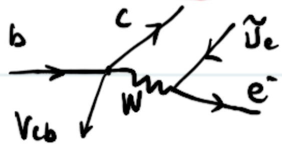
2. 算符乘积展开 (operator product expansion)

2-1 四夸克有效算符

粒子物理电弱标准模型中 $b \to c \ell \tilde{\nu}$



$$iM = \bar{u}(p_c)\,\frac{ig_2}{2\sqrt{2}}\gamma_\mu(1-\gamma_5)V_{cb}\,u(p_b)$$

$$\times\ \bar{u}(p_{e^-})\,\frac{ig_2}{2\sqrt{2}}\gamma_\nu(1-\gamma_5)\,v(p_{\tilde{\nu}_e}) \times \frac{-ig^{\mu\nu}}{k^2-m_W^2}$$

其中 $k^\mu = (p_b - p_c)^\mu$

(*) m_b : 4.8 GeV

m_c : 1.5 GeV $\qquad\ll\quad m_W = 81.4$ GeV

$m_e \sim m_{\tilde{\nu}_e} \sim 0$

$$k^2 = (p_b-p_c)^2 = p_b^2 - 2p_b\cdot p_c + p_c^2 \underset{\text{b夸克静止系}}{\leq} p_b^2 = m_b^2 \ll m_W^2$$

$$iM \approx \frac{-ig_2^2}{8m_W^2} V_{cb}\,\bar{u}(p_c)\gamma_\mu(1-\gamma_5)u(p_b)\ \ \bar{u}(p_e)\gamma^\mu(1-\gamma_5)v(p_{\tilde{\nu}_e})$$

$$\frac{G_F}{\sqrt{2}} = \frac{g_2^2}{8m_W^2} \qquad\qquad iM \sim \langle c e \tilde{\nu}|\exp[i\int d^4x\, H_{eff}]|b\rangle$$

① 有效理论: $\quad H_{eff} = \frac{G_F}{\sqrt{2}} V_{cb}\ \bar{c}\gamma^\mu(1-\gamma_5)b\ \ \bar{e}\gamma_\mu(1-\gamma_5)\tilde{\nu}$ $\quad\triangle$

c. b. e. ν 均为费米子场算符.

S矩阵元:

$$\langle c e \tilde{\nu}_e| -i\int d^4x\, H_{eff}(x)|b\rangle = (2\pi)^4\delta^4(\ldots)\cdot iM$$

如何从完整理论得到该有效理论?

(*) 算符分析：

$$S = T\exp\left[i\int d^4x\, \mathcal{L}_I\right] \qquad \mathcal{H}_I = -\mathcal{L}_I$$

$$= T\left\{ \underline{\frac{ig_2}{2\sqrt{2}} V_{cb} \int d^4x\, \bar{c}\gamma_\mu(1-\gamma_5) b\, W^{+\mu}} \right. \qquad i\int d^4x\, \mathcal{L}_I$$

$$\left. \times \frac{ig_2}{2\sqrt{2}} \int d^4y\, \bar{e}\gamma_\nu(1-\gamma_5)\tilde{\nu}_e W^{-\nu}(y) \right\} \qquad i\int d^4y\, \mathcal{L}_I$$

$$T\{W^{+\mu}(x) W^{-\nu}(y)\} = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \underline{\frac{-ig^{\mu\nu}}{k^2-m_W^2+i\varepsilon}}$$

$$\overset{k \ll m_W}{=\!=} \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{-ig^{\mu\nu}}{-m_W^2+i\varepsilon}$$

$$= \frac{-ig^{\mu\nu}}{-m_W^2} \delta^4(x-y)$$

则 $$S \cong \frac{-ig_2^2}{8m_W^2} T\left\{\int d^4x\, \bar{c}\gamma^\mu(1-\gamma_5) b \int d^4y\, \bar{e}\gamma^\nu(1-\gamma_5)\tilde{\nu}_e(y) \cdot \delta^4(x-y)\right\}$$

$$= \frac{-iG_F}{\sqrt{2}} T\left\{ \underline{\int d^4x\, \bar{c}\gamma^\mu(1-\gamma_5) b\, \bar{e}\gamma_\mu(1-\gamma_5)\tilde{\nu}_e} \right\}$$

有效理论的形成，相当于

$$\underset{y\sim x}{\underline{O_1(x)\, O_2(y)}} = O_1(x)\, O_2(x+y-x) = \underbrace{O_1(x)\, O_2(x)}_{\hat{O}_1(x)} + (y-x)\, O_1(x)\, O_2'(x) + \cdots$$

$$O_1(x)\, O_2(y) = \hat{O}_1(x) + \cdots$$

把算符的乘积变成了新的定域算符，这就是算符乘积展开，也是有效理论前提.

