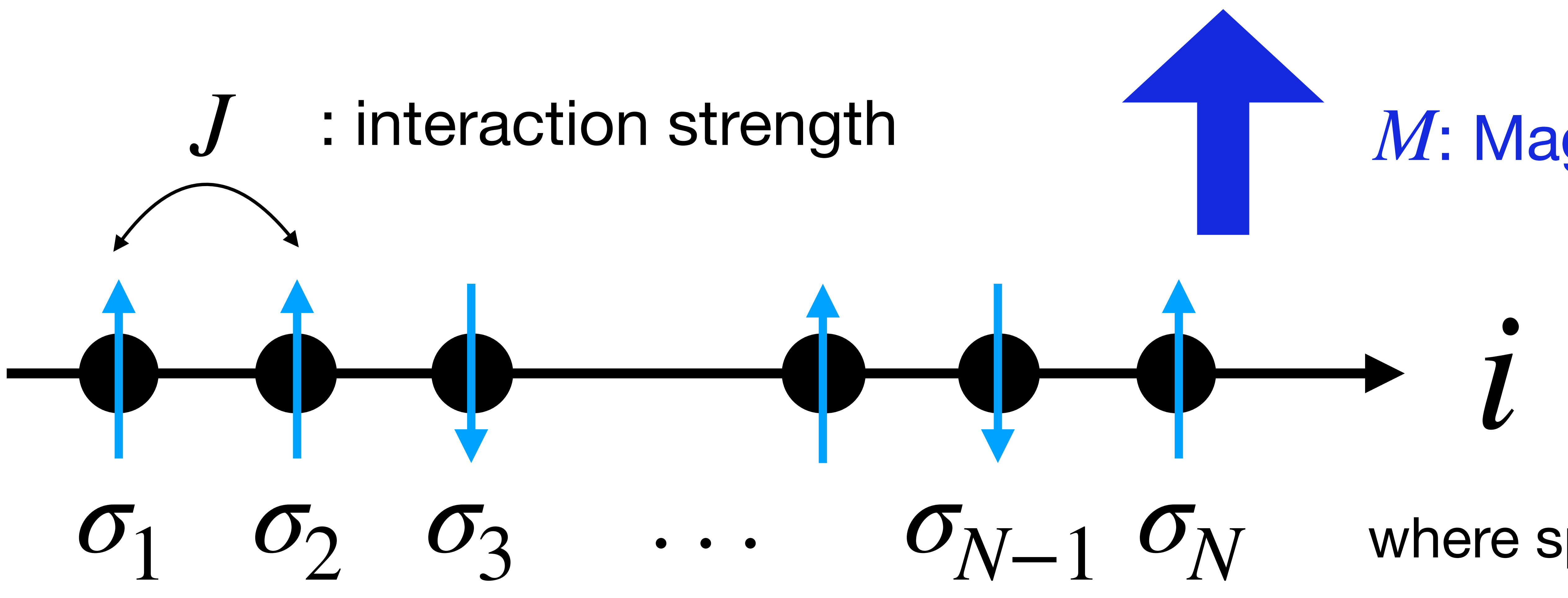
Interaction b/w dynamical spin and BG magnetic field
(Heavy-like d.o.f.) (Light-like d.o.f.)

Example: Ising model

Ex. One dimension

J : interaction strength



M: Magnetic field

where spin of i: $\sigma_i = \pm 1$

Theory	Hamiltonian	Probability	Partition function
A magnetic field $= 0$	$H_0 = -J\sum_{i=1}^{N}\sigma_i\sigma_{i+1}$	$\rho_A = e^{-\beta \cdot H_0}/Z_0$	$Z_0 = \mathrm{Tr}[e^{-\beta H_0}]$
B magnetic field $\neq 0$	$H_g = -J\sum_{i=1}^{N}\sigma_i\sigma_{i+1} - g \cdot M\sum_{j=1}^{N}\sigma_j$	$\rho_B = e^{-\beta H_g}/Z_g$	$Z_g = \mathrm{Tr}[e^{-\beta H_g}]$

Interaction b/w dynamical spin and BG magnetic field
(Heavy-like d.o.f.) (Light-like d.o.f.)

