Application of Generalized Partial Wave Amplitude in EFT

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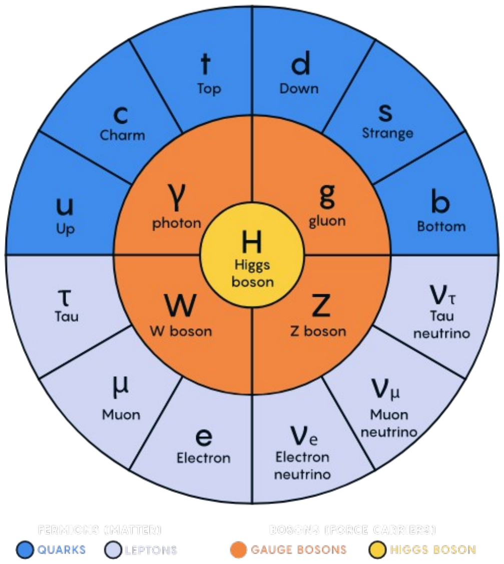
- Introduction of SMEFT and Young Tensor Method
- J-Basis (partial wave) amplitude
- Application 1: Partial wave unitarity bound of EFT operators
- Application 2: Finding Tree-level UV origin of EFT operators
- Conclusion

SM is not complete

Standard Model

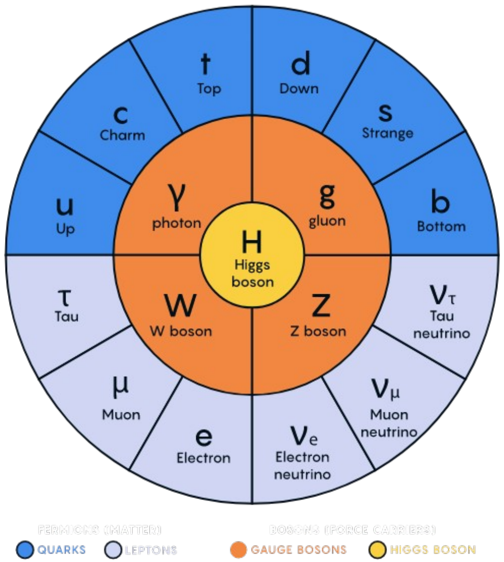


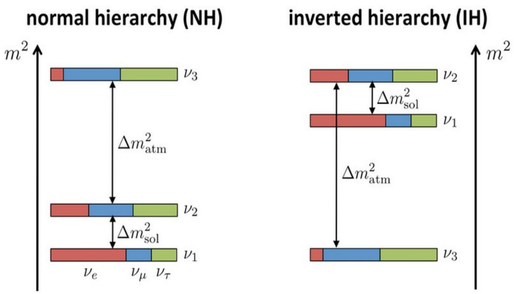
- In 1979, Sheldon Glashow, Abdus Salam, and Steven Weinberg shared the Nobel prize for their contributions to the unification of the electromagnetic and weak forces [1] .
- In 1984, Carlo Rubbia and Simon van der Meer shared the Nobel prize for their decisive contributions to the discovery of the W and Z bosons, the carriers of the weak force [1] .
- In 1999, Gerard 't Hooft and Martinus Veltman shared the Nobel prize for their elucidation of the quantum structure of the electroweak interactions [1] .
- In 2004, David Gross, Hugh David Politzer, and Frank Wilczek shared the Nobel prize for their discovery of asymptotic freedom, the property that explains the behavior of the strong force [1] .
- In 2008, Yoichiro Nambu, Makoto Kobayashi, and Toshihide Maskawa shared the Nobel prize for their discoveries of the mechanisms of spontaneous symmetry breaking and CP violation in the Standard Model [1] .
- In 2013, François Englert and Peter Higgs shared the Nobel prize for their theoretical discovery of the Higgs mechanism, which gives mass to the particles in the Standard Model [2] [1] .

Nobel prices related to the Standard Model

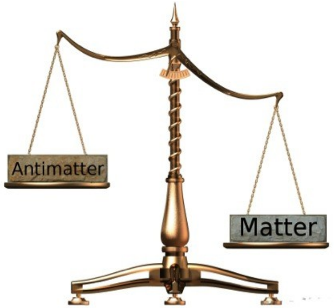
SM is not complete

Standard Model

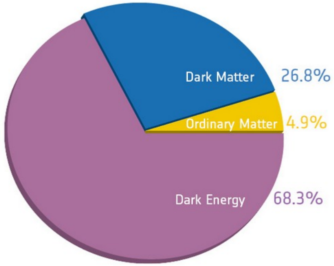




Neutrino Mass



Matter-Antimatter Asymmetry



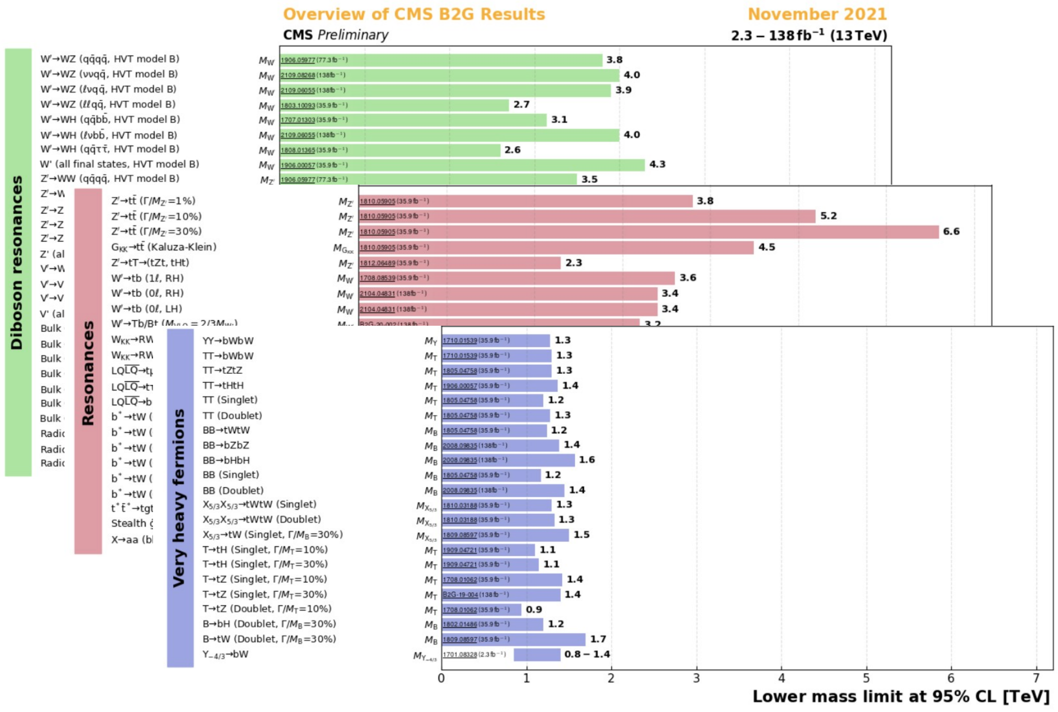
Dark Matter and Dark Energy

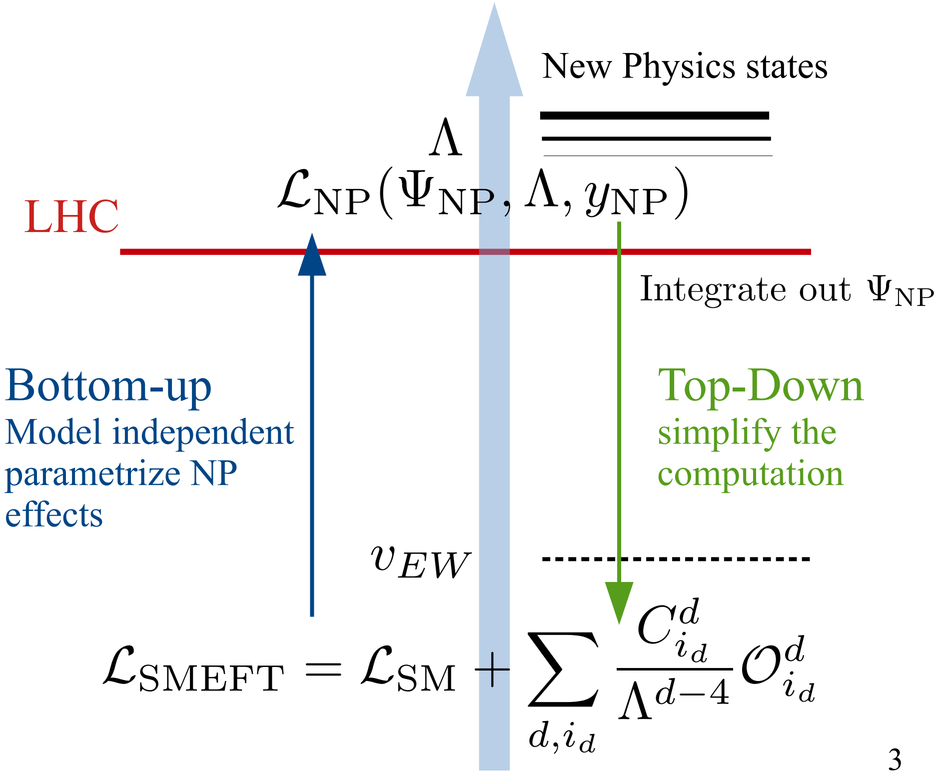
New Physics Must Exist

$$\mathcal{L}_{\rm NP}(\phi_{\rm SM}(x), \psi_{\rm NP}(x), \ldots; g_i, y_i, \tilde{g}_i, \tilde{y}_i, \Lambda, \ldots)$$