有漏洞么？

ⓐ $y-x$ 很大呢？ 幂次压低

ⓑ 复合算符重整化呢？

$$O_1(x)\, O_2(y) = \sum_i C_i\, \hat{O}_i(x)$$

弥补 UV 之间的差别，称之为短程系数，或 Wilson 系数

(四) 路径积分方法:

$$Z_W = \int [dW^+][dW^-]\ \exp\left[i\int d^4x\, \mathcal{L}_I\right]$$

$$\mathcal{L}_W = -\frac{1}{2}(\partial_\mu W^+_\nu - \partial_\nu W^+_\mu)(\partial^\mu W^{-\nu} - \partial^\nu W^{-\mu}) + m_W^2 W^+_\mu W^-_\nu$$

$$+ \underbrace{\frac{g_2}{\sqrt{2}}(J^+_\mu W^{+\mu} + J^-_\mu W^{-\mu})}_{\text{相互作用}}$$

利用路径积分中方法，可将生成泛函写为:

$$Z_W = \int [dW^+][dW^-]\ \exp\left\{\left[i\int d^4x\, d^4y\, W^+_\mu(x)\, K^{\mu\nu}(x,y)\, W^-_\nu(y)\right] + \frac{ig_2}{2\sqrt{2}}\int d^4x \left[J^{+\mu}W^+_\mu + J^{-\mu}W^-_\mu\right]\right\}$$

$$K_{\mu\nu}(x,y) = \delta^4(x-y)\left[g_{\mu\nu}(\partial^2+m_W^2) - \partial_\mu\partial_\nu\right]$$

它的逆为 $\Delta_{\mu\nu}(x,y) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, \frac{-i}{k^2-m_W^2}\left[g_{\mu\nu} - \frac{k_\mu k_\nu}{m_W^2}\right]$

① 积分出 $W^\pm$，则生成泛函为

$$Z_W \sim \exp\left[-\frac{ig_2^2}{8}\int d^4x\, d^4y\ J^-_\mu(x)\, \Delta^{\mu\nu}(x,y)\, J^+_\nu(y)\right]$$

② 对 $\Delta_{\mu\nu}(x,y)$ 取截断，即

$$\Delta_{\mu\nu}(x,y) = \frac{g_{\mu\nu}}{M_W^2}\ \delta^4(x-y)$$

③ 代入后得

$$Z_W \approx \exp\left[-i\frac{G_F}{\sqrt{2}}\int d^4x\, J^+_\mu(x)\, J^{-\mu}(x)\right]$$

这与算符分析方法一致.

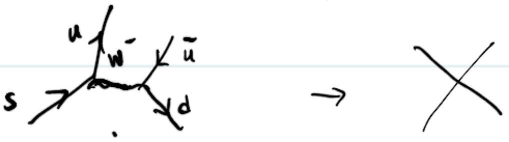
④ 一般地，需要考虑QCD修正:

$$H_{eff} = \frac{G_F}{\sqrt{2}}\left\{V_{ub}V^*_{ud}\left(C_1\hat{O}_1 + C_2\hat{O}_2\right) - V_{tb}V^*_{td}\sum_{i=3}^{10} C_i O_i\right\}$$

② 对于半轻衰，复合算符重整化系数为1.

③ 对于非轻衰，则需计算演化。见 Peskin P607.

