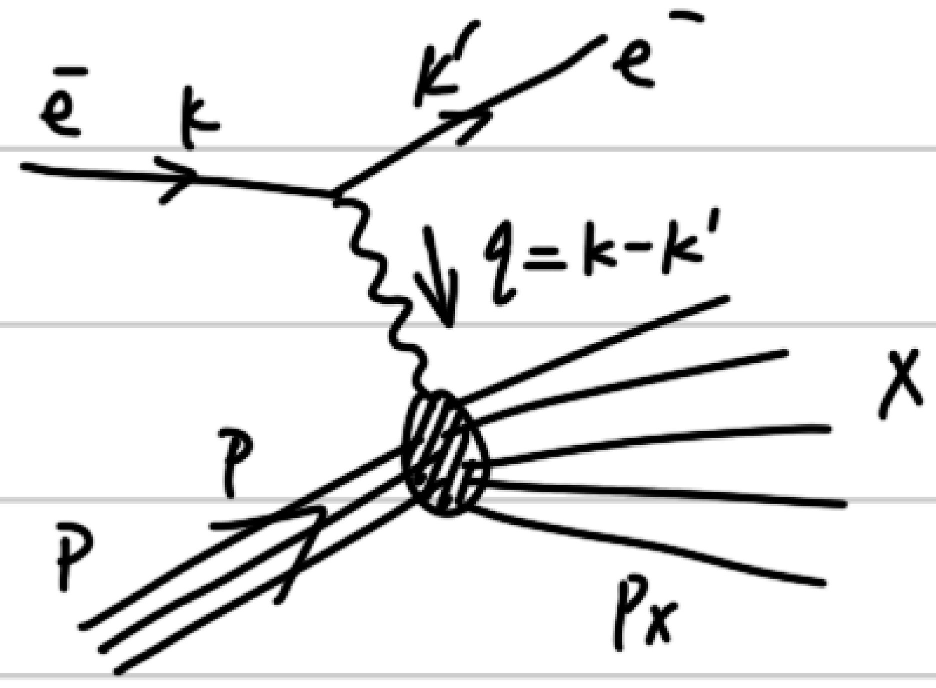
§1. DIS运动学:



(到 QED 领头阶)

深度: $Q^2=-q^2$ 很大 动量转移

非弹性: $m_X^2 = p_X^2 = (p+q)^2 \neq p^2 = m_p^2$

$\Rightarrow 2p\cdot q \neq Q^2$ (通常, p_X^0 很大)

① DIS: $\bar e(k)\, p(p) \to e^-(k')\, X(p_X)$

振幅

$$iM = \bar u(k')(-ie\gamma^\mu)u(k)\,\frac{-i}{q^2}\,(ie)\,\langle X|J^\mu_{em}(0)|p\rangle$$

截面

$$\sigma \sim |iM|^2 \sim \sum_X (2\pi)^4\delta^4(p+q-p_X)\langle p|J^\mu_{em}(0)|X\rangle\langle X|J^\nu_{em}(0)|p\rangle$$

where $\sum_X$: i) 末态粒子种类

ii) 末态粒子数

iii) 末态相空间积分 $\int \widetilde{dp_X}$

完备性关系: $\sum_X |X\rangle\langle X| = \mathbb{1}$ OPE会用到, 再说

② 矩阵元张量 (将矩阵元改写为更协变形式 —— 解析地在 $w<1$ 和 $w>1$ 连续延拓过.

$$W^{\mu\nu}(p,q) = \frac{1}{4\pi}\int d^4z\, e^{iqz}\,\langle p|[J^\mu_{em}(z), J^\nu_{em}(0)]|p\rangle$$
$$= \frac{1}{4\pi}\int d^4z\, e^{iqz}\sum_X\langle p|J^\mu_{em}(z)|X\rangle\langle X|J^\nu_{em}(0)|p\rangle$$
$$-\frac{1}{4\pi}\int d^4z\, e^{iqz}\sum_X\langle p|J^\nu_{em}(0)|X\rangle\langle X|J^\mu_{em}(z)|p\rangle$$