Why SMEFT

- New Physics scale might be large compared to the SM Electroweak scale
- SMEFT provides a Universal way to parameterize the all kinds of new physics effects
- SMEFT simplifies and better organizes the theoretical calculation





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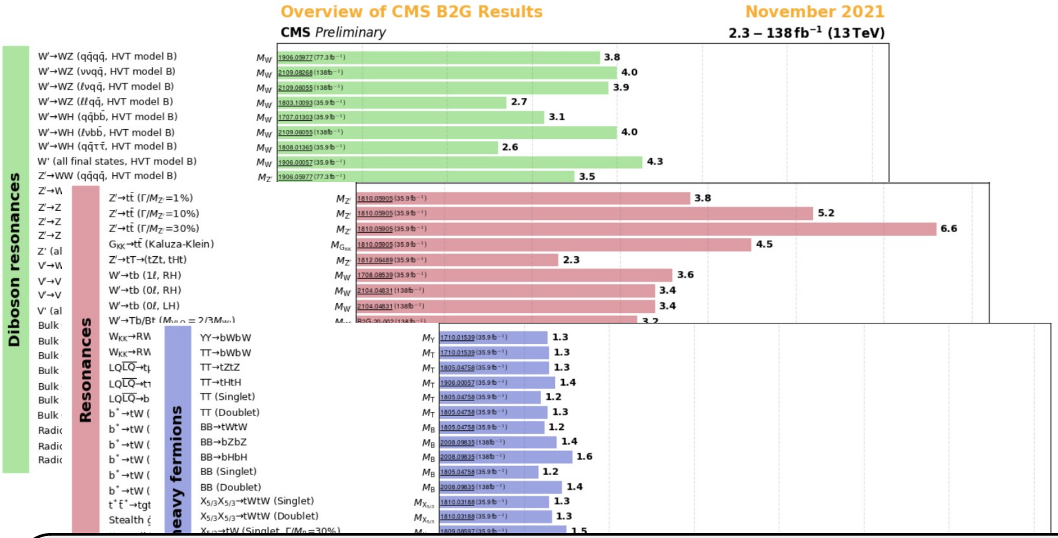
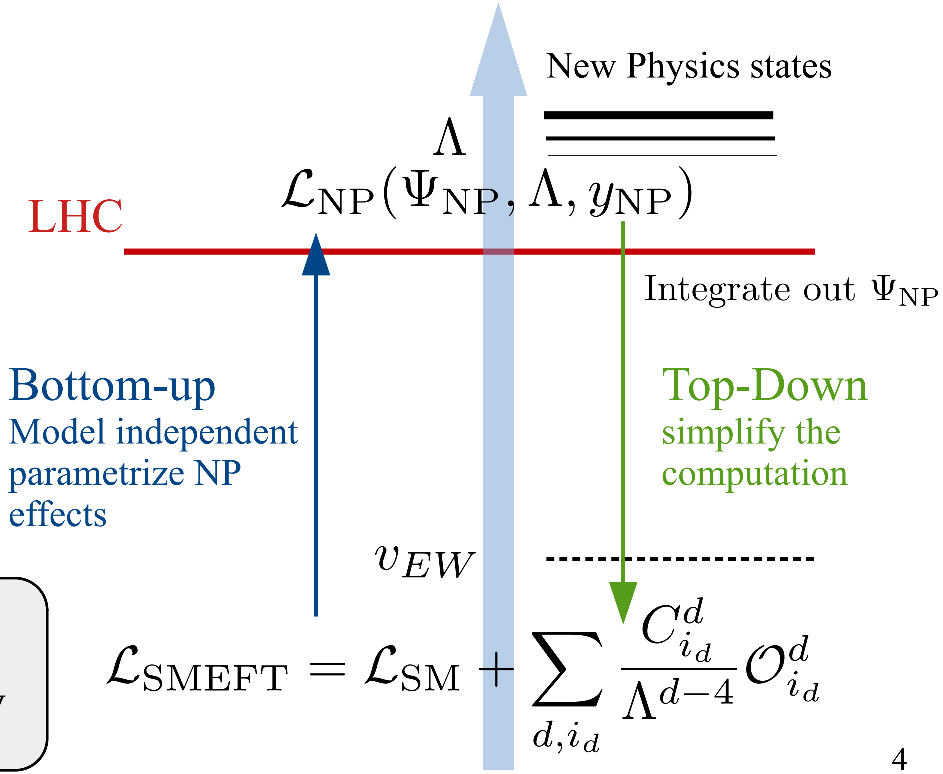
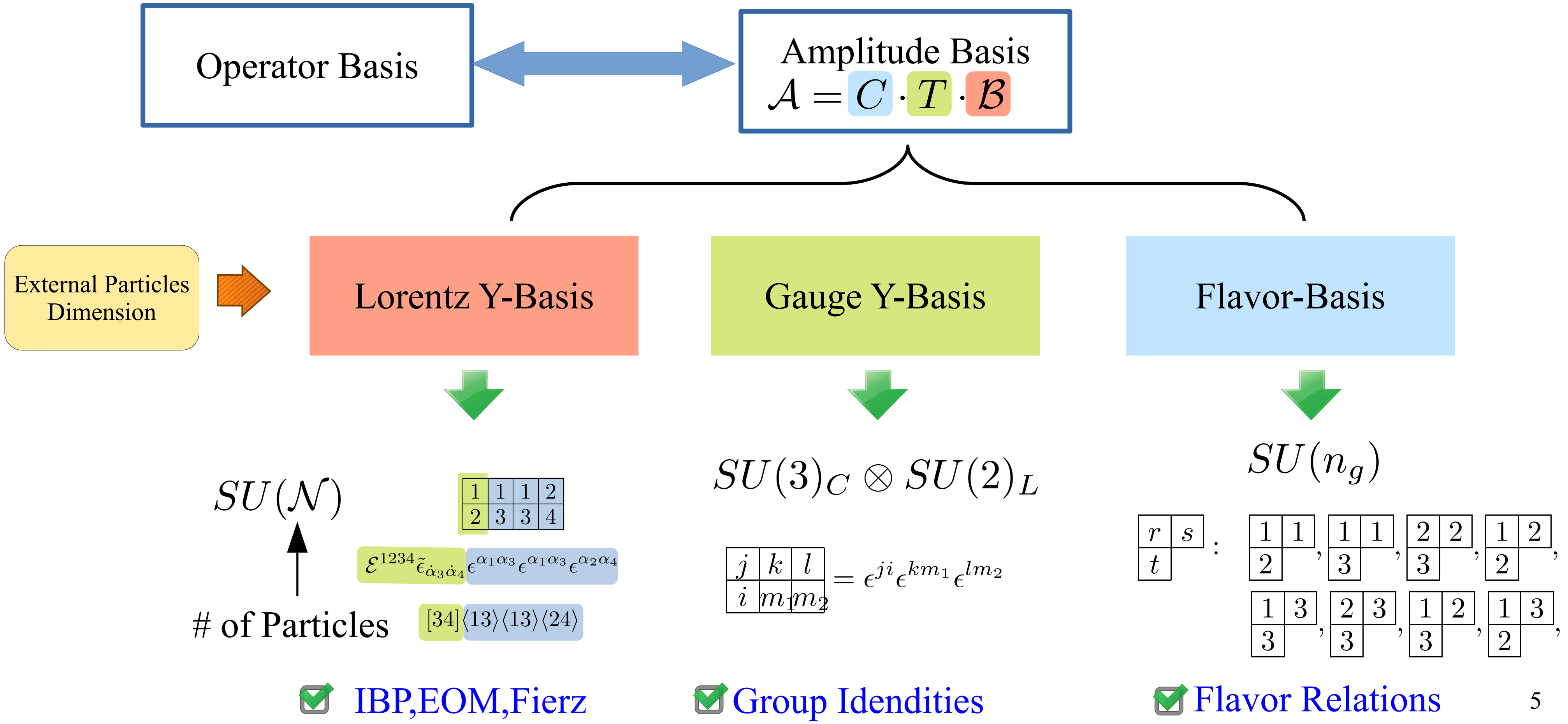




First Task: Define the EFT – Construct Operator Basis
non-trivial: # of operator is large, subject to redundancy