非轻：



$s \to u\bar u d$

树图：$H_{eff}=\frac{G_F}{\sqrt{2}}\underbrace{(\bar d\gamma^\mu(1-\gamma_5)u)(\bar u\gamma_\mu(1-\gamma_5)s)}_{O_1}$ $\qquad k^2 \ll m_W^2$

加入QCD修正后：

$$H_{eff}=\frac{G_F}{\sqrt{2}}(C_1O_1+C_2O_2)\cdot V_{ud}^* V_{us}$$

$$O_1=(\bar d\gamma^\mu(1-\gamma_5)u)(\bar u\gamma_\mu(1-\gamma_5)s)$$

$$O_2=(\bar u\gamma^\mu(1-\gamma_5)u)(\bar d\gamma_\mu(1-\gamma_5)s) = (\bar d^\alpha\gamma^\mu(1-\gamma_5)u^\beta)(\bar u^\beta\gamma_\mu(1-\gamma_5)s^\alpha)$$

颜色不一样

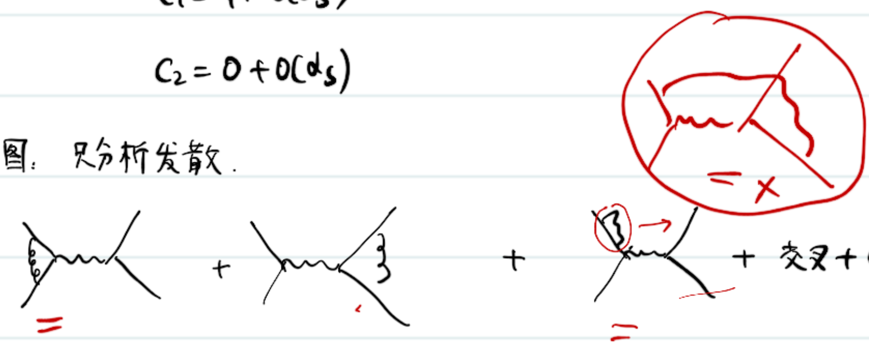
C_1与C_2为短程系数

$$C_1=1+O(\alpha_s)$$

$$C_2=0+O(\alpha_s)$$

一单圈图：只分析发散.

完整：



+ 交叉 + ct

(*) 其实不用加ct，因为矢量流守恒，所有UV发散都抵消

有效

+ 复合算符重整化.

自动抵消

(*) O_1  $= -g_s^2 \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^2}\left[\frac{1}{2}\bar{u}\gamma^\mu(1-\gamma_5)u\,\bar{d}\gamma_\mu(1-\gamma_5)s - \frac{1}{6}\bar{d}\gamma^\mu(1-\gamma_5)u\,\bar{u}\gamma_\mu(1-\gamma_5)s\right]$

 O_1 $= +g_s^2 \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^2}\frac{1}{4}\left[\frac{1}{2}\bar{u}\gamma^\mu(1-\gamma_5)u\,\bar{d}\gamma_\mu(1-\gamma_5)s - \frac{1}{6}\bar{d}\gamma^\mu(1-\gamma_5)u\,\bar{u}\gamma_\mu(1-\gamma_5)s\right]$

四张图相加 $i\mathcal{M} = -\frac{3}{4}g_s^2 \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^2}\left[\bar{u}\gamma^\mu(1-\gamma_5)u\,\bar{d}\gamma_\mu(1-\gamma_5)s - \frac{1}{3}\bar{d}\gamma^\mu(1-\gamma_5)u\,\bar{u}\gamma_\mu(1-\gamma_5)s\right]$

(*) 对于O_2也是类似地，需交换 $\langle O_1\rangle$ 与 $\langle O_2\rangle$

可以类似于复合算符重整化：

$$O_1^0 = \underline{Z_{11}O_1 + Z_{12}O_2} \qquad O_2^0 = Z_{21}O_1 + Z_{22}O_2$$

$$\begin{pmatrix} O_1^0 \\ O_2^0 \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}\begin{pmatrix} O_1 \\ O_2 \end{pmatrix} \qquad \begin{pmatrix} O_1 \\ O_2 \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}^{-1}\begin{pmatrix} O_1^0 \\ O_2^0 \end{pmatrix}$$

$$= \begin{pmatrix} 1+\delta_{11} & \delta_{12} \\ \delta_{21} & 1+\delta_{22} \end{pmatrix}\begin{pmatrix} O_1 \\ O_2 \end{pmatrix}$$

$\downarrow$

逆矩阵为：$\begin{pmatrix} 1-\delta_{11} & -\delta_{12} \\ -\delta_{21} & 1-\delta_{22} \end{pmatrix}$