Using: $J^\mu_{em}(z) = e^{ipz}J^\mu_{em}(0)e^{-ipz}$, then

$$\langle p|J^\mu_{em}(z)|X\rangle = \langle p|J^\mu_{em}(0)|X\rangle e^{i(p-p_X)z}$$
$$\langle X|J^\mu_{em}(z)|p\rangle = \langle X|J^\mu_{em}(0)|p\rangle e^{i(p_X-p)z}$$

then

$$W^{\mu\nu} = \frac{1}{4\pi}\sum_X(2\pi)^4\delta^4(p+q-p_X)\langle p|J^\mu_{em}(0)|X\rangle\langle X|J^\nu_{em}(0)|p\rangle$$
$$\underline{-\frac{1}{4\pi}\sum_X(2\pi)^4\delta^4(p_X+q-p)\langle p|J^\mu_{em}(0)|X\rangle\langle X|J^\nu_{em}(0)|p\rangle}$$ 重子数守恒 ($p_X^0>p^0$)

最终

$$\sigma(ep\to ep) = \frac{e^4}{2s}\int\frac{d^3k'}{(2\pi)^3 2E_{k'}}\,\frac{1}{2}\sum_s \bar u(k)\gamma_\mu u(k')\,\bar u(k')\gamma_\nu u(k)\,\frac{1}{(Q^2)^2}\,4\pi W^{\mu\nu}$$

电子部分:

$$\frac{1}{2}\sum_s \bar u(k)\gamma_\mu u(k')\,\bar u(k')\gamma_\nu u(k) = 2(k_\mu k'_\nu - k\cdot k' g_{\mu\nu} + k_\nu k'_\mu)$$

③ 现有σ变量: k', $\cos\theta$, 要换成 x, y.

常用运动学变量: $Q^2 = -q^2 = -(k'-k)^2$: 动量转移 (Mandelstam $\hat{t}$)

dimensionless $Q^2 = xys$

$$\begin{cases} x = \frac{Q^2}{2p\cdot q} = \frac{2kk'(1-\cos\theta)}{2m(k-k')} & \text{部分子动量分数} \\ y = \frac{2p\cdot q}{2p\cdot k} = \frac{k-k'}{k} & \text{在质子静系, } y\text{: 入射e转移走的能量比数} \end{cases}$$

$$\frac{\partial(x,y)}{\partial(k',\cos\theta)} = \frac{2k'}{2m(k-k')} = \frac{2k'}{ys} \qquad \int\frac{d^3k'}{(2\pi)^3 2k'} = \int\frac{2\pi k'\,dk'\,d\cos\theta}{(2\pi)^3\cdot 2} = \int dx\,dy\,\frac{ys}{(4\pi)^2}$$

$$\Rightarrow \frac{d^2\sigma}{dx\,dy} = \frac{4\pi\alpha^2 y}{(Q^2)^2}\left(k_\mu k'_\nu + k_\nu k'_\mu - kk' g_{\mu\nu}\right) W^{\mu\nu}$$