Young Tensor Method

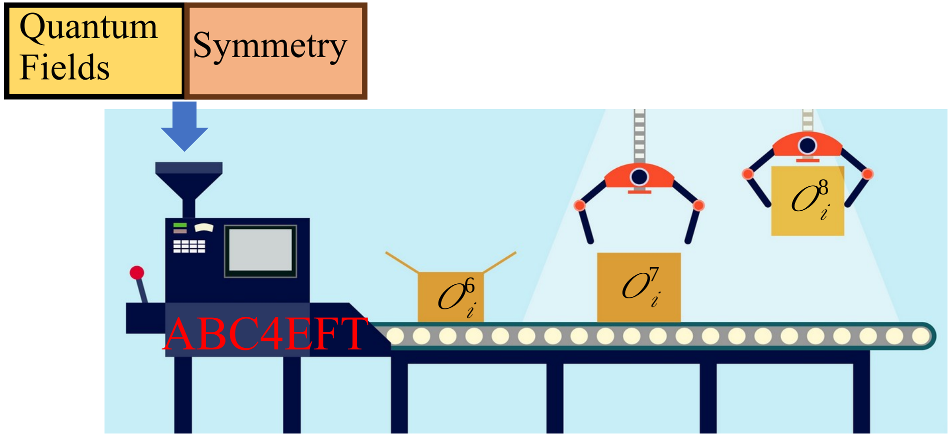
[**HLL**, et.al. 2005.00008, 2201.04639]



Young Tensor Method

[HLL, et.al. 2005.00008, 2201.04639]

Mathemtaica program ABC4EFT:
automated the basis construction



SMEFT dim-8	Phys. Rev. D 104, 015026
SMEFT dim-9	Phys. Rev. D 104, 015025
LEFT dim<=9	JHEP 06 (2021)
LEFT dim<=9	JHEP 11 (2021)
GRSMEFT dim<=9	JHEP 10 (2023)

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ABC4EFT 1.1.0
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A Mathematica Package for
Amplitude Basis Construction for Effective Field Theories

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The package is available at hepforge
For the latest version, see the GitHub
If you use this package in your research,
Please cite: arXiv: 2201.04639, 2005.00008, 2007.07899
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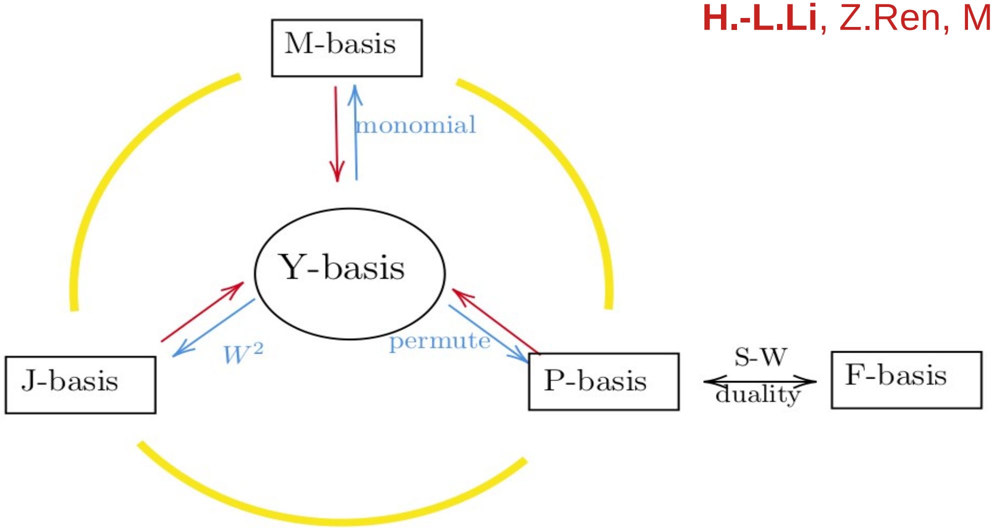
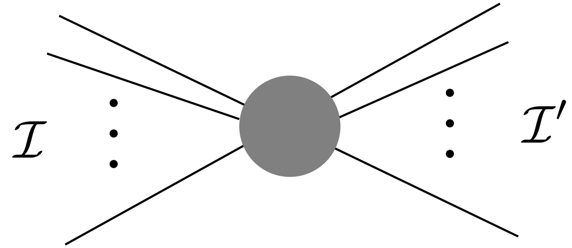
Different Operator Basis

Operator Type: Fixed field contents and the number of derivative $W_L W_L H H^\dagger D,\ Q^3 L$

H.-L.Li, Z.Ren, M.-L.Xiao, J.-H.Yu, Y.-H. Zheng, 2201.04639

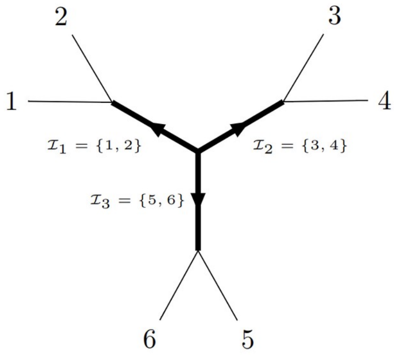


- **Y-basis**: obtained with Young tensor method and amplitude operator correspondence.
- **M-basis**: independent monomial operators
- **P-basis**: irrep of symmetric group of repeated fields—also irrep of $SU(n_f)$
- **F-basis**: independnent flavor tensor spaces – eliminate the improper and redundant flavor tensors
- **J-basis: Eigen-basis of Casimirs**



$$\mathcal{O}^{J,\mathbf{R}}_{\mathcal{I}\to\mathcal{I}'} \sim \mathcal{T}(\mathbf{R})\bar{B}^J\ (\mathcal{I}\to\mathcal{I}')$$

$$\begin{cases} W^2_{\mathcal{I}}\bar{B}^J = -s_{\mathcal{I}}J(J+1)\bar{B}^J \\ \mathbb{C}_{\mathcal{I}}\mathcal{T}(\mathbf{R}) = C(\mathbf{R})\mathcal{T}(\mathbf{R}) \end{cases}$$



$$\mathcal{O}^{J_1\mathbf{R}_1, J_2\mathbf{R}_2, J_3\mathbf{R}_3} \sim \mathcal{T}(\mathbf{R}_1,\mathbf{R}_2,\mathbf{R}_3)B^{J_1,J_2,J_3}$$

J-Basis as Generalized Partial-Wave Basis

Pioncare Casimir Operator acting on amplitude: [M.-Y. Jiang, J.Shu, M.-L.Xiao, Y.-H.Zheng, 2001.04481]

$$W_{\mathcal{I}}{}^2 \bar{B}^J(\mathcal{I} \to \mathcal{I}') = -s_{\mathcal{I}} J(J+1) \bar{B}^J(\mathcal{I} \to \mathcal{I}')$$

$$M_{\mathcal{I},\alpha\beta} = i \sum_{i\in\mathcal{I}} \left(\lambda_{i\alpha} \frac{\partial}{\partial \lambda_i^\beta} + \lambda_{i\beta} \frac{\partial}{\partial \lambda_i^\alpha} \right)$$

$$\tilde{M}_{\mathcal{I},\dot{\alpha}\dot{\beta}} = i \sum_{i\in\mathcal{I}} \left(\tilde{\lambda}_{i\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\beta}}} + \tilde{\lambda}_{i\dot{\beta}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}} \right)$$

$$W_{\mathcal{I}}^2 = \frac{1}{8} P^2 \left(\mathrm{Tr}\left[M_{\mathcal{I}}^2\right] + \mathrm{Tr}\left[\tilde{M}_{\mathcal{I}}^2\right] \right) - \frac{1}{4} \mathrm{Tr}\left[P^\top M_{\mathcal{I}} P \widetilde{M}_{\mathcal{I}}\right]$$

Partial wave basis is **eigen-basis of the Casimir W^2**

Given an amplitude basis, one can find the **representation matrix** of the Casimir operator

Dim[amplitudes] $\{h_1, h_2 \ldots, h_N\}$ → $\{\mathcal{B}_i^y\}$ → $W_{\mathcal{I}}^2 \mathcal{B}_i^y = -s_{\mathcal{I}} \mathcal{W}_i^{2\,j} \mathcal{B}_j^y$

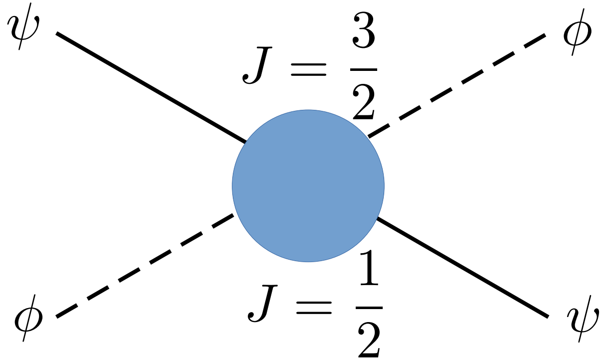
J-Basis as Generalized Partial-Wave Basis

Take $L_1 L_2 H_3 H_4 D^2$ as an example:

$\mathcal{W}^2$

$$\mathcal{B}^{y}_{\psi^2\phi^2D^2} = \begin{pmatrix} s_{34}\langle 12\rangle \\ [34]\langle 13\rangle\langle 24\rangle \end{pmatrix}, \quad W^2_{\{13\}}\mathcal{B}^y = s_{13}\begin{pmatrix} -\frac{15}{4} & 2 \\ 0 & -\frac{3}{4} \end{pmatrix}\mathcal{B}^y, \quad \mathcal{K}^{jy}_{\mathcal{B}} = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \mathcal{B}^j = \mathcal{K}^{jy}_{\mathcal{B}}\mathcal{B}^y = \begin{cases} 3s_{34}\langle 12\rangle + 2[34]\langle 13\rangle\langle 24\rangle & J=\frac{3}{2} \\ \langle 13\rangle\langle 24\rangle & J=\frac{1}{2} \end{cases}$$



J-Basis as Generalized Partial-Wave Basis

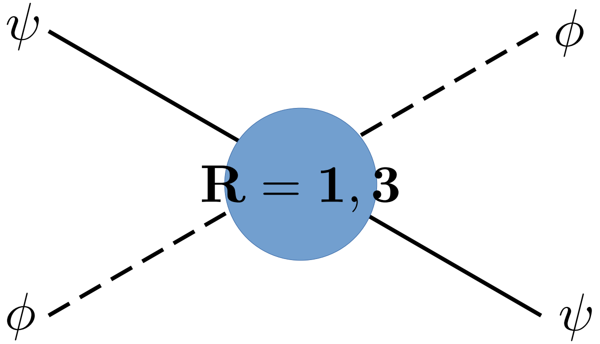
Take $L_1 L_2 H_3 H_4 D^2$ as an example:

$$\mathcal{T}^m_{LLHH} = \begin{pmatrix} \epsilon^{ik}\epsilon^{jl} \\ \epsilon^{ij}\epsilon^{kl} \end{pmatrix}, \quad \underset{\{13\}}{\mathbb{C}_2} \circ \mathcal{T}^m = (\underset{\{13\}}{C_2})^{\mathrm{T}}.\mathcal{T}^m = \begin{pmatrix} 0 & 0 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} \epsilon^{ik}\epsilon^{jl} \\ \epsilon^{ij}\epsilon^{kl} \end{pmatrix}.$$

$C_2(\mathbf{1})$ $\quad$ $C_2(\mathbf{3})$

$$\mathcal{K}^{jm}_G.(\underset{\{13\}}{C_2})^{\mathrm{T}}(\mathcal{K}^{jm}_G)^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix} \text{ with } \mathcal{K}^{jm}_G = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix}$$

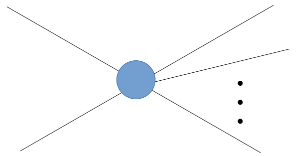
$$\Rightarrow \mathcal{T}^j = \mathcal{K}^{jm}_G \mathcal{T}^m = \begin{cases} \epsilon^{ik}\epsilon^{jl} & \mathbf{R} = \mathbf{1} \\ \epsilon^{ik}\epsilon^{jl} - 2\epsilon^{ij}\epsilon^{kl} & \mathbf{R} = \mathbf{3} \end{cases}$$



Application 1: Partial-Wave Unitarity Bound

[C. Degrande **HLL**, L.-X. Xu. 2506.xxxxx]

- EFT amplitudes scale as Energy with positive power → Unitarity violation
- Partial-wave unitarity bound → Consistency check for the perturbativity of EFT expansion
  – important for constraints from collider experiments
  – provides theoretical prior for a Bayesian global fit
- Traditional unitarity bound based on the 2-to-2 scattering
  – good for operators with equal or less than 4 fields
  – beyond dimension-6 more operator contain more than 4 fields



- Generalized partial-wave amplitude for 2-to-N scattering is needed
  – can be derived from 2-to-2 scattering amplitude
  – needs to know the partial-wave basis for 2-to-N: J-Basis!
  – needs to know the normalization for the basis: N-body phase-space integral

Application 1: Partial-Wave Unitarity Bound

Unitarity of S-matrix: $\mathcal{T} - \mathcal{T}^\dagger = i\mathcal{T}^\dagger\mathcal{T}$

Sandwich by two 2 particle states:

$$M_{i\to f} - M^*_{f\to i} = i\sum_X \int d\Pi_X M^*_{f\to X} M_{i\to X} (2\pi)^4 \delta^4(p_i - p_X)$$

Taking forward scattering limit:

$$2\mathrm{Im}M(0) = \sum_{X\neq i}\int d\Pi_X |M_{i\to X}|^2 (2\pi)^4\delta^4(p_X - p_i) + \int \frac{\sin\theta d\theta}{16\pi}|M(\theta)|^2$$

$$2\sum_J 16\pi(J+1/2)\mathrm{Im}T^J(s) = \sum_{X\neq i}\int d\Pi_X |M_{i\to X}|^2(2\pi)^4\delta^4(p_X - p_i) + \sum_J 16\pi(J+1/2)|T^J(s)|^2$$

Assuming the expansion for 2-to-N amplitude:

$$M_{i\to X} = \sum_{J,a} C^{Ja}_{i\to X} B^{Ja}_{i\to X} \qquad \int d\Pi_X B^{Ja}_{i\to X}(B^{J'a'}_{i\to X})^*(2\pi)^4\delta^4(p_X - p_i) = g^{Ja}_{i\to X}(s)\delta_{aa'}\delta_{JJ'}$$

Master formula:

$$\frac{\sum_{a,X\neq i} g^{Ja}_{i\to X}(s)|C^{Ja}_{i\to X}|^2}{16\pi(J+1/2)} \leq 1$$

Application 1: Partial-Wave Unitarity Bound

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Master formula:

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Outstanding problem:
computing *g(s)* analytically
Why? Numerical hard; Exact zero

Application 1: Partial-Wave Unitarity Bound

N-body massless phase-space integral

$$\prod_{i=1}^{N} \frac{d^3 p_i}{(2\pi)^3 2E_i} \; (2\pi)^4 \delta^4 \left(P - \sum_{i=1}^{N} p_i \right)$$

Parameterize the final momenta with spinor helicity variables: $\lambda^\alpha(p_i) = u_i \lambda^\alpha(k_1) + v_i \lambda^\alpha(k_2)$

$$\begin{pmatrix} \lambda^\alpha(p_1) & \lambda^\alpha(p_2) & \cdots & \lambda^\alpha(p_N) \end{pmatrix}^T = \begin{pmatrix} u_1 & u_2 & \cdots & u_N \\ v_1 & v_2 & \cdots & v_N \end{pmatrix}^T \begin{pmatrix} \lambda^\alpha(k_1) \\ \lambda^\alpha(k_2) \end{pmatrix}$$

Spinor variables for two Initial state particles

u, *v* are two complex variables

$$d\Pi_N = (2\pi)^{4-3N} s^{N-2} \frac{d^N u d^N v}{U(1)^N} \delta(1 - |\vec{u}|^2) \delta(1 - |\vec{v}|^2) \delta^2(\vec{u}^\dagger \vec{v})$$

Momentum conservation

Little group redundancy for an overall phase of the spinor variables can be used to fix the phase of *u* to zero

$p^\mu (\sigma_\mu)^{\alpha\dot{\alpha}} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}$ Invariant under $\lambda \to e^{i\phi}\lambda, \tilde{\lambda} = \lambda^*$

Application 1: Partial-Wave Unitarity Bound

Processing *u* integral: $u_i = r_i e^{-i\phi_i}$

$$\frac{d^N u}{U(1)^N}\delta(1-|\vec{u}|^2) = \frac{\prod_{i=1}^N r_i dr_i d\phi_i}{\boxed{U(1)^N}}\delta\left(1-\sum_{i=1}^N r_i^2\right) = \int \prod_{i=1}^N [r_i dr_i d\phi_i \boxed{\delta(\phi_i)}]\,\delta\left(1-\sum_{i=1}^N r_i^2\right)$$

r_i can be parameterized on the spherical coordinate of S^{N-1}

$$u_i = r_i$$

$$\begin{aligned} r_N &= \cos\theta_{N-1}\ , \\ r_{N-1} &= \sin\theta_{N-1}\cos\theta_{N-2}\ , \\ r_{N-2} &= \sin\theta_{N-1}\sin\theta_{N-2}\cos\theta_{N-3}\ , \\ &\vdots \\ r_2 &= \sin\theta_{N-1}\ldots\sin\theta_2\cos\theta_1\ , \\ r_1 &= \sin\theta_{N-1}\ldots\sin\theta_2\sin\theta_1 \end{aligned}$$

$$\frac{d^N u}{U(1)^N}\delta(1-|\vec{u}|^2) = \frac{1}{2}\left(\prod_{i=1}^{N-1}\sin^{2i-1}\theta_i\cos\theta_i\right)d\theta_1\ldots d\theta_{N-1}$$

