Lecture 4:
Variational principle for fluid dynamics

Variational principle for perfect fluid

Variational principle for the dynamics of the perfect fluid

- has a long history
- is nontrivial
- has many versions

The geometric version we adopt here

- is close to the one used by Carter and by Arnold's school
- involves the use of "restricted variations"
- suitable for both relativistic and nonrelativistic hydro

.

Action for a barotropic fluid

Action for barotropic fluid dynamics is a spacetime integral of fluid pressure

$$S[\pi_\nu] = \int d^4x\, P(\pi_\nu)\,, \tag{67}$$

where pressure is understood as a function of fluid 4-momentum π_ν.

Admissible variations. To obtain equations of motion one should vary action over π_ν but only as

$$\delta\pi_\nu = \epsilon^\mu \partial_\mu \pi_\nu + \pi_\mu \partial_\nu \epsilon^\mu\,, \tag{68}$$

where ϵ^μ is an arbitrary 4-vector field.

This prescription is the variation as a Lie derivative of a one-form along ϵ, that is $\delta\pi = \mathcal{L}_\epsilon \pi$.

Equations of motion from action

Exercise 22

Vary the action (67) as prescribed by (68) and obtain the equation of motion

$$J^\nu(\partial_\nu \pi_\mu - \partial_\mu \pi_\nu) + \pi_\mu \partial_\nu J^\nu = 0\,, \tag{69}$$

where we defined

$$J^\mu \equiv -\frac{\delta S}{\delta \pi_\mu} = -\frac{\partial P}{\partial \pi_\mu}\,. \tag{70}$$

Exercise 23

Show that (69) leads to

$$\partial_\mu J^\mu = 0\,, \tag{71}$$

$$J^\nu(\partial_\nu \pi_\mu - \partial_\mu \pi_\nu) = 0\,. \tag{72}$$

Hint: multiply (69) by J^μ.

Admissible variations

Vary the action $S[\pi]$ over admissible variations

$$\delta\pi_\nu = \epsilon^\mu \partial_\mu \pi_\nu + \pi_\mu \partial_\nu \epsilon^\mu \,, \qquad \delta\pi = \mathcal{L}_\epsilon \pi$$

and obtain

$$\delta S = \int d^4x \, \frac{\delta S}{\delta \pi_\nu} (\epsilon^\mu \partial_\mu \pi_\nu + \pi_\mu \partial_\nu \epsilon^\mu) = 0 \,.$$

Introducing "particle current" $\mathcal{J}^\nu \equiv -\frac{\delta S}{\delta \pi_\nu}$

$$\mathcal{J}^\nu (\partial_\nu \pi_\mu - \partial_\mu \pi_\nu) + \pi_\mu \partial_\nu \mathcal{J}^\nu = 0 \,.$$

Multiply by $\mathcal{J}^\mu$

$$\mathcal{J}^\nu (\partial_\nu \pi_\mu - \partial_\mu \pi_\nu) = 0 \,, \qquad \partial_\nu \mathcal{J}^\nu = 0 \,.$$

Notice, that the conservation of $\partial_\nu \mathcal{J}^\nu = 0$ follows from diffeos.

A few remarks

- For a given fluid we have $P(\mu)$ – equation of state
- For a given space time symmetry we express μ as a scalar made out of π_μ.

$$\mu = -\pi_0 - \frac{\pi_i^2}{2}\,, \qquad \text{- Galilean}\,, \tag{73}$$

$$\mu = \sqrt{-\pi_\mu^2}\,, \qquad \text{- relativistic}\,. \tag{74}$$

- Equations are the same for $S[\pi_\nu - A_\nu]$ as long as variation is done only over π_μ.
- One can generalize the action by adding derivative terms. As long as $S = S[\pi_\nu]$ the derivation holds.

Conserving forms of e quations of motion

Variational principle gives equations of motion

$$\partial_\mu T^\mu_\nu = 0\,, \qquad\qquad T^\mu_\nu = J^\mu \pi_\nu + \delta^\mu_\nu P\,,$$

$$\partial_\mu J^\mu = 0\,, \qquad\qquad J^\mu = -\frac{\partial P}{\partial \pi_\mu}\,,$$

Identify:

- T^μ_ν - energy-momentum tensor
- J^μ - charge/number current
- π_μ - specific momentum
- P - pressure

Energy-momentum tensor directly from the action

Exercise 24*

Introduce metric $g_{\mu\nu}$ into the action (67). Vary the obtained action over the metric and derive the form of the energy momentum tensor.

Hint: the metric enters into the measure of integration but also in a formula for a chemical potential in terms of π_μ. It is easier to do this exercise in relativistic case.