$\langle O_1\rangle$的矩阵元为： $\langle O_1^t\rangle + \langle O_1^{1\text{-loop}}\rangle + \text{c.t.} \to$ 有限

$$-\frac{3}{4}g_s^2\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^2}\left[\langle O_2^t\rangle - \frac{1}{3}\langle O_1^t\rangle\right] - \delta_{11}\langle O_1^t\rangle - \delta_{12}\langle O_2^t\rangle = \text{有限}$$

$$\delta_{11} = \frac{1}{4}g_s^2\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^2} \qquad \delta_{12} = -\frac{3}{4}g_s^2\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^2}$$

类似地 $\delta_{21} = \delta_{12} \qquad \delta_{22} = \delta_{11}$

$$\gamma = \frac{1}{Z}\frac{\partial Z}{\partial \ln\mu} = \mu\frac{\partial}{\partial\mu}\begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22}\end{bmatrix}$$

$$= \frac{g_s^2}{(4\pi)^2}\begin{pmatrix} -2 & 6 \\ 6 & -2\end{pmatrix}$$

(4) 对角化算符：

$$O^{1/2} = \frac{1}{2}(\underbrace{\bar d\gamma^\mu(1-\gamma_5)u\,\bar u\gamma_\mu(1-\gamma_5)s}_{O_1} - \underbrace{\bar u\gamma^\mu(1-\gamma_5)u\,\bar d\gamma_\mu(1-\gamma_5)s}_{O_2})$$

$$O^{3/2} = \frac{1}{2}(\bar d\gamma^\mu(1-\gamma_5)u\,\bar u\gamma_\mu(1-\gamma_5)s + \bar u\gamma^\mu(1-\gamma_5)u\,\bar d\gamma_\mu(1-\gamma_5)s)$$

$$\gamma_{1/2} = -8\frac{g_s^2}{(4\pi)^2} \qquad \gamma_{3/2} = 4\frac{g_s^2}{(4\pi)^2}$$

$$\langle O_1\rangle\Big|_{\mu=m_W} = \langle O^{1/2}\rangle_{\mu=m_W} + \langle O^{3/2}\rangle_{\mu=m_W}$$

$$= \left(\frac{\log m_W^2/\Lambda^2}{\log m_K^2/\Lambda^2}\right)^{4/b_0}\langle O^{1/2}\rangle\Big|_{\mu=m_K} \qquad b_0 = 11-\frac{2}{3}n_f$$

$$+ \left(\frac{\log m_W^2/\Lambda^2}{\log m_K^2/\Lambda^2}\right)^{-2/b_0}\langle O^{3/2}\rangle\Big|_{\mu=m_K}$$

$$= 2.1\,\langle O^{1/2}\rangle\Big|_{\mu=m_K} + 0.7\,\langle O^{3/2}\rangle\Big|_{\mu=m_K}$$

$$= 2.1\left(\frac{1}{2}\langle O_1\rangle + \frac{1}{2}\langle O_2\rangle\right)\Big|_{\mu=m_K} + 0.7\left(\frac{1}{2}\langle O_1\rangle - \frac{1}{2}\langle O_2\rangle\right)\Big|_{\mu=m_K}$$

$$= 1.4\,\langle O_1\rangle\Big|_{\mu=m_K} + 0.7\,\langle O_2\rangle\Big|_{\mu=m_K}$$

$$H = \frac{G_F}{\sqrt{2}}\left(C_1(\mu)\,O_1(\mu) + C_2(\mu)\,O_2(\mu)\right)$$

$$\to \quad C_1(m_W) = 1 \qquad C_2(m_W) = 0$$

$$C_1(m_K) = 1.4 \qquad C_2(m_K) = 0.7.$$

$\downarrow$ Wilson 系数的演化.