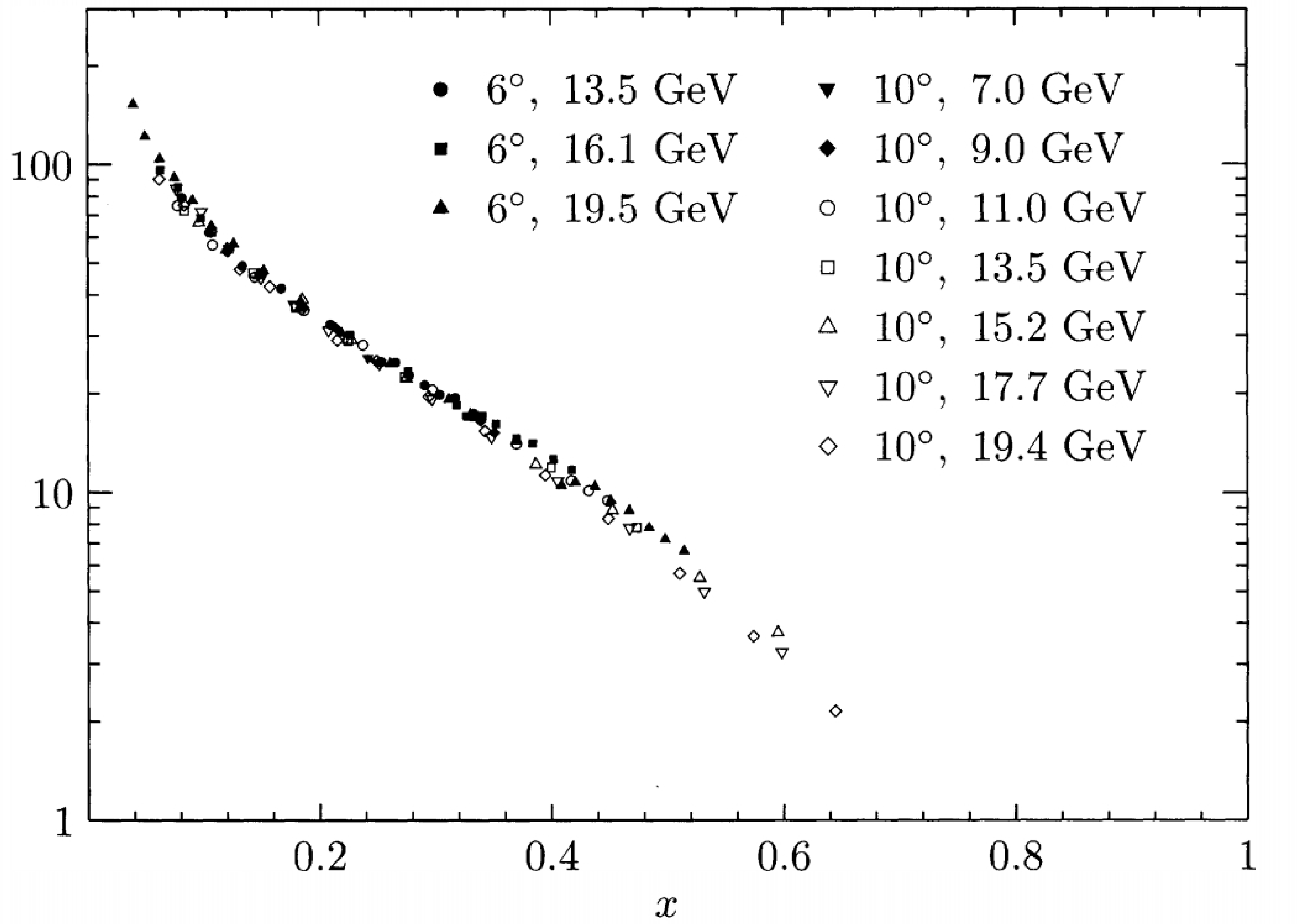


Figure 14.2. Test of Bjorken scaling using the $e^- p$ deep inelastic scattering cross sections measured by the SLAC-MIT experiment, J. S. Poucher, et. al., *Phys. Rev. Lett.* **32**, 118 (1974). We plot $d^2\sigma/dxdQ^2$ divided by the factor (14.9) against x, for the various initial electron energies and scattering angles indicated. The data span the range 1 GeV$^2 < Q^2 <$ 8 GeV^2.

② 分析 $W^{\mu\nu}$ 结构: $g^{\mu\nu}$, $p^\mu p^\nu$, $q^\mu q^\nu$, $p^\mu q^\nu$, $q^\mu p^\nu$

$$q_\mu W^{\mu\nu} = 0 \Rightarrow -g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}, \quad p^\nu - \frac{p\cdot q}{q^2} q^\nu$$

$$q_\nu W^{\mu\nu} = 0 \Rightarrow -g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}, \quad p^\mu - \frac{p\cdot q}{q^2} q^\mu$$

$$\Rightarrow W^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) F_1 + \left(p^\mu - \frac{p\cdot q}{q^2} q^\mu\right)\left(p^\nu - \frac{p\cdot q}{q^2} q^\nu\right)\frac{F_2}{p\cdot q}$$

$$\frac{d^2\sigma}{dx\,dy} = \frac{4\pi\alpha^2 y}{Q^4}\left[\frac{2p\cdot k\, p\cdot k'}{p\cdot q} F_2 + 2kk' F_1\right]$$

$$= \frac{4\pi\alpha^2 y}{Q^4}\left[s(1-y)F_2 + xy^2 s F_1\right] \quad (\text{using } 2kk' = Q^2 = xys)$$

§2. OPE in DIS.