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Interaction b/w dynamical spin and BG magnetic field

- Relative entropy:

Example: Ising model

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Interaction b/w dynamical spin and BG magnetic field

- Relative entropy:

$$S(\rho_A||\rho_B) = W_0 - W_g + g\cdot\left(dW_g/dg\right)_{g=0}$$

Example: Ising model

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$$W_g \equiv -\ln Z_g$$

$$W_g = W_0 + g\cdot(dW_g/dg)_{g=0} + g^2\cdot(d^2W_g/dg^2)_{g=0} + \mathcal{O}(g^3)$$

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Cancel

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- Relative entropy:

$$S(\rho_A||\rho_B) = W_0 - W_g + g\cdot\left(dW_g/dg\right)_{g=0}$$

$$= -g^2\cdot(d^2W_g/dg^2)_{g=0}/2 + \mathcal{O}(g^3)$$

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- Relative entropy:

$$S(\rho_A || \rho_B) = W_0 - W_g + g \cdot \left(dW_g/dg \right)_{g=0}$$

$\propto$ Magnetic susceptibility

$$= -g^2 \cdot (d^2 W_g/dg^2)_{g=0}/2 + \mathcal{O}(g^3)$$

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- Relative entropy:

$$S(\rho_A||\rho_B) = W_0 - W_g + g\cdot\left(dW_g/dg\right)_{g=0}$$

Magnetic susceptibility

$$= M^2\beta\cdot g^2\cdot\beta e^{\beta J}/2 + \mathcal{O}(g^3) \geq 0$$

Example: Ising model

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- Relative entropy:

$$S(\rho_A||\rho_B) = W_0 - W_g + g\cdot\left(dW_g/dg\right)_{g=0}$$

Magnetic susceptibility

$$= M^2\beta\cdot g^2\cdot\beta e^{\beta J}/2 + \mathcal{O}(g^3) \geq 0 \Rightarrow M^2\cdot g^2\cdot\beta/2 + \mathcal{O}(g^3) \geq 0$$

High temperature: $\beta J \ll 1$

Example: Ising model

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- Relative entropy:

$$S(\rho_A||\rho_B) = W_0 - W_g + g\cdot\left(dW_g/dg\right)_{g=0}$$

Magnetic susceptibility

Curies's law

$$= M^2\beta\cdot g^2\cdot\beta e^{\beta J}/2 + \mathcal{O}(g^3) \geq 0 \Rightarrow M^2\cdot g^2\cdot\beta/2 + \mathcal{O}(g^3) \geq 0$$

High temperature: $\beta J \ll 1$

Example: Ising model

Theory	Hamiltonian	Probability	Partition function
A magnetic field $= 0$	$H_0 = -J\sum_{i=1}^{N}\sigma_i\sigma_{i+1}$	$\rho_A = e^{-\beta\cdot H_0}/Z_0$	$Z_0 = \mathrm{Tr}[e^{-\beta H_0}]$
B magnetic field $\neq 0$	$H_g = -J\sum_{i=1}^{N}\sigma_i\sigma_{i+1} - g\cdot M\sum_{j=1}^{N}\sigma_j$	$\rho_B = e^{-\beta H_g}/Z_g$	$Z_g = \mathrm{Tr}[e^{-\beta H_g}]$

- Relative entropy:

$$S(\rho_A||\rho_B) = W_0 - W_g + g\cdot\left(dW_g/dg\right)_{g=0}$$

Magnetic susceptibility

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High temperature: $\beta J \ll 1$

Non-negativity of relative entropy explains why the magnetic susceptibility is positive