正负电子湮灭与算符乘积展开:

① 运动学

② 无相互作用 OPE

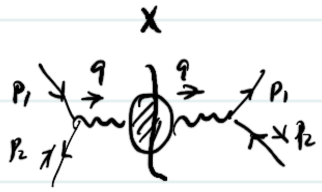
③ 领头项近似

④ OPE 的一般形式

⑤ OPE 适用范围与解析延拓

1. $e^+e^-\to X$ 的散射截面:

a. 图像: $s=(p_1+p_2)^2=q^2$



光学定理:

$$\sigma(e^+e^-\to X) = \frac{1}{2s}\,\mathrm{Im}\,\mathcal{M}(e^+e^-\to e^+e^-)$$ $e^-(p_1)e^+(p_2)\to e^-(p_1)e^+(p_2)$ 向前散射

$$i\mathcal{M}(e^+e^-\to e^+e^-) = (-ie)^2\,\bar u(p_1)\gamma_\mu v(p_2)\cdot\frac{-i}{s}\cdot i\Pi_h^{\mu\nu}(q^2)\,\frac{-i}{s}\,\bar v(p_2)\gamma_\nu u(p_1)$$

$$\Pi_h^{\mu\nu}(q^2)=(q^2g^{\mu\nu}-q^\mu q^\nu)\,\Pi_h(q^2)$$

其中 $q^\mu q^\nu$ 项不产生贡献. (EOM 或矢量流守恒)

$$\frac14\sum_{spin}\bar u(p_1)\gamma_\mu v(p_2)\,\bar v(p_2)\gamma^\mu u(p_1) = \frac14\,\mathrm{tr}[\not p_1\gamma_\mu\not p_2\gamma^\mu] = \frac14(-2)4(p_1\cdot p_2) = -s$$

所以 $\sigma(e^+e^-\to X) = -\frac{4\pi\alpha}{s}\,\mathrm{Im}\,\Pi_h(q^2)$

b. 算符分析:

不变振幅为: $i\mathcal{M}(e^+e^-\to X) = \bar v(p_2)(-ie\gamma_\mu)u(p_1)\cdot\frac{-i}{s}\cdot\langle X|(-ieJ^\mu_{em})|0\rangle$

散射截面为:

$$\sigma=\frac{1}{2s}\,\frac14\sum_{spin}\sum_X(2\pi)^4\delta^4(p_X-q)\,|i\mathcal{M}|^2$$

注意此处的求和也包含动量空间积分 $\frac{d^3p_1}{(2\pi)^3 2E_1}\cdots\frac{d^3p_X}{(2\pi)^3 2E_X}$

化简振幅模方:

$$\frac{1}{4}\sum_{spin}|iM|^2 = \frac{1}{4}\frac{e^4}{s^2}\operatorname{tr}[\not p_1\gamma_\mu \not p_2\gamma_\nu]\cdot\langle 0|J^\mu_{em}|X\rangle\langle X|J^\nu_{em}|0\rangle$$

$$= \frac{e^4}{s^2}(p_{2\mu}p_{1\nu} - g_{\mu\nu}p_1\cdot p_2 + p_{1\mu}p_{2\nu})\langle 0|J^\mu_{em}(x)\langle X|J^\nu_{em}|0\rangle$$

则 $\quad a = \frac{e^4}{2s^3}\,l^{\mu\nu}W_{\mu\nu}$

其中 $\quad l^{\mu\nu} = p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - \frac{q^2}{2}g^{\mu\nu}$

$$\underline{W^{\mu\nu} = \sum_X (2\pi)^4\delta^4(p_X-q)\langle 0|J^\mu_{em}(x)\langle X|J^\nu_{em}|0\rangle}$$

可以将张子矩阵之重新表述为 $\longrightarrow$ $\langle 0|J^\mu_{em}(x)|X\rangle = \langle 0|J^\mu_{em}(0)|X\rangle\cdot e^{-ip_X\cdot x}$

$$W^{\mu\nu} = \int d^4x\, e^{iq\cdot x}\langle 0|J^\mu_{em}(x)J^\nu_{em}(0)|0\rangle$$

$I = \sum_X |X\rangle\langle X|$ 这里求和包含 $\frac{d^3p_1}{(2\pi)^3 2E_1}\cdots\frac{d^3p_X}{(2\pi)^3 2E_X}$

由于 $q^0>0$, 则可以改写为 $\quad J^\mu_{em}(x)J^\nu_{em}(0) - J^\nu_{em}(0)J^\mu_{em}(x)$

$$\underline{W^{\mu\nu} = \int d^4x\, e^{iq\cdot x}\langle 0|[J^\mu_{em}(x), J^\nu_{em}(0)]|0\rangle}, \qquad \underline{q^0\sim Q} \quad x^0\sim\frac{1}{Q}$$