to use QCD, introduce:

$$t^{\mu\nu} = i\int d^4z \, e^{iqz} \, \mathbb{T}\left[J^\mu_{em}(z) J^\nu_{em}(0)\right]$$

$$T^{\mu\nu} = \langle p| t^{\mu\nu} |p\rangle$$

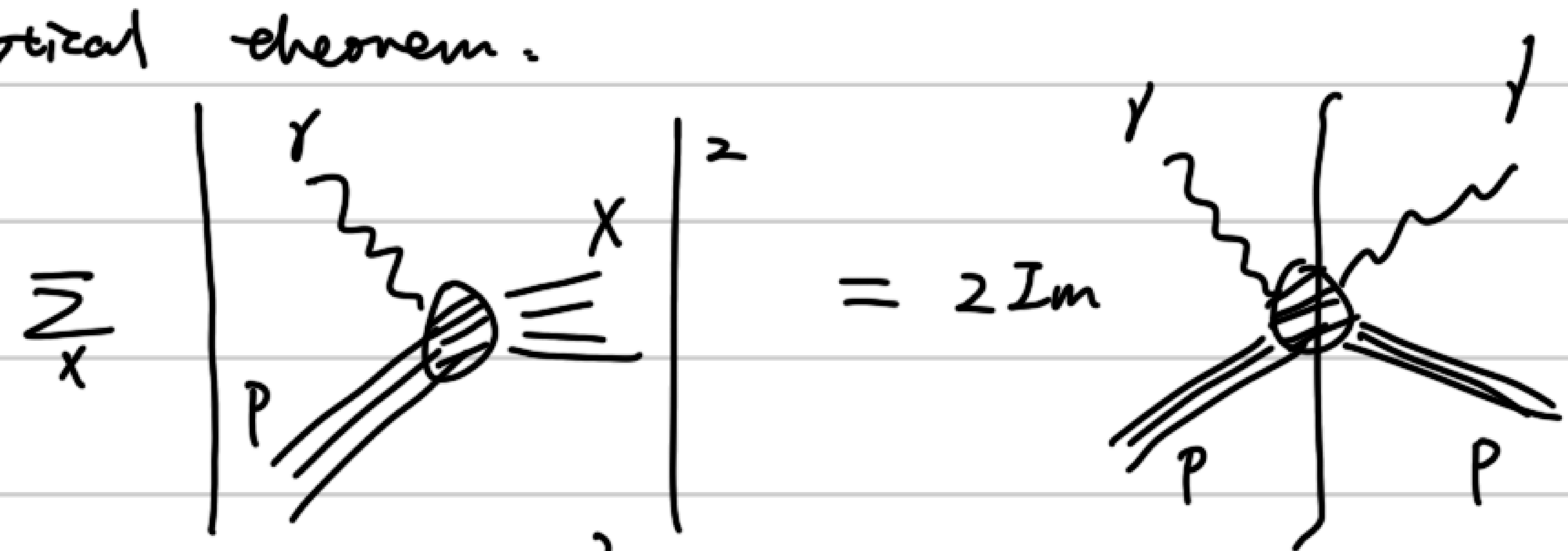
$$= \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) T_1 + \left(p^\mu - \frac{p\cdot q}{q^2} q^\mu\right)\left(p^\nu - \frac{p\cdot q}{q^2} q^\nu\right)\frac{T_2}{p\cdot q}$$

Using LSZ, Schwinger-Dyson:

$$\langle x| J^\mu_{em}(0) |p\rangle \Rightarrow \mathcal{M}(\gamma p \to X)$$

$$\langle p| \mathbb{T}\left[J^\mu_{em}(z) J^\mu_{em}(0)\right] |p\rangle \Rightarrow \mathcal{M}(\gamma p \to \gamma p)$$

Optical theorem.



$|\mathcal{M}(\gamma p \to X)|^2$ ~ $W^{\mu\nu}$ $\mathcal{M}(\gamma p \to \gamma p)$ ~ $T^{\mu\nu}$

$$2\pi F_{1,2}(\omega, Q^2) = \mathrm{Im}\, T_{1,2}(\omega + i\epsilon, Q^2) \qquad \omega = \frac{1}{x} = \frac{2p\cdot q}{Q^2}$$

$T^{\mu\nu}$ 是算符编时乘积的矩阵元 —— OPE.