※ This result holds in general dimensions

Example: Euler-Heisenberg theory

- Consider the U(1) gauge field A_μ coupled to a charged fermion ψ

Theory	Action in Minkowski space	Probability	Partition function
A	$I_0 = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\gamma_\mu \partial^\mu - m)\psi \right)$		
B	$I_g = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\gamma_\mu \partial^\mu - m)\psi - g \cdot e(\bar{\psi}\gamma_\mu \psi) A^\mu \right)$		

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Interaction b/w heavy filed ψ and light field A^μ

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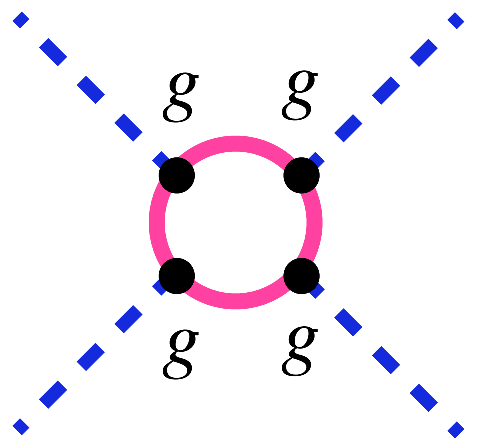

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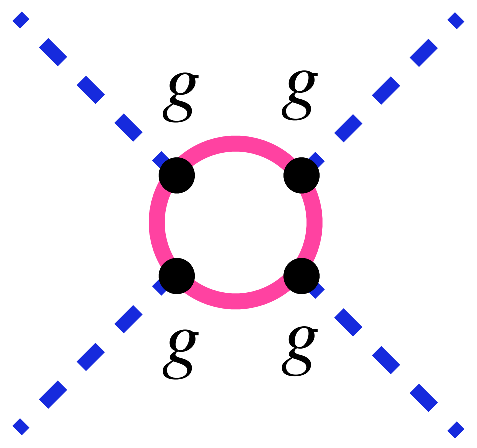

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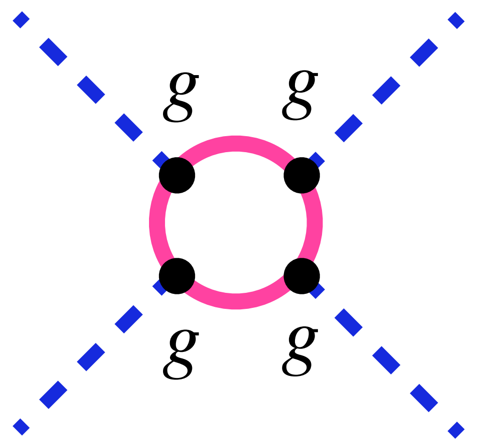

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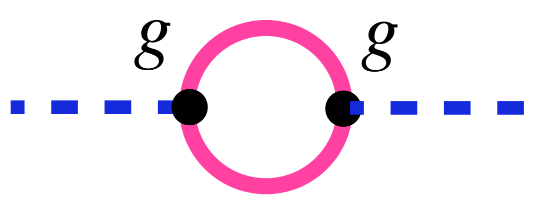

$$\supset g^2\cdot\int (d^4x)_E(\partial^2\bar{F}\bar{F}), \cdots \Rightarrow 0, \text{ for } \partial\bar{F} = \text{const}.$$

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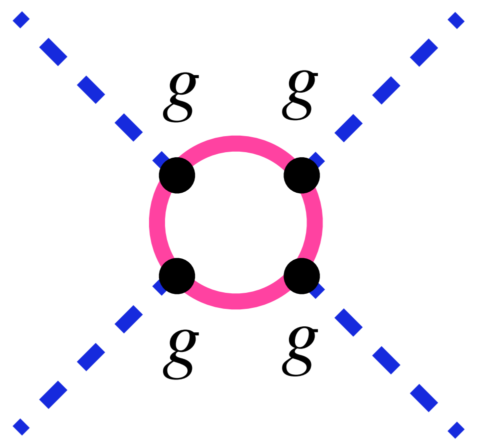