Application 1: Partial-Wave Unitarity Bound

Processing *v* integral:

$$d^N v \delta(1-|\vec{v}|^2)\boxed{\delta^2(\vec{u}^\dagger \vec{v})} = \frac{d^{N-1}v}{|u_N|^2}\;\delta\left(1-\sum_{i=1}^{N-1}|v_i|^2-|v_N|^2\right)$$

$$\boxed{v_N = -\frac{\sum_{i=1}^{N-1} u_i^* v_i}{u_N^*} = -\frac{\sum_{i=1}^{N-1} r_i v_i}{r_N}}$$

Change integral variables $v = Ov'$ $\quad (O^{-1})^2 = I + \frac{1}{r_N^2}\mathbf{r}^T\mathbf{r} \quad \mathbf{r} = (r_1, r_2, \ldots, r_{N-1})$

$$d^N v \delta(1-|\vec{v}|^2)\delta^2(\vec{u}^\dagger \vec{v}) = d^{N-1}v'\;\delta\left(1-\sum_{i=1}^{N-1}|v_i'|^2\right)$$

Embedding of S^{2N-3} in $\mathbb{C}^{N-1}$

$$\begin{aligned}
v_1' &= e^{-i\xi_1}\cos\eta_1\\
v_2' &= e^{-i\xi_2}\sin\eta_1\cos\eta_2\\
&\vdots\\
v_{N-2}' &= e^{-i\xi_{N-2}}\sin\eta_1\ldots\sin\eta_{N-3}\cos\eta_{N-2}\\
v_{N-1}' &= e^{-i\xi_{N-1}}\sin\eta_1\ldots\sin\eta_{N-3}\sin\eta_{N-2}.
\end{aligned}$$

$$d^{N-1}v'\;\delta\left(1-\sum_{i=1}^{N-1}|v_i'|^2\right) = \left(\prod_{k=1}^{N-2}\cos\eta_k\sin^{2(N-2-k)+1}\eta_k\right)d\xi_i\ldots d\xi_{N-1}d\eta_i\ldots d\eta_{N-2}$$

Application 1: Partial-Wave Unitarity Bound

Processing *v* integral:

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$$\boxed{v_N = -\frac{\sum_{i=1}^{N-1}u_i^* v_i}{u_N^*} = -\frac{\sum_{i=1}^{N-1}r_i v_i}{r_N}}$$

Change integral variables $v = Ov'$ $\quad (O^{-1})^2 = I + \frac{1}{r_N^2}\mathbf{r}^T\mathbf{r} \quad \mathbf{r} = (r_1, r_2, \ldots, r_{N-1})$

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\end{aligned}$$

$$d^{N-1}v'\,\delta\left(1-\sum_{i=1}^{N-1}|v_i'|^2\right) = \left(\prod_{k=1}^{N-2}\cos\eta_k\sin^{2(N-2-k)+1}\eta_k\right)d\xi_i\ldots d\xi_{N-1}d\eta_i\ldots d\eta_{N-2}$$

Key point: *O* is also analytically solvable

$(O^{-1})^2$ is rank-1 update of identity matrix using Sherman–Morrison Formula:

$$O = I - \left(\frac{\sqrt{1+\beta}-1}{\beta\sqrt{1+\beta}}\right)\frac{\mathbf{r}^T\mathbf{r}}{r_N^2},\quad \beta = \frac{|\mathbf{r}|^2}{r_N^2}$$

Application 1: Partial-Wave Unitarity Bound

Summarize: *u* and *v* completely expressed with angular and phase parameters, and all the delta functions are resolved

$$v_i = O_{ij} v'_j \;\; (i, j \in 1, 2, \ldots, N-1) \qquad v_N = -\frac{\sum_{i=1}^{N-1} u_i^* v_i}{u_N^*} = -\frac{\sum_{i=1}^{N-1} r_i v_i}{r_N}$$

$$O = I - \left(\frac{\sqrt{1+\beta}-1}{\beta\sqrt{1+\beta}}\right)\frac{\mathbf{r}^T\mathbf{r}}{r_N^2}, \quad \beta = \frac{|\mathbf{r}|^2}{r_N^2}$$

$$u_i = r_i$$

$$\begin{aligned} r_N &= \cos\theta_{N-1} \;, \\ r_{N-1} &= \sin\theta_{N-1}\cos\theta_{N-2} \;, \\ r_{N-2} &= \sin\theta_{N-1}\sin\theta_{N-2}\cos\theta_{N-3} \;, \\ &\vdots \\ r_2 &= \sin\theta_{N-1}\ldots\sin\theta_2\cos\theta_1 \;, \\ r_1 &= \sin\theta_{N-1}\ldots\sin\theta_2\sin\theta_1 \end{aligned}$$

$$\begin{aligned} v'_1 &= e^{-i\xi_1}\cos\eta_1 \\ v'_2 &= e^{-i\xi_2}\sin\eta_1\cos\eta_2 \\ &\vdots \\ v'_{N-2} &= e^{-i\xi_{N-2}}\sin\eta_1\ldots\sin\eta_{N-3}\cos\eta_{N-2} \\ v'_{N-1} &= e^{-i\xi_{N-1}}\sin\eta_1\ldots\sin\eta_{N-3}\sin\eta_{N-2}. \end{aligned}$$

Application 1: Partial-Wave Unitarity Bound

For 3-body final state:

$$u_1 = \sin\theta_2 \sin\theta_1,$$
$$u_2 = \cos\theta_1 \sin\theta_2,$$
$$u_3 = \cos\theta_2.$$

An equivalent parameterization [2005.06983]

$$v_1 = e^{-i\xi_1}\cos\eta_1\left(\cos^2\theta_1 + \cos\theta_2\sin^2\theta_1\right) + e^{-i\xi_2}\sin\eta_1(\cos\theta_2 - 1)\cos\theta_1\sin\theta_1 \ ,$$
$$v_2 = e^{-i\xi_1}\cos\eta_1(\cos\theta_2 - 1)\cos\theta_1\sin\theta_1 + e^{-i\xi_2}\sin\eta_1\left(\cos\theta_2\cos^2\theta_1 + \sin^2\theta_1\right) \ ,$$
$$v_3 = -\sin\theta_2(e^{-i\xi_1}\cos\eta_1\sin\theta_1 + e^{-i\xi_2}\cos\theta_1\sin\eta_1).$$

For 4-body final state:
Our new result

$$u_1 = \sin\theta_3\sin\theta_2\sin\theta_1,$$
$$u_2 = \sin\theta_3\sin\theta_2\cos\theta_1,$$
$$u_3 = \sin\theta_3\cos\theta_2,$$
$$u_4 = \cos\theta_3,$$

$$\begin{aligned}
v_1 \;=\; & e^{-i\xi_2}\sin\eta_1\cos\eta_2(\cos\theta_3 - 1)\sin^2\theta_2\sin\theta_1\cos\theta_1 \\
& + e^{-i\xi_3}\sin\eta_1\sin\eta_2(\cos\theta_3 - 1)\sin\theta_2\cos\theta_2\sin\theta_1 \\
& + e^{-i\xi_1}\cos\eta_1\left(\sin^2\theta_2\left(\cos\theta_3\sin^2\theta_1 + \cos^2\theta_1\right) + \cos^2\theta_2\right), \\
v_2 \;=\; & e^{-i\xi_2}\sin\eta_1\cos\eta_2\left(\sin^2\theta_2\left(\cos\theta_3\cos^2\theta_1 + \sin^2\theta_1\right) + \cos^2\theta_2\right) \\
& + e^{-i\xi_3}\sin\eta_1\sin\eta_2(\cos\theta_3 - 1)\sin\theta_2\cos\theta_2\cos\theta_1 \\
& + e^{-i\xi_1}\cos\eta_1(\cos\theta_3 - 1)\sin^2\theta_2\sin\theta_1\cos\theta_1, \\
v_3 \;=\; & e^{-i\xi_2}\sin\eta_1\cos\eta_2(\cos\theta_3 - 1)\sin\theta_2\cos\theta_2\cos\theta_1 \\
& + e^{-i\xi_3}\sin\eta_1\sin\eta_2\left(\cos\theta_3\cos^2\theta_2 + \sin^2\theta_2\right) \\
& + e^{-i\xi_1}\cos\eta_1(\cos\theta_3 - 1)\sin\theta_2\cos\theta_2\sin\theta_1, \\
v_4 \;=\; & -\sin\theta_3\left[\sin\theta_2\left(e^{-i\xi_2}\sin\eta_1\cos\eta_2\cos\theta_1 + e^{-i\xi_1}\cos\eta_1\sin\theta_1\right)\right. \\
& \left. + e^{-i\xi_3}\sin\eta_1\sin\eta_2\cos\theta_2\right].
\end{aligned}$$

Generalization to N body is straightforward!