when z small. C^k_{ab} 发散

OPE: $$\mathbb{T}\left[O_a(z) O_b(0)\right] = \sum_k C^k_{ab}(z)\, O^k(0)$$

In k space: $$\int d^4z \, e^{iqz} \, \mathbb{T}\left[O_a(z) O_b(0)\right] = \sum_k C^k_{ab}(q)\, O^k(0)$$ z small, q large. qz 成立

$$T^{\mu\nu}(p,q) = \int d^4z \, e^{iqz} \langle p| \mathbb{T}\left[J^\mu_{em}(z) J^\nu_{em}(0)\right] |p\rangle$$

$$= \sum_{\dots} C(q) \langle O\rangle(p)$$

① O 的指标：$\underbrace{\mu, \nu}_{\text{open}}$；$\underbrace{\mu_1, \mu_2, \cdots, \mu_n}_{\text{contract with } C}$

考虑：全部指标都与 C 收缩　（对称化，$\underbrace{\text{没有 } g^{\mu\nu}\text{、trace part}}_{\text{否则会是 higher twist}}$）

☆：$C_{\mu_1\mu_2\cdots\mu_n} \; O_{d,n}^{\mu_1\mu_2\cdots\mu_n}$　　d：量纲　n：对称指标数

$$\underbrace{\langle O_{d,n}^{\mu_1\mu_2\cdots\mu_n} \rangle}_{[d-2]} \sim m_p^{d-2-n} \underbrace{p^{\mu_1} p^{\mu_2} \cdots p^{\mu_n}}_{[n]}$$

$$\underbrace{C_{\mu_1\mu_2\cdots\mu_n}}_{[2-d]} \sim Q^{2-d-n} \underbrace{q_{\mu_1} q_{\mu_2} \cdots q_{\mu_n}}_{[n]}$$

$$\underset{[0]}{T^{\mu\nu}} \sim \underset{[2-d]}{C}\,\underset{[d-2]}{\langle O \rangle}$$

$$\Rightarrow \frac{q_{\mu_1} q_{\mu_2} \cdots q_{\mu_n}}{Q^{2n}} Q^{2-d+n} m_p^{d-2-n} p^{\mu_1} p^{\mu_2} \cdots p^{\mu_n}$$

$$\sim \omega^n \left(\frac{Q}{m_p}\right)^{2-\overbrace{(d-n)}^{t}}$$

$$= \omega^n \left(\frac{Q}{m_p}\right)^{2-t} \qquad \left(\omega = \frac{2p\cdot q}{Q^2}\right)$$

② 确定 $\hat{O}$ 基本形式：

	$q/\bar{q}$	$G_{\mu\nu}$	∂_μ
d	$3/2$	2	1
n	$1/2$	1	1
t	1	1	0 ☆

$t^{\mu\nu} \sim \mathbb{T}\left[J_{em}^{\mu}(x) J_{em}^{\nu}(0)\right]$，　$J^\mu \to \bar{q}\gamma^\mu q$，

OPE后手征不变的 Lorentz structure：　$\bar{q}\gamma^\mu q$　$\bar{q}\gamma^\mu\gamma_5 q$

leading twist 2：

$$O_{q,V}^{\mu_1\cdots\mu_n} = \frac{1}{2}\left(\frac{i}{2}\right)^{n-1} S\left\{\bar{q}\gamma^{\mu_1} \overleftrightarrow{D}^{\mu_2} \cdots \overleftrightarrow{D}^{\mu_n} q\right\}$$

$$O_{q,A}^{\mu_1\cdots\mu_n} = \frac{1}{2}\left(\frac{i}{2}\right)^{n-1} S\left\{\bar{q}\gamma^{\mu_1} \overleftrightarrow{D}^{\mu_2} \cdots \overleftrightarrow{D}^{\mu_n} \gamma_5 q\right\}$$

$$O_{g,V}^{\mu_1\cdots\mu_n} = -\frac{1}{2}\left(\frac{i}{2}\right)^{n-2} S\left\{G_a^{\mu_1\rho} \overleftrightarrow{D}^{\mu_2} \cdots \overleftrightarrow{D}^{\mu_{n-1}} G_{a\,\rho}^{\mu_n}\right\}$$

local 矩阵元：$\langle q(p)|\bar{\psi}\gamma^\mu\psi|q(p)\rangle = \frac{1}{2}\sum_s \bar{u}\gamma^\mu u = \frac{1}{2}\mathrm{tr}[\not{p}\gamma^\mu] = 2p^\mu$

$$\langle q(p)|\bar{\psi}\gamma^\mu \overleftrightarrow{D}^\nu \psi|q(p)\rangle = -2ip^\nu \frac{1}{2}\sum_s \bar{u}\gamma^\mu u = -4i\,p^\mu p^\nu$$