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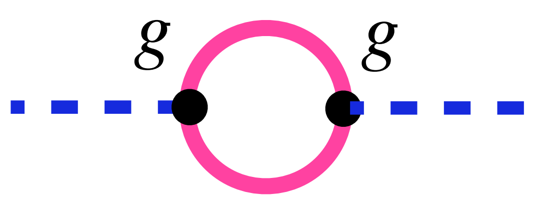

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dim-8 operators

Relative entropy constrains Wilson coefficients of dim-8 operator

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⇒ Similar results for SU(N) gauge fields are obtained when dim-8 operators are generated through the interaction between heavy and light fields.

Example: SMEFT SU(N) gauge bosonic operators

- Relative entropy when dim-8 operators are generated by **interaction** b/w heavy and light fields:

※ assume the interaction doesn't involve higher-derivative terms

$$S(P_A||P_B) = W_0 - W_g + g\cdot(dW_g/dg)_{g=0} = \int (d^4x)_{\rm E}\frac{1}{\Lambda^4}\sum_i c_i\mathcal{O}_i \geq 0$$

$$\mathcal{O}_1^{F^4} = (F^a_{\mu\nu}F^{a,\mu\nu})(F^b_{\rho\sigma}F^{b,\rho\sigma})$$

$$\mathcal{O}_2^{F^4} = (F^a_{\mu\nu}\tilde{F}^{a,\mu\nu})(F^b_{\rho\sigma}\tilde{F}^{b,\rho\sigma})$$

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T^a : generator of $SU(N)$ Lie algebra

$$[T^a, T^b] = if^{abc}T^c$$

$$\{T^a, T^b\} = \delta^{ab}\hat{1}/N + d^{abc}T^c$$

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※ assume the interaction doesn't involve higher-derivative terms

- Classical solution of $\partial^\mu F^a_{\mu\nu} + g f^{abc} A^{\mu,b} F^c_{\mu\nu} = 0$: $A^a_\mu = u^a_1 \epsilon_{1\mu} w_1 + u^a_2 \epsilon_{2\mu} w_2$ with $f^{abc} u^a_1 u^b_2 = 0$, $\partial_\mu w_1 = l_\mu$, and $\partial_\mu w_2 = k_\mu$

※ l_μ, k_μ : constant vectors

- $U(1)_Y$: $c_1^{B^4} \geq 0, \quad c_2^{B^4} \geq 0, \quad 4c_1^{B^4} c_2^{B^4} \geq (\tilde{c}_1^{B^4})^2,$
- $SU(2)_L$: $c_1^{W^4} + c_3^{W^4} \geq 0, \quad c_2^{W^4} + c_4^{W^4} \geq 0, \quad 4(c_1^{W^4} + c_3^{W^4})(c_2^{W^4} + c_4^{W^4}) \geq (\tilde{c}_1^{W^4} + \tilde{c}_2^{W^4})^2,$

U(1) and SU(2) bounds are the same as positivity bounds from unitarity and causality

[G.N. Remmen, and N.L. Rodd, arXiv:1908.09845]

- $SU(3)_C$: $2c_1^{G^4} + c_3^{G^4} \geq 0, \quad 3c_2^{G^4} + 2c_5^{G^4} \geq 0, \quad 3c_2^{G^4} + 3c_4^{G^4} + c_6^{G^4} \geq 0, \quad 3c_4^{G^4} + 2c_6^{G^4} \geq 0,$

$$4(3c_1^{G^4} + 3c_3^{G^4} + c_5^{G^4})(3c_2^{G^4} + 3c_4^{G^4} + c_6^{G^4}) \geq (3\tilde{c}_1^{G^4} + 3\tilde{c}_2^{G^4} + \tilde{c}_3^{G^4})^2$$