Lecture 5:
Anomalous action for fluid dynamics

Adler-Bell-Jackiw Anomaly

Massless Dirac fermions in 3+1:

$$Z = \int D\psi\, e^{i\int d^4x\,\mathcal{L}}\,, \qquad \mathcal{L} = \bar{\psi}\, i\partial\!\!\!/\, \psi\,,$$

Global symmetries of classical Lagrangian $\mathcal{L}$ with corresponding Noether currents (vector and axial)

$$\psi \to e^{i\lambda + i\gamma^5\tilde{\lambda}}\psi\,, \qquad j^\mu = \langle\bar{\psi}\gamma^\mu\psi\rangle\,, \qquad j_A^\mu = \langle\bar{\psi}\gamma^\mu\gamma^5\psi\rangle\,.$$

However, the complete quantum theory Z does NOT have both symmetries. For example, in the presence of the vector gauge field

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - A\!\!\!/)\psi\,,$$

we have

$$\partial_\mu j^\mu = 0\,, \qquad \partial_\mu j_A^\mu = 2\boldsymbol{E}\cdot\boldsymbol{B}\,\frac{e^3}{4\pi^2\hbar^2 c^2}\,.$$

Mixed $U(1)_V \times U(1)_A$ anomaly

- 3+1 massless Dirac fermion has $G = U(1)_V \times U(1)_A$ global symmetry
- Gauging this symmetry, i.e., introducing dynamic gauge fields coupled to G-currents makes theory inconsistent - anomaly
- In the presence of background gauge field the theory is consistent but charges do not conserve - mixed t'Hooft anomaly
- This anomaly can be interpreted as a system's coupling to a charge reservoir (5d topological insulator, 4d spectators ...)
- For any system with global symmetry one can ask whether there is an anomaly

Motivation: role of chiral anomaly in fluids

- Quantum anomalies play important role in QFT with Dirac fermions
- Sometimes, Dirac fermions behave as fluids
- What are the effects of the anomalies on hydrodynamics?
- Pioneered by:
  Alekseev, Cheianov, Fröhlich; Son, Surowka; Haehl, Loganayagam, Rangamani; ...
- Hydrodynamics (bosonization) captures anomalies in 1D. What are anomalies in conventional 3D (Euler) fluids?
- Geometrical fluid dynamics:
  Lichnerowitz, Carter, Arnold, Marsden, Holm, ...
- Generalization of some of QHE physics to 4+1 dimensions with 3+1 dimensional boundary

Geometric transport

The Lichnerowicz-Carter equation:

$$\mathcal{J}^\nu(\partial_\nu \pi_\mu - \partial_\mu \pi_\nu) = 0\,, \quad \text{or in form notation} \quad i_{\mathcal{J}} d\pi = 0\,.$$

Then

$$i_{\mathcal{J}}(d\pi \wedge d\pi) = 2(i_{\mathcal{J}} d\pi) \wedge d\pi = 0\,.$$

But $d\pi \wedge d\pi$ is a 4-form and is proportional to the volume form and should be zero. Or we can conclude that $d\pi$ has a rank 2 and, we have a fundamental property of barotropic fluid

$$\boxed{d\pi \wedge d\pi = 0\,.}$$

In components:

$$\boxed{\partial_\mu(\epsilon^{\mu\nu\lambda\rho}\pi_\nu \partial_\lambda \pi_\rho) = 0\,.}$$

Chiral current

In external e/m field $S[\pi] \to S[\pi - A]$ but still

$$\partial_\mu(\epsilon^{\mu\nu\lambda\rho}\pi_\nu\partial_\lambda\pi_\rho) = 0\,.$$

However, the helicity density is NOT gauge invariant. Introduce the chiral current:

$$J_A^\mu = \epsilon^{\mu\nu\lambda\rho}(\pi_\nu - A_\nu)\partial_\lambda(\pi_\rho + A_\rho) = \epsilon^{\mu\nu\lambda\rho}p_\nu(\partial_\nu p_\rho + F_{\nu\rho})\,, \qquad p_\mu \equiv \pi_\mu - A_\mu\,.$$

The chiral current is gauge invariant but it is NOT conserved!

$$\boxed{\partial_\mu J_A^\mu = 2\boldsymbol{E}\cdot\boldsymbol{B}\,,} \qquad d(\pi - A)\wedge d(\pi + A) = -dA\wedge dA\,.$$

Axial/chiral anomaly!