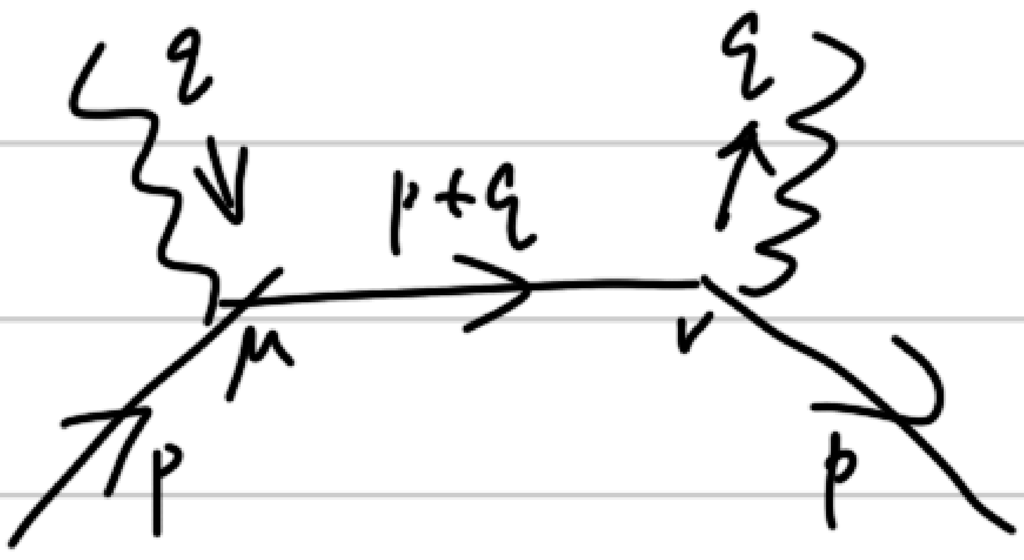
③ $\langle q|JJ|q\rangle \sim C_q \langle q|O_q|q\rangle + C_g \langle q|O_g|q\rangle$

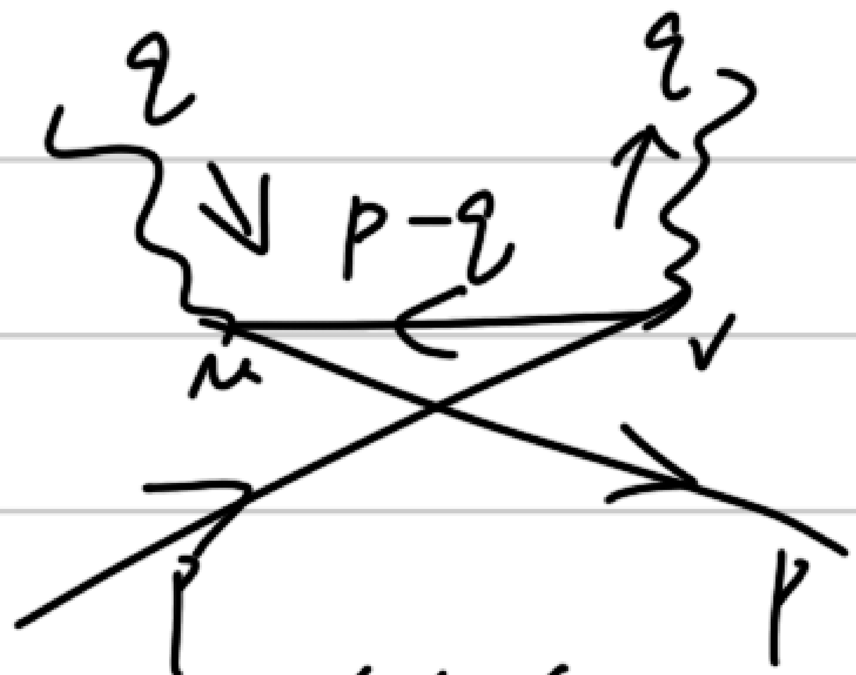
$\quad\ \alpha_s^0 \qquad\qquad\qquad \sim \alpha_s^0 \qquad\qquad\qquad \sim \alpha_s^1$

At $O(\alpha_s^0)$ of JJ：最重要的来自相同 flavor 乘积.