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SU(3) bounds are stronger than positivity bounds from unitarity and causality

Example: a single scalar field EFT

- Consider a single scalar field EFT:

$$\int (d^4x)_{\mathrm{E}}\sqrt{g}\left[-\frac{1}{2}(\partial_\mu\phi)^2 - \sum_{j=1} c_{j+1}\left(-\frac{1}{2}(\partial_\mu\phi)^2\right)^{j+1}\right]$$

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- Relative entropy when higher dimensional operators are generated by **interaction** b/w heavy and light fields:

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- up to the **third** order of $(\partial_\mu\phi)^2/2$: $c_2\int(d^4x)_{\rm E}(-(\partial_\mu\phi)^2/2)^2 + c_3\int(d^4x)_{\rm E}(-(\partial_\mu\phi)^2/2)^3 \geq 0 \quad \Rightarrow \boxed{c_2 + \frac{c_3}{2}(-(\partial_\mu\phi)^2/2) \geq 0}$

⋮

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- Relative entropy when higher dimensional operators are generated by **interaction** b/w heavy and light fields:

$$S(P_A||P_B)=W_0-W_g+g\cdot(dW_g/dg)_{g=0}=\int(d^4x)_{\mathrm{E}}\sum_{j=1}c_{j+1}(-(\partial_\mu\phi)^2/2)^j\geq 0$$

- up to the **second** order of $(\partial_\mu\phi)^2/2$: $c_2\int(d^4x)_{\mathrm{E}}(-(\partial_\mu\phi)^2/2)^2\geq 0 \quad \Rightarrow \boxed{c_2\geq 0}$

- up to the **third** order of $(\partial_\mu\phi)^2/2$: $c_2\int(d^4x)_{\mathrm{E}}(-(\partial_\mu\phi)^2/2)^2+c_3\int(d^4x)_{\mathrm{E}}(-(\partial_\mu\phi)^2/2)^3\geq 0 \quad \Rightarrow \boxed{c_2+\frac{c_3}{2}(-(\partial_\mu\phi)^2/2)\geq 0}$