参数化: $\quad W^{\mu\nu} = (q^\mu q^\nu - q^2 g^{\mu\nu})\frac{1}{6\pi}W(s)$

则 $\quad a = \frac{e^4}{2s^3}(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - \frac{q^2}{2}g^{\mu\nu})\cdot(-q^2 g_{\mu\nu})\frac{1}{6\pi}W(s)$

$$= \frac{(4\pi\alpha)^2}{2s^3}(-q^2)(\overset{q^2}{2p_1\cdot p_2} - \frac{q^2}{2}\cdot 4)\frac{1}{6\pi}W(s)$$

$$= \frac{4\pi\alpha^2}{3s}W(s)$$

$$\int d^4x\, e^{iq\cdot x}\langle 0|J^\nu_{em}(0)J^\mu_{em}(x)|0\rangle$$

$$= \int d^4x\, e^{iq\cdot x}\sum_X\langle 0|J^\nu_{em}(0)|X\rangle\langle X|J^\mu_{em}(x)|0\rangle$$

$$= \int d^4x\, e^{iq\cdot x}\sum_X\langle 0|J^\nu_{em}(0)|X\rangle\langle X|J^\mu_{em}(0)|0\rangle e^{ip_X\cdot x}$$

$$= \sum_X(2\pi)^4\delta^4(p_X+q)$$

$\times$

(六) 无相互作用分析:

假设忽略 QCD 相互作用，分析下面组合

$$T[J^\mu(x)J^\nu(0)] = -\mathrm{Tr}[\langle 0|T[\psi(0)\bar\psi(x)]|0\rangle \gamma^\mu_x \langle 0|T[\psi(x)\bar\psi(0)]|0\rangle \gamma^\nu]$$

$$+N[\bar\psi(x)\gamma^\mu[\langle 0|T[\psi(x)\bar\psi(0)]|0\rangle]\gamma^\nu\psi(0)]$$

$$+N[\bar\psi(0)\gamma_\nu[\langle 0|T[\psi(0)\bar\psi(x)]|0\rangle]\gamma^\mu\psi(x)]$$

$$+N[\bar\psi(x)\gamma^\mu\psi(x)\bar\psi(0)\gamma^\nu\psi(0)]$$

利用坐标空间传播子的表达式

$$\Delta(x) = \int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot x}\frac{i}{p^2-m^2+i\varepsilon}$$

$$= \frac{-1}{4\pi^2}\frac{1}{x^2-i\varepsilon} + \text{less singular term}$$

$$S_F(x) = \int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot x}\frac{i(\not p+m)}{p^2-m^2+i\varepsilon}$$

$$= (i\not\partial+m)\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot x}\frac{i}{p^2-m^2+i\varepsilon}$$

$$= (i\not\partial+m)\left(\frac{-1}{4\pi^2}\frac{1}{x^2-i\varepsilon}\right)$$

$$= \cdots$$

$$x^{-6} \qquad\qquad x^{-3}$$

代入可得: $T[J_\mu(x)J_\nu(0)] = \dfrac{x^2g_{\mu\nu}-2x_\mu x_\nu}{\pi^4(x^2-i\varepsilon)^4} + \dfrac{ix^\lambda}{2\pi^2(x^2-i\varepsilon)^2}\Lambda_{\mu\lambda\nu\rho}O_V^\rho(x,0)$

$$+\frac{x^\lambda}{2\pi^2(x^2-i\varepsilon)^2}\varepsilon_{\mu\lambda\nu\rho}O_A^\rho(x,0) + O_{\mu\nu}(x,0)$$

$$O_V^\mu(x,y) = \bar\psi(x)\gamma^\mu\psi(y) - \bar\psi(y)\gamma^\mu\psi(x)$$

$$O_A^\mu(x,y) = \bar\psi(x)\gamma^\mu\gamma_5\psi(y) - \bar\psi(y)\gamma^\mu\gamma_5\psi(x)$$

$$O_{\mu\nu}(x,y) = \bar\psi\gamma^\mu\psi(x)\,\bar\psi\gamma^\nu\psi(y)$$

$$\Lambda_{\mu\lambda\nu\rho} = g_{\mu\lambda}g_{\nu\rho} + g_{\mu\rho}g_{\nu\lambda} - g_{\mu\nu}g_{\lambda\rho}$$