$\bar q \gamma^\mu q(x)\ \bar q \gamma^\nu q(0)$

$$= \bar q(x)\gamma^\mu \overbrace{q(x)\ \bar q(0)}\gamma^\nu q(0) + \overbrace{\bar q(x)\gamma^\mu q(x)\ \bar q(0)\gamma^\nu q(0)} + \cdots$$





$$iM^{\mu\nu} = i\left\{\bar u \gamma^\mu \frac{i(\not p+\not q)}{(p+q)^2}\gamma^\nu u + \bar u \gamma^\nu \frac{i(\not p-\not q)}{(p-q)^2}\gamma^\mu u\right\}$$

$\downarrow$ (p^2 on shell)

$$2p\cdot q + q^2 = q^2\left(1+\frac{2p\cdot q}{q^2}\right) = q^2(1-\omega) \qquad \left(\omega = \frac{2p\cdot q}{Q^2}\right)$$

$$\bar u \gamma^\mu(\not p+\not q)\gamma^\nu u = \bar u\Big[(p+q)^\mu\gamma^\nu + (p+q)^\nu\gamma^\mu - g^{\mu\nu}(\not p+\not q) + \underbrace{i\epsilon^{\mu\nu\rho\sigma}(p+q)_\rho\gamma_\sigma\gamma_5}_{=0}\Big]u$$

$$\frac{1}{2}\sum_s \bar u \gamma_\sigma u = 2p_\sigma .$$

$$\frac{1}{2}\sum_s \bar u \gamma_\sigma\gamma_5 u = 2hp_\sigma \rightarrow \text{螺旋度求和} = 0$$

$$\Rightarrow \bar u\gamma^\mu(\not p+\not q)\gamma^\nu u = 2(p+q)^\mu p^\nu + 2(p+q)^\nu p^\mu - 2g^{\mu\nu}(p\cdot q)$$

于是

$$iM^{\mu\nu} = \left[-\frac{2}{q^2}\left((p+q)^\mu p^\nu + (p+q)^\nu p^\mu - g^{\mu\nu}(p\cdot q)\right) + \left(\mu\leftrightarrow\nu,\ q\to -q,\ \omega\to-\omega\right)\right]\frac{1}{1-\omega}$$

$$\frac{1}{1-\omega} \underset{\omega<1}{=\!=} \sum_{n=0}^{\infty}\omega^n$$

$$= -\frac{2}{q^2}\sum_{n=0}^{\infty}\omega^n\left[(p+q)^\mu p^\nu + (p+q)^\nu q^\mu - g^{\mu\nu}(p\cdot q)\right] + \left(\mu\leftrightarrow\nu,\ q\to-q,\ \omega\to-\omega\right)$$

$$= -\frac{4}{q^2}\sum_{n\ \mathrm{odd}}^{\infty}\omega^n\left(q^\mu p^\nu + q^\nu p^\mu - g^{\mu\nu}(p\cdot q)\right) - \frac{\ }{q^2}\sum_{n\ \mathrm{even}}^{\infty}\omega^n 2p^\mu p^\nu$$

$$= -\frac{4}{q^2}\sum_{1,3,5\cdots}^{\infty}\left[2^n\frac{(p\cdot q)^n}{-q^{2n}}\underbrace{(q^\mu p^\nu+p^\mu q^\nu)}_{\downarrow\ \text{凑 } T^{\mu\nu}\text{ 参数化形式}} - 2^n\frac{(p\cdot q)^{n+1}}{-q^{2n}}g^{\mu\nu}\right]$$

$$-\frac{4}{q^2}\sum_{0,2,4}^{\infty} 2^n\frac{(p\cdot q)^n}{-q^{2n}}2p^\mu p^\nu$$

$$= -\frac{4}{q^2}\sum_{0,2,4}^{\infty} 2^n\frac{(p\cdot q)^n}{-q^{2n}}\left[2p^\mu p^\nu - \frac{2p\cdot q}{q^2}(q^\mu p^\nu+p^\mu q^\nu)+2\left(\frac{p\cdot q}{q^2}\right)^2 q^\mu q^\nu\right]$$

$$-\frac{4}{q^2}\sum_{1,3,5}^{\infty} 2^n\frac{(p\cdot q)^{n+1}}{-q^{2n}}\left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)$$

$$= -\frac{8}{q^2}\sum_{0,2,4\cdots}^{\infty}\frac{2^n(p\cdot q)^n}{-q^{2n}}\left(p^\mu-\frac{p\cdot q}{q^2}q^\mu\right)\left(p^\nu-\frac{p\cdot q}{q^2}q^\nu\right)$$

$$-\frac{4}{q^2}\sum_{1,3,5\cdots}^{\infty}\frac{2^n(p\cdot q)^{n+1}}{-q^{2n}}\left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)$$