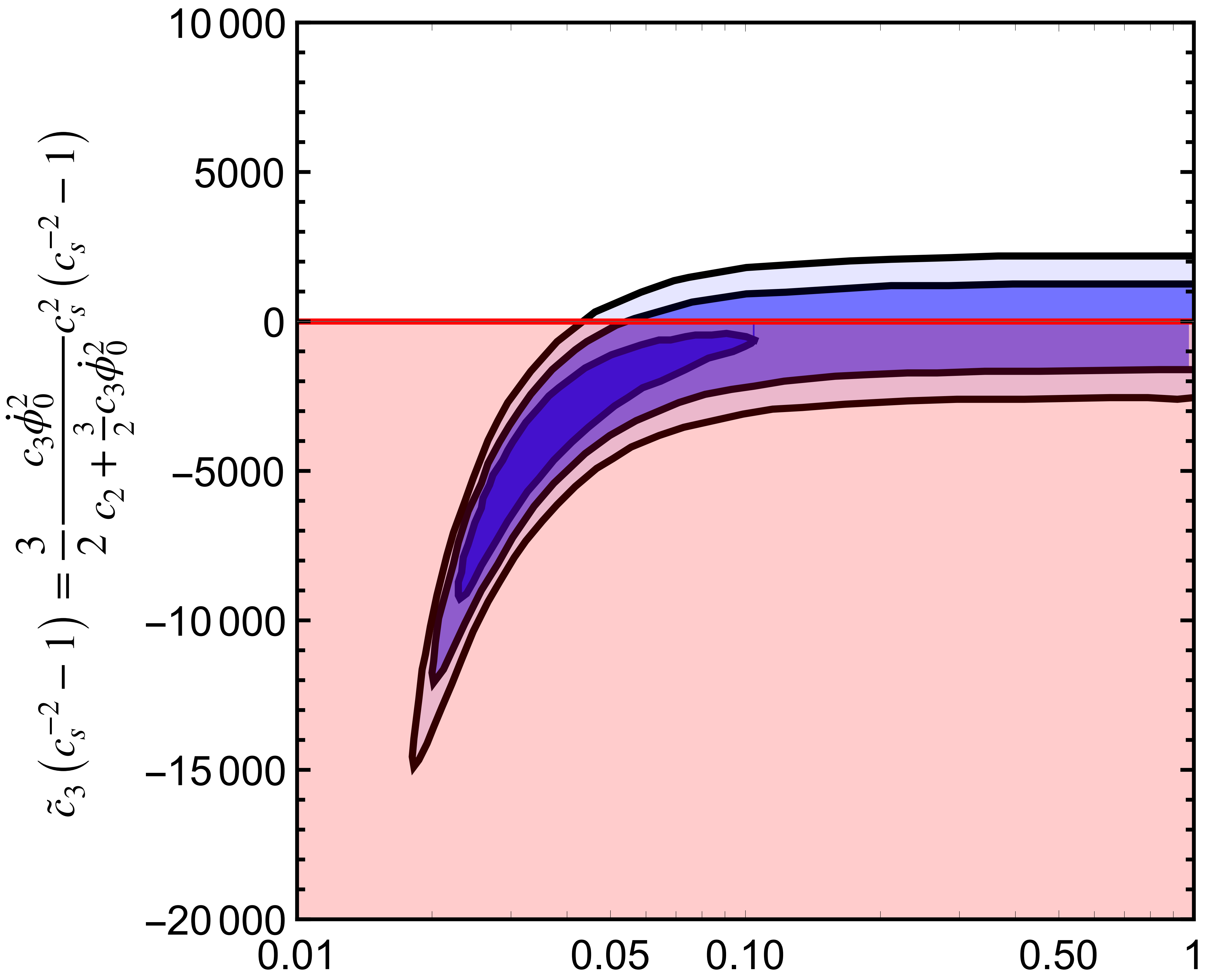
⋮

Imposing validity of EFT expansion, relative entropy also yields constraints on higher dimension operators than dim-8, e.g., $c_3, c_4, \cdots$

An application: non-Gaussianity in a single scalar field EFT

- Consider a single scalar field EFT up to dim-12:

$$\int (d^4x)_{\rm E}\sqrt{g}\left[-\frac{1}{2}(\partial_\mu\phi)^2 - c_2\left(-\frac{1}{2}(\partial_\mu\phi)^2\right)^2 - c_3\left(-\frac{1}{2}(\partial_\mu\phi)^2\right)^3 - \cdots\right]$$



Speed of sound: $c_s = \left[1 + 2\dot{\phi}_0(c_2 + \frac{3}{2}c_3\dot{\phi}_0^2)/(1 + c_2\dot{\phi}_0^2 + \frac{3}{4}c_3\dot{\phi}_0^4)\right]^{-1/2}$

- primordial fluctuation described by this EFT yields **non-Gaussianity**
- blue regions are the observationally allowed regions with statistics of $1\,\sigma$, $2\,\sigma$, and $3\,\sigma$
- red region is **prohibited by relative entropy**, i.e., non-negativity of relative entropy is violated, under the assumption of validity of EFT expansion

※ Note that red region is allowed if EFT expansion is invalid, i.e., higher dimensional operators than dim-12 are not negligible

Summary

- Relative entropy quantifies differences between theories with and without interaction

- We found that the positive of relative entropy yields a unified understanding of various phenomena, e.g.,

  - Positive magnetic susceptibility in the Ising model

  - Positivity bounds on the SMEFT SU(N) gauge bosonic operators

- Relative entropy provides a new approach to constraining EFTs

  and the scope of application must be studied in the future