Application 1: Partial-Wave Unitarity Bound

Example: $M = \langle 14\rangle[45]$

$$|4\rangle = u_2(\theta_1,\theta_2)|1\rangle + v_2(\theta_1,\theta_2,\xi_1,\xi_2,\eta_2)|2\rangle$$
$$|5\rangle = u_3(\theta_1,\theta_2)|1\rangle + v_3(\theta_1,\theta_2,\xi_1,\xi_2,\eta_2)|2\rangle$$

$$|4] = |4\rangle^*,\ |5] = |5\rangle^*$$

$$M = (|v_2|^2 u_3^* - v_2 u_2^* v_3^*)\boxed{\langle 12\rangle[21]}$$

← Center of mass energy square s

For on-shell local amplitudes the integral factorize, thus can always be done analytically

$$\int |M|^2 \mathrm{dPS}_3 = \int f_1(\theta_1)d\theta_1 \int f_2(\theta_2)d\theta_2 \int f_3(\eta_1)d\eta_1 \int f_4(\xi_1)d\xi_1 \int f_5(\xi_2)d\xi_2$$

We provide the Mathematica code to compute the integral for 3- and 4-body final state

M_1 M_2 $\#FS$

```
PSIntAMPUser[ab[1, 4] × sb[4, 5], ab[1, 4] × sb[4, 5], 3, {2, 3}]
```

incoming label

$$\frac{s^3}{3072\,\pi^3}$$

$$\int d\Pi_{k\notin\{i,j\}}(2\pi)^4\delta^4(p_i+p_j-\sum_{k\notin\{i,j\}}p_k)\mathrm{M}_1^*\mathrm{M}_2$$

Application 1: Partial-Wave Unitarity Bound

A SMEFT dim-8 example: $C_{f1_f6}|H|^2 H^\dagger \overleftrightarrow{D}_\mu H(\overline{e_R}_{f_6}\gamma^\mu e_{Rf_1})$

1. The corresponding local on-shell amplitude is:

$$
\begin{aligned}
M^{f_1 f_6}_{i_2 i_3 i_4 i_5} = & \quad C_{f_1 f_6}\Big\{ \big(\delta^{i_2}_{i_4}\delta^{i_3}_{i_5} 15[56] + \mathrm{sym}(45)\big) + \mathrm{sym}(23) \\
- & \quad \big(\delta^{i_2}_{i_4}\delta^{i_3}_{i_5} 13[36] + \mathrm{sym}(23)\big) + \mathrm{sym}(45)\Big\},
\end{aligned}
$$

2. For the channel: $H_{i_2}(p_2)H_{i_3}(p_3) \to e^+(-p_1)e^-(-p_6)H^{\dagger i_4}(-p_4)H^{\dagger i_5}(-p_5)$
Derive the J-basis and normalization factors

$$
\begin{aligned}
B^{J=1} &= 2\langle 13\rangle[36] + \langle 14\rangle[46] + \langle 15\rangle[56], \quad g^{J=1} = \frac{s^4}{184320\pi^5} \\
B^{J=0}_1 &= \frac{\langle 14\rangle[46] + \langle 15\rangle[56]}{\sqrt{2}}, \quad g^{J=1}_1 = \frac{s^4}{737280\pi^5}, \\
B^{J=0}_2 &= \frac{-\langle 14\rangle[46] + \langle 15\rangle[56]}{\sqrt{2}}, \quad g^{J=1}_2 = \frac{s^4}{1474560\pi^5}
\end{aligned}
$$

$\Rightarrow$

$$\frac{\sum_{f_1 f_6} |C_{f_1 f_6}|^2 s^4}{737280\pi^6} \leq 1$$

3. Iterate over all possible scattering channels and find out the strongest bound

Application 2: Tree-level UV origins of operators

- The ultimate goal is to determine the concerte UV theory
- If the non-zero Wilson coefficient were to be measured,
  – Restricted to a subset of UV theories responsible for that operator
- Outstanding problem: UV – EFT correspondence

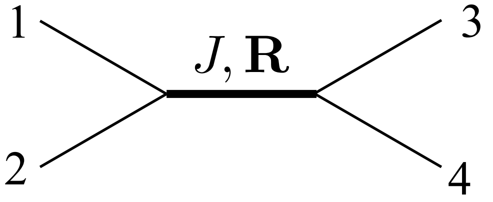
"Top down": Enumerate all possible $\mathcal{L}_{\text{UV}}$ and perform matching:
– Tree-Level SMEFT dim-6 [J. de Blas, et.al. 1711.10391]
– One-Loop SMEFT dim-6 [G. Guedes, et.al 2303.16965]
– partial SMEFT dim-8 [J.Chakrabortty, et.al 2210.14761, 2306.09103, 2308.03849]

"Bottom up": **J-Basis/UV correspondence**
– Tree-Level SMEFT dim-6,7 [**HLL**, et.al. 2204.03660; Xu-Xiang Li, et.al 2307.1038]
– Tree-Level SMEFT dim-8 [**HLL**, et.al. 2309.15933]

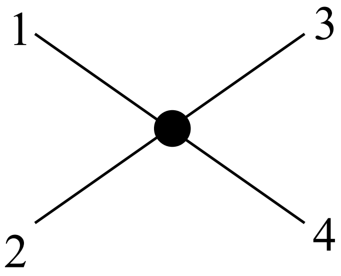
Application 2: Tree-level UV origins of operators

- Intuition – diagrammatic matching

UV Resonance Φ — spin J, gauge rep: $\mathbf{R}$ $\qquad$ $\psi_1\psi_2\Phi, \psi_3\psi_4\Phi$



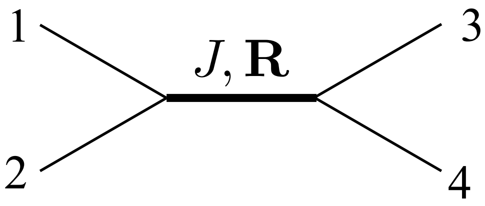
$\mathcal{A}_{\text{UV}}^{J_{12}=J;\mathbf{R}_{12}=\mathbf{R}}$



Application 2: Tree-level UV origins of operators

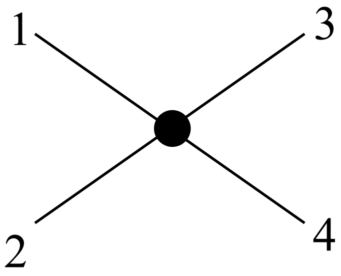
- Intuition – diagrammatic matching

UV Resonance Φ — spin J, gauge rep: $\mathbf{R}$ $\quad \psi_1\psi_2\Phi, \psi_3\psi_4\Phi$



$$\mathcal{A}_{\mathrm{UV}}^{J_{12}=J;\mathbf{R}_{12}=\mathbf{R}}$$

Expanding the propagator

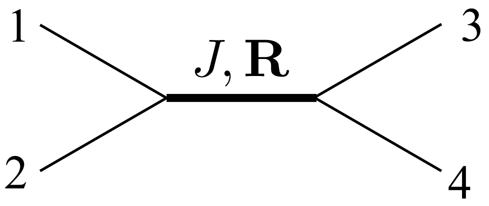


$$\mathcal{A}_{\mathrm{EFT}}^{J_{12}=J;\mathbf{R}_{12}=\mathbf{R}}$$ → J-basis amplitude: An extension of partial wave amplitude

Application 2: Tree-level UV origins of operators

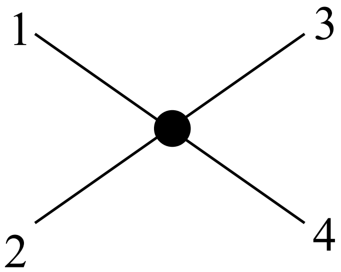
- Intuition – diagrammatic matching

UV Resonance Φ — spin J, gauge rep: $\mathbf{R}$ $\qquad \psi_1\psi_2\Phi, \psi_3\psi_4\Phi$



$$\mathcal{A}_{\mathrm{UV}}^{J_{12}=J;\mathbf{R}_{12}=\mathbf{R}}$$

Expanding the propagator



$$\mathcal{A}_{\mathrm{EFT}}^{J_{12}=J;\mathbf{R}_{12}=\mathbf{R}}$$

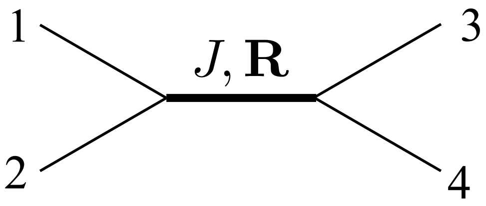
→ J-basis amplitude:
An extension of partial wave amplitude

$$\mathbb{C}_{12}\mathcal{A}_{\mathrm{EFT}}^{\mathbf{R}_{12}=\mathbf{R}} = C(\mathbf{R})\mathcal{A}_{\mathrm{EFT}}^{\mathbf{R}_{12}=\mathbf{R}}$$
$$\mathcal{W}_{12}^2\mathcal{A}_{\mathrm{EFT}}^{J_{12}=J} = sJ(J+1)\mathcal{A}_{\mathrm{EFT}}^{J_{12}=J}$$