拆分 $p\cdot q$.

$$= -\frac{8}{q^2}\sum_{0,2,4\cdots}^{\infty}\frac{2^n q_{\mu_3}q_{\mu_4}\cdots q_{\mu_{n+2}}}{Q^{2n}}\left(g^{\mu\mu_1}-\frac{q^\mu q^{\mu_1}}{q^2}\right)\left(g^{\nu\mu_2}-\frac{q^\nu q^{\mu_2}}{q^2}\right)p^{\mu_1}p^{\mu_2}\cdots p^{\mu_{n+2}}$$

$$-\frac{4}{q^2}\sum_{1,3,5\cdots}^{\infty}\frac{2^n q^{\mu_1}q^{\mu_2}\cdots q^{\mu_{n+1}}}{Q^{2n}}\left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)p^{\mu_1}p^{\mu_2}\cdots p^{\mu_{n+1}}$$

对应的 OPE 算符: (外态平均 $\Longleftrightarrow \gamma_5=0$, 只有 V 算符)

$$-\frac{8}{q^2}\sum_{0,2,4\cdots}^{\infty}\frac{2^n q^{\mu_3}q^{\mu_4}\cdots q^{\mu_{n+2}}}{Q^{2n}}\left(g^{\mu\mu_1}-\frac{q^\mu q^{\mu_1}}{q^2}\right)\left(g^{\nu\mu_2}-\frac{q^\nu q^{\mu_2}}{q^2}\right)\langle q|O_{q,V}^{\mu_1,\mu_2\cdots\mu_{n+2}}|q\rangle$$

$$-\frac{4}{q^2}\sum_{1,3,5\cdots}^{\infty}\frac{2^n q^{\mu_1}q^{\mu_2}\cdots q^{\mu_{n+1}}}{Q^{2n}}\left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)\langle q|O_{q,V}^{\mu_1\cdots\mu_{n+1}}|q\rangle$$

$\Rightarrow$

$$t^{\mu\nu} = \sum_{2,4,6\cdots}^{\infty}\frac{2^{n+1}q^{\mu_3}q^{\mu_4}\cdots q^{\mu_n}}{(Q^2)^{n-1}}\left(g^{\mu\mu_1}-\frac{q^\mu q^{\mu_1}}{q^2}\right)\left(g^{\nu\mu_2}-\frac{q^\nu q^{\mu_2}}{q^2}\right)O_{q,V}^{\mu_1\cdots\mu_n}$$

$$+\sum_{2,4,6\cdots}^{\infty}\frac{2^{n+1}q^{\mu_1}q^{\mu_2}\cdots q^{\mu_n}}{(Q^2)^n}\left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)O_{q,V}^{\mu_1\cdots\mu_n}$$

$$\Rightarrow \quad C^{1,2}_{q,n} = 1 + O(\alpha_s^1)$$

$$C^{1,2}_{g,n} = 0 + O(\alpha_s^1)$$

$$\Rightarrow \quad \langle p | O^{\mu_1 \cdots \mu_n}_{q,n} | p \rangle = 2A^n \, p^{\mu_1} \cdots p^{\mu_n}$$ (非微扰)

$$T_1 = \sum_f Q_f^2 \sum_{2,4,6\cdots} \frac{2^{n+1} (p\cdot q)^n}{(Q^2)^n} A_f^n$$

$$T_2 = \sum_f Q_f^2 \sum_{2,4,6\cdots} \frac{2^{n+1} (p\cdot q)^{n-2}}{(Q^2)^{n-1}} A_f^n$$

$$2\pi F_{1,2}(\omega, Q^2) = \operatorname{Im} T_{1,2}(\omega + i\epsilon, Q^2)$$