AGA, P.B. Wiegmann, PRL 128, 054501 (2022) [arXiv:2110.11480]

One-dimensional action (reminder)

In 1+1 dimensions with $\pi_\mu = \partial_\mu \phi$ we had

$$S_{1+1} = \int d^2x \, P(\pi - A) + \tilde{A} \wedge (\pi - A) - \int_{M_3} \tilde{A} \wedge dA \tag{75}$$

and as a consequence

$$\partial j = [d\tilde{A}] \,, \qquad \partial \tilde{j} = -[dA] \,, \tag{76}$$

$$j^\mu = -\partial P / \partial \pi_\mu \,, \qquad \tilde{j} = [\pi - A] \,. \tag{77}$$

Can we introduce $\tilde{A}$ into the 3+1 action similarly?

3+1 action with axial gauge field

In 3+1 dimensions

$$S_{3+1} = \int d^4x\, P(\pi - A) + \tilde{A} \wedge (\pi - A) \wedge d(\pi + A) - \int_{M_5} \tilde{A} \wedge dA \wedge dA\,. \qquad (78)$$

Varying over π by diffeos (admissible variations) we derive

$$\partial j = -2[dA \wedge d\tilde{A}]\,, \qquad \partial \tilde{j} = -[dA \wedge dA]\,, \qquad (79)$$

$$j^\mu = -\partial P/\partial \pi_\mu + [(\pi - A) \wedge d\tilde{A}]\,, \qquad \tilde{j} = [(\pi - A) \wedge d(\pi + A)]\,, \qquad (80)$$

and

$$\partial_\nu T^\nu_\mu = F_{\mu\nu} j^\nu + \tilde{F}_{\mu\nu} \tilde{j}^\nu\,, \qquad (81)$$

$$T^\nu_\mu = J^\nu p_\mu + \delta^\nu_\mu P\,. \qquad (82)$$

These currents and anomalies are known as covariant current and covariant anomalies.

However, this is not the most general form of anomalies.

Generalization

In 3+1 dimensions

$$S_{3+1}[\pi,\psi] = \int d^4x\, P(\pi - A) + \tilde{A}\wedge(\pi - A)\wedge d(\pi + A) + \psi\Big[d\pi\wedge d\pi + \alpha dA\wedge dA\Big] \quad (83)$$

$$- \int_{M_5} \tilde{A}\wedge dA\wedge dA + \alpha\tilde{A}\wedge d\tilde{A}\wedge d\tilde{A}\,. \quad (84)$$

Varying over π and ψ freely, we obtain

$$\partial j = -2[dA\wedge d\tilde{A}]\,, \qquad \partial\tilde{j} = -[dA\wedge dA] - 3\alpha[d\tilde{A}\wedge d\tilde{A}]\,. \quad (85)$$

Exercise 25*

Show that this action is axial and vector gauge invariant. Derive expressions for $j, \tilde{j}, T$ for this action.

Hint: see arXiv:2403.12360

Further generalization

In the curved spacetime

$$S_{3+1}[\pi,\psi] = \int d^4x\sqrt{g}\,P(\pi - A) + \tilde{A}\wedge(\pi - A)\wedge d(\pi + A)$$
$$+\,\psi\Big[d\pi\wedge d\pi + \alpha dA\wedge dA + \beta\,\mathrm{tr}(R\wedge R)\Big]$$
$$-\int_{M_5}\tilde{A}\wedge dA\wedge dA + \alpha\tilde{A}\wedge d\tilde{A}\wedge d\tilde{A} + \beta\tilde{A}\wedge\mathrm{tr}(R\wedge R)\,. \qquad (86)$$

Varying over π and ψ freely, we obtain mixed axial-gravitational anomaly

$$D_\nu j^\nu = -2[dA\wedge d\tilde{A}]\,, \qquad D_\nu \tilde{j}^\nu = -\Big[dA\wedge dA + 3\alpha[d\tilde{A}\wedge d\tilde{A} + \beta\,\mathrm{tr}(R\wedge R)\Big]\,. \qquad (87)$$

Exercise 26*

Derive expressions for $j, \tilde{j}, T$ *for this action.*

Hint: see arXiv:2403.12360

Conclusion: five lectures in three sessions!

1. Lecture 1: Introduction to Euler fluid dynamics
2. Lecture 2: Introduction to chiral anomaly
3. Lecture 3: Covariant form of Euler equation
4. Lecture 4: Variational principle for fluid dynamics
5. Lecture 5: Anomalous action for fluid dynamics

+ many more details in arXiv:2403.12360 and references therein
+ many generalizations producing nondissipative fluids with anomalies and assortments of conservation laws
+ dissipative versions using similar tricks but in Keldysh formalism (work in progress)