Application 2: Tree-level UV origins of operators

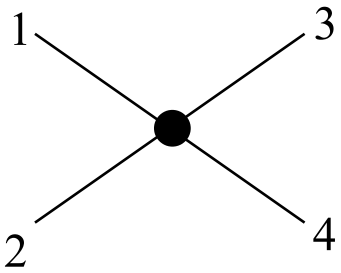
➢ Intuition – diagrammatic matching

UV Resonance Φ — spin J, gauge rep: $\mathbf{R}$ $\quad\psi_1\psi_2\Phi, \psi_3\psi_4\Phi$



$$\mathcal{A}_{\rm UV}^{J_{12}=J;\mathbf{R}_{12}=\mathbf{R}}$$

↓ Expanding the propagator



$$\mathcal{A}_{\rm EFT}^{J_{12}=J;\mathbf{R}_{12}=\mathbf{R}}$$

→ J-basis amplitude: An extension of partial wave amplitude

$$\mathbb{C}_{12}\mathcal{A}_{\rm EFT}^{\mathbf{R}_{12}=\mathbf{R}} = C(\mathbf{R})\mathcal{A}_{\rm EFT}^{\mathbf{R}_{12}=\mathbf{R}}$$

$$\mathcal{W}_{12}^2\mathcal{A}_{\rm EFT}^{J_{12}=J} = sJ(J+1)\mathcal{A}_{\rm EFT}^{J_{12}=J}$$

↓ Amplitude-Operator Correspondence

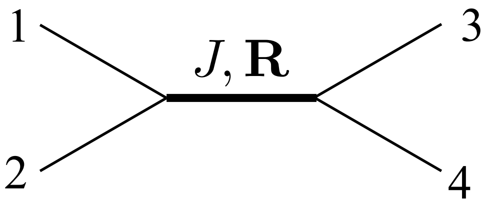
$$\mathcal{O}^{J_{12}=J;\mathbf{R}_{12}=\mathbf{R}}$$

→ J-basis operator:

Application 2: Tree-level UV origins of operators

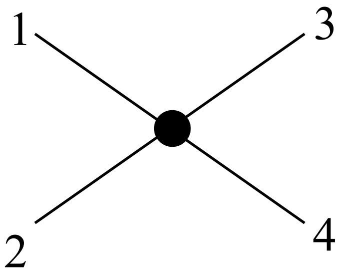
- Intuition – diagrammatic matching

UV Resonance Φ — spin J, gauge rep: $\mathbf{R}$ $\psi_1\psi_2\Phi, \psi_3\psi_4\Phi$



$$\mathcal{A}_{\text{UV}}^{J_{12}=J;\mathbf{R}_{12}=\mathbf{R}}$$

Expanding the propagator



$$\mathcal{A}_{\text{EFT}}^{J_{12}=J;\mathbf{R}_{12}=\mathbf{R}}$$

J-basis amplitude:
An extension of partial wave amplitude

$$\mathbb{C}_{12}\mathcal{A}_{\text{EFT}}^{\mathbf{R}_{12}=\mathbf{R}} = C(\mathbf{R})\mathcal{A}_{\text{EFT}}^{\mathbf{R}_{12}=\mathbf{R}}$$
$$\mathcal{W}_{12}^2\mathcal{A}_{\text{EFT}}^{J_{12}=J} = sJ(J+1)\mathcal{A}_{\text{EFT}}^{J_{12}=J}$$

Amplitude-Operator Correspondence

$$\mathcal{O}^{J_{12}=J;\mathbf{R}_{12}=\mathbf{R}}$$

J-basis operator:

$$\mathcal{O}^{J,\mathbf{R}}|J',\mathbf{R}'\rangle_{12} \sim \delta^{JJ'}\delta_{\mathbf{R}\mathbf{R}'}$$

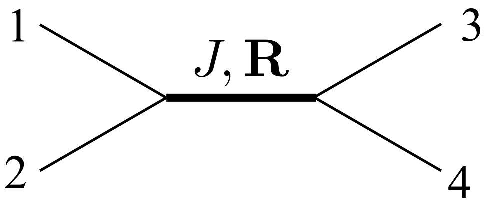
$\mathcal{K}^{jp}$

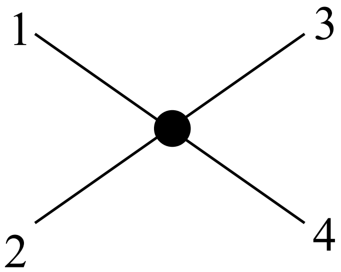
Physical Basis: $\{\mathcal{O}^p\}$

Application 2: Tree-level UV origins of operators

- Intuition – diagrammatic matching

UV Resonance Φ — spin J, gauge rep: $\mathbf{R}$ $\quad \psi_1\psi_2\Phi, \psi_3\psi_4\Phi$





Reorganize into j-basis according to certain scattering channel

$$\mathcal{A}_{\mathrm{UV}}^{J_{12}=J;\mathbf{R}_{12}=\mathbf{R}}$$

Expanding the propagator

$$\mathcal{A}_{\mathrm{EFT}}^{J_{12}=J;\mathbf{R}_{12}=\mathbf{R}}$$

→ J-basis amplitude: An extension of partial wave amplitude

$$\mathbb{C}_{12}\mathcal{A}_{\mathrm{EFT}}^{\mathbf{R}_{12}=\mathbf{R}} = C(\mathbf{R})\mathcal{A}_{\mathrm{EFT}}^{\mathbf{R}_{12}=\mathbf{R}}$$
$$\mathcal{W}_{12}^2\mathcal{A}_{\mathrm{EFT}}^{J_{12}=J} = sJ(J+1)\mathcal{A}_{\mathrm{EFT}}^{J_{12}=J}$$

Amplitude-Operator Correspondence

$$\mathcal{O}^{J_{12}=J;\mathbf{R}_{12}=\mathbf{R}}$$

→ J-basis operator:

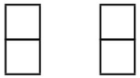
$$\mathcal{O}^{J,\mathbf{R}}|J',\mathbf{R}'\rangle_{12} \sim \delta^{JJ'}\delta_{\mathbf{R}\mathbf{R}'}$$

$\mathcal{K}^{jp}$

Physical Basis: $\{\mathcal{O}^p\}$

Application 2: Tree-level UV origins of operators

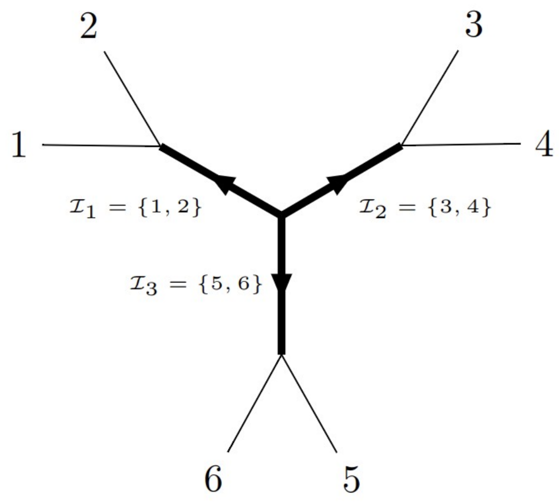
- A quick example: SMEFT dim-6: H^4D^2, for $H_1^\dagger H_2 \to H_3^\dagger H_4$

J	$\mathbf{R}$	J-basis	$\mathcal{K}^{\mathrm{jp}}$	P-basis	Sym$_{H,H^\dagger}$
0	$\mathbf{1}$	$(H_1^\dagger H_2)D^2(H_3^\dagger H_4)$	$1 \quad 0 \quad 1 \quad 0$	$Q_{\varphi\Box}$	
	$\mathbf{3}$	$(H_1^\dagger \tau^I H_2)D^2(H_3^\dagger \tau^I H_4)$	$-1 \quad 4 \quad -1 \quad 4$	$Q_{\varphi D}$	
1	$\mathbf{1}$	$(H_1^\dagger i\overleftrightarrow{D}_\mu H_2)(H_3^\dagger i\overleftrightarrow{D}^\mu H_4)$	$-1 \quad -4 \quad -1 \quad 4$	$Q'_{\varphi\Box}$	
	$\mathbf{3}$	$(H_1^\dagger i\tau^I\overleftrightarrow{D}_\mu H_2)(H_3^\dagger i\tau^I\overleftrightarrow{D}^\mu H_4)$	$-3 \quad 0 \quad 5 \quad -8$	$Q'_{\varphi D}$	

$$Q_{H\Box} \sim S(\mathbf{1}), S(\mathbf{3}), V(\mathbf{1}), V(\mathbf{3}) \qquad Q_{HD} \sim S(\mathbf{3}), V(\mathbf{1})$$

Application 2: Tree-level UV origins of operators

Directly extend to multi-partition for operators with more than 4 fields



$$[W^2_{\mathcal{I}_1}, W^2_{\mathcal{I}_2}] = [W^2_{\mathcal{I}_1}, W^2_{\mathcal{I}_3}] = [W^2_{\mathcal{I}_3}, W^2_{\mathcal{I}_2}] = 0$$

$$[\mathbb{C}_{\mathcal{I}_1}, \mathbb{C}_{\mathcal{I}_2}] = [\mathbb{C}_{\mathcal{I}_1}, \mathbb{C}_{\mathcal{I}_3}] = [\mathbb{C}_{\mathcal{I}_3}, \mathbb{C}_{\mathcal{I}_2}] = 0$$