$\varepsilon_{0123} = +1$

利用编时乘积组合：　　　　$\theta(x^0)$

$$T[J_\mu(x)J_\nu(0)] - T[J_\mu(x)J_\nu(0)]^\dagger = \varepsilon(x_0)\,[J_\mu(x), J_\nu(0)]$$

$$\varepsilon(x_0) = \frac{x_0}{|x_0|}$$

且　$\frac{1}{x^2 - i\varepsilon} = \frac{P}{x^2} + i\pi\,\delta(x^2)$

$$\frac{1}{(x^2-i\varepsilon)^n} = \frac{P}{(x^2)^n} + i\pi\,\frac{(-1)^{n-1}}{(n-1)!}\,\delta^{(n-1)}(x^2)$$

可以得到：

$$\begin{aligned}\varepsilon(x_0)[J_\mu(x), J_\nu(0)] &= \frac{i}{3\pi^3}\left(2x_\mu x_\nu - x^2 g_{\mu\nu}\right)\delta^{(3)}(x^2) \\ &+ \frac{1}{\pi}\,x^\lambda\, a_{\mu\lambda\nu\rho}\,\delta^{(1)}(x^2)\,O_V^\rho(x,0) \\ &- \frac{i}{\pi}\,x^\lambda\,\delta^{(1)}(x^2)\,\varepsilon_{\mu\lambda\nu\rho}\,O_A^\rho(x,0) \\ &+ O_{\mu\nu}(x,0) - O_{\nu\mu}(0,x)\end{aligned}$$

(4). 无相互作用. 领头阶近似 即 parton model. 只取 $\delta^{(3)}(x^2)$ 项：

$$W_{\mu\nu} = \sum e_q^2\,\frac{i}{3\pi^2}\left(g_{\mu\nu}\frac{\partial}{\partial q}\cdot\frac{\partial}{\partial q} - 2\frac{\partial}{\partial q^\mu}\frac{\partial}{\partial q^\nu}\right)I_3$$

$$I_n = \int d^4x\, e^{iq\cdot x}\,\varepsilon(x_0)\,\delta^{(n)}(x^2)$$

可在动量空间化简得：

$$I_n = \frac{i\pi^2}{4^{n-1}(n-1)!}\,(q^2)^{n-1}\,\varepsilon(q_0)\,\theta(q^2)$$

代入矩阵元： $W_{\mu\nu} = \sum e_q^2 \frac{1}{6\pi} (q_\mu q_\nu - q^2 g_{\mu\nu}) \varepsilon(q_0)\theta(q^2)$

则 $\sigma = \frac{4\pi\alpha^2}{3s} N_c \sum e_q^2$.

$$\sigma(e^+e^- \to X) = \sum_q \sigma(e^+e^- \to q\bar{q})$$

(六) 一般性理论：包含相互作用之后：

x很小. $\underline{J_\mu(x) J_\nu(0)} \sim C^1_{\mu\nu}(x)\cdot 1 + C^{q\bar{q}}_{\mu\nu}(x)\ \bar{q}q(0) + C^{F^2}_{\mu\nu}(x)\cdot (F^a_{\alpha\beta})^2(0) + \cdots \ \underline{\bar{q}\overleftrightarrow{D}q}$

(+6) $\qquad$ +4 $\qquad$ +4

$\bar{q}\overleftrightarrow{D}q$

量纲分析： $C^1_{\mu\nu}(x) \sim x^{-6}$

$C^{q\bar{q}}_{\mu\nu} \sim m x^{-2}$

$C^{F^2}_{\mu\nu} \sim x^{-2}$

做傅立叶变换

$$-e^2 \int d^4x\ e^{iq\cdot x}\ J_\mu(x) J_\nu(0)$$
$$= -ie^2 (q^2 g_{\mu\nu} - q_\mu q_\nu) [C^1(q^2)\cdot 1 + C^{q\bar{q}}(q^2)\, m\bar{q}q + C^{F^2}(q^2)(F^a_{\alpha\beta})^2 + \cdots]$$

$$C^1 \sim q^0 \qquad C^{\bar{q}q} \sim (q^2)^{-2} \qquad (C^{F^2}) \sim (q^2)^{-2}$$