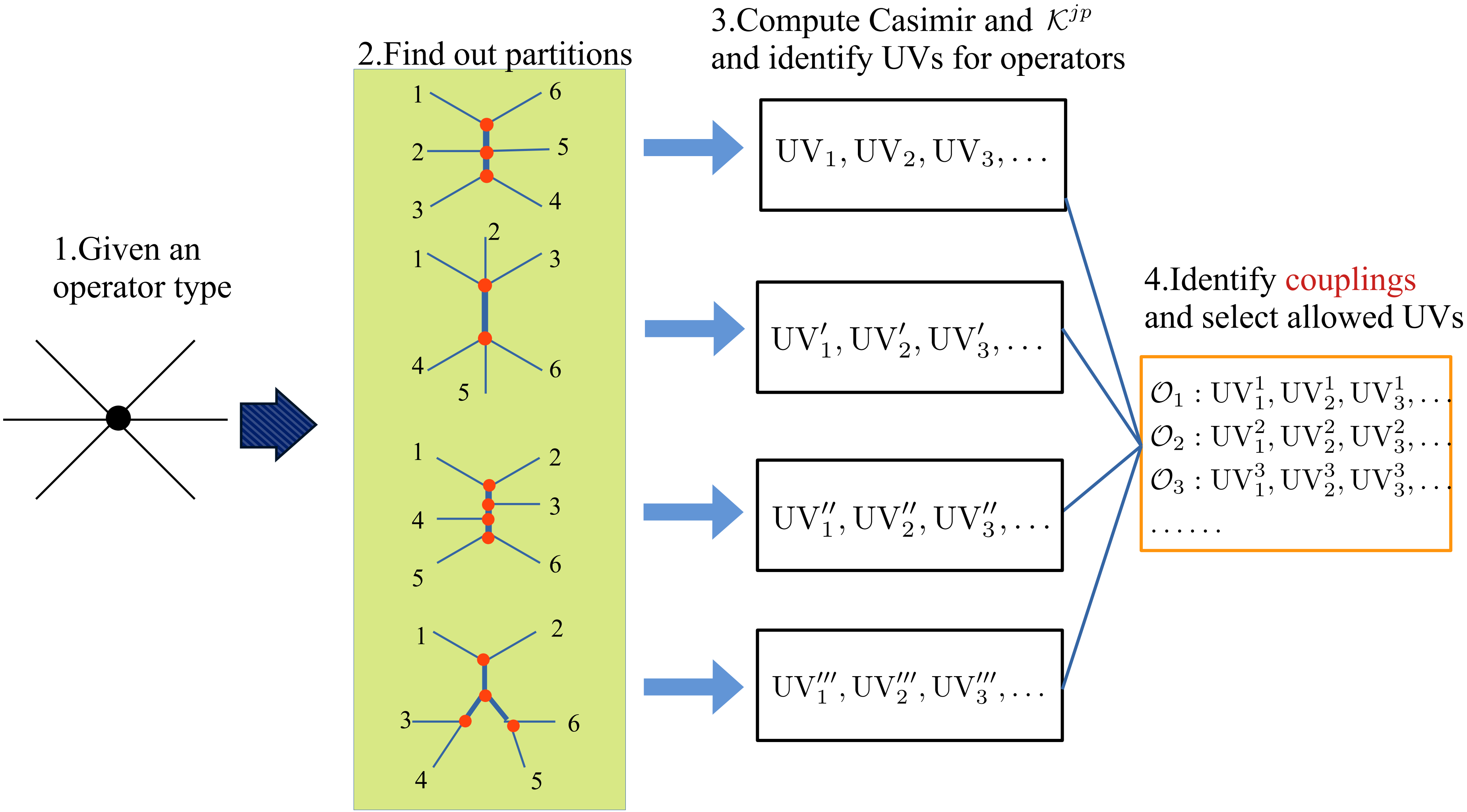
Can Find simultanous eigenbasis for each Casimir operator

$$\mathbf{W}^2_{\mathcal{I}_i} \mathcal{A}^{\{J_i\},\{\mathbf{R}_i\}} = -s_{\mathcal{I}_i} J_i (J_i + 1) \mathcal{A}^{\{J_i\},\{\mathbf{R}_i\}}$$

$$\mathbb{C}_{\mathcal{I}_i} \mathcal{A}^{\{J_i\},\{\mathbf{R}_i\}} = C(\mathbf{R}_i) \mathcal{A}^{\{J_i\},\{\mathbf{R}_i\}}$$

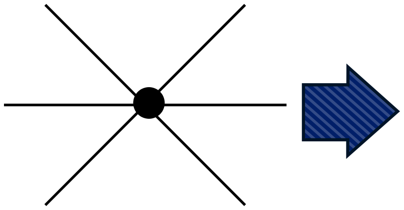
In this way we have systematically obtain the J-basis amplitudes

Short introduction to J-basis UV correspondence

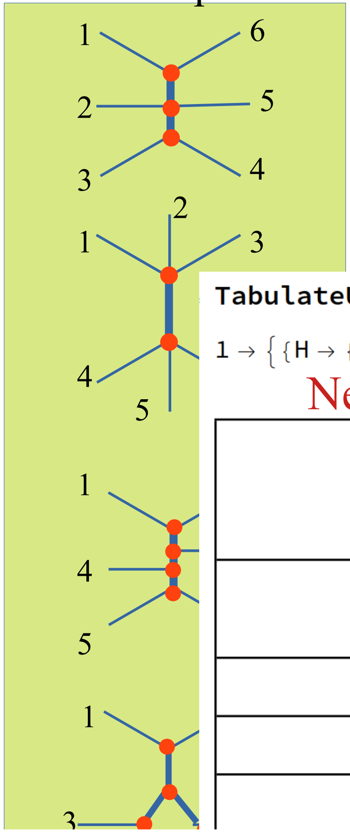


Short introduction to J-basis UV correspondence

1.Given an operator type



2.Find out partitions



3.Compute Casimir and $\mathcal{K}^{jp}$ and identify UVs for operators

$\mathrm{UV}_1, \mathrm{UV}_2, \mathrm{UV}_3, \ldots$

4.Identify couplings

```
TabulateUVs["H"^3 "H†"^3, 1]
```

$1 \to \{\{H \to \{3\}, H^\dagger \to \{3\}\}, H_i H_j H_k H^{\dagger i} H^{\dagger j} H^{\dagger k}\}$

New Fields Lists	New Vertices Lists
S1(1,1,0)	(H,H†,S1) (H,H†,S1,S1) (H,H†,S1) (H,H†,S1) (S1,S1,S1)
S7(1,3,0)	(H,H†,S7) (H,H†,S7,S7) (H,H†,S7)
S4(1,2,1/2)	(H,H†,H†,S4)
S9(1,4,1/2)	(H,H†,H†,S9)
S8(1,3,1)	(H†,H†,S8) (H,H†,S8†,S8) (H,H,S8†)
	(H†,H†,H†,S10)

Tree-Level SMEFT dim-8 [HLL, et.al. 2309.15933]

Summary:

- J-basis as generalized partial-wave basis can be derived systematically with Casimir operator method

- The delta-function of momentum conservation in N-body massless phase space is completely solvable using the spinor variable technique .

- The normalization factor for J-basis amplitudes can be obtained analytically, which enable the analytical derivation of the partial-wave unitarity bound of effective operators.

- The J-basis/UV correspondence can help to find the tree-level UV origin of effective operators, and the UV origins of SMEFT dim-6, dim-7, dim-8 operators are tabulated.

Sherman–Morrison Formula

$$\sqrt{1+\mathbf{u}^T\mathbf{u}} = 1 + \left(\frac{\sqrt{1+|\mathbf{u}|^2}-1}{|\mathbf{u}|^2}\right)\mathbf{u}^T\mathbf{u},$$

$$\left(1+\mathbf{u}^T\mathbf{u}\right)^{-1} = 1 - \frac{1}{1+|\mathbf{u}|^2}\mathbf{u}^T\mathbf{u},$$

$$\begin{aligned} \mathbb{C}_2 &= \mathbb{T}^a\mathbb{T}^a, \text{ for both } SU(2) \text{ and } SU(3), \\ \mathbb{C}_3 &= d^{abc}\mathbb{T}^a\mathbb{T}^b\mathbb{T}^c, \text{ for } SU(3) \text{ only}, \end{aligned}$$

$$\mathbb{T}^A_{\otimes\{\mathbf{r}_i\}} = \sum_{i=1}^{N} E_{\mathbf{r}_1} \times E_{\mathbf{r}_2} \times \cdots \times T^A_{\mathbf{r}_i} \times \ldots E_{\mathbf{r}_N}$$

$$\underset{\$}{\mathbb{T}}^A \circ \Theta_{I_1 I_2 \ldots I_N} = \sum_{i\in\$}^{N} (T^A_{r_i})^Z_{I_i} \Theta_{I_1\ldots I_{i-1} Z I_{i+1} I_N}\,.$$