在领头阶

$$C^1(q^2) = -\left(3\sum_f Q_f^2\right) \frac{\alpha}{3\pi} \log(-q^2 + i\varepsilon)$$

$$\sigma(e^+e^- \to \text{hadrons}) = \frac{4\pi\alpha^2}{s}\left\{ \mathrm{Im}\, C^1(q^2) + \mathrm{Im}\, C^{q\bar{q}}(q^2)\ \langle 0|\bar{q}q|0\rangle + \mathrm{Im}\, C^{F^2}(q^2) \langle 0|(F^a_{\alpha\beta})^2|0\rangle + \cdots \right\}$$

领头阶： $\sigma(e^+e^- \to \text{hadrons}) = \frac{4\pi\alpha^2}{s} \sum_q e_q^2$.

(*) 适用区域问题：

在OPE使用中需要保证小x为主(类空)。但对于类时区域，中间产生了大量强子，不一定能保障适用性！

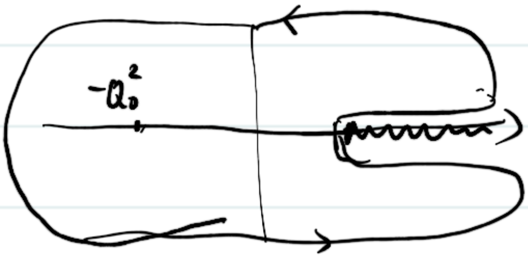
不过人们可以通过色散积分做延拓进行联系.

定义n阶矩

$$I_n = -4\pi\alpha \oint \frac{dq^2}{2\pi i} \frac{1}{(q^2+Q_0^2)^{n+1}} \Pi_h(q^2)$$

如果 contract the contour on the pole. we find

$$I_n = -4\pi\alpha \frac{1}{n!} \frac{d^n}{d(q^2)^n} \Pi_h(q^2)\Big|_{q^2=-Q_0^2} \quad ①$$



选择如上围道 可得

$$I_n = -4\pi\alpha \int \frac{dq^2}{2\pi i} \frac{1}{(q^2+Q_0^2)^{n+1}} \, \mathrm{Disc}\, \Pi_h(q^2)$$

$$= -4\pi\alpha \int \frac{dq^2}{2\pi} \frac{1}{(q^2+Q_0^2)^{n+1}} \frac{1}{i} 2i\, \mathrm{Im}\, \Pi_h(q^2)$$

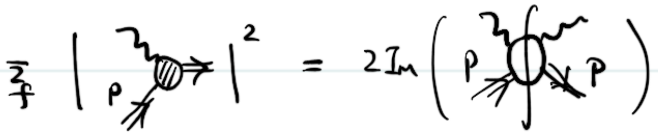
$$= \frac{1}{\pi} \int_0^\infty ds \frac{s}{(s+Q_0^2)^{n+1}} \sigma(s)$$

由公式①可得：

$$\int_0^\infty ds \frac{s}{(s+Q_0^2)^{n+1}} \sigma(s) = \frac{4\pi\alpha^2}{n(Q_0^2)^n} \sum_f Q_f^2 + \mathcal{O}(\alpha_s(Q_0^2)) + \mathcal{O}((Q_0^2)^{2})$$

在$q^2=-Q_0^2$的OPE，与散射截面的积分有关.

DIS:



运动学:

(+) DIS 为电子与核子深度非弹性散射:

$$e(k)+p(p)\to e(k')+X(p_X) \qquad q=k-k'$$

$$i\mathcal{M}(ep\to ef) = -ie\,\bar{u}(k')\gamma_\mu u(k)\,\frac{-i}{q^2}\,(ie)\langle X|J^\mu(0)|p\rangle$$

散射截面与下面物理量有关:

$$\sum_X (2\pi)^4\delta^4(q+p-p_X)\langle p|J^\mu_{em}(0)|X\rangle\langle X|J^\nu_{em}(0)|p\rangle$$ 这里 $\sum_X$ 包括动量求和

(+) 引入矩阵元张量:

$$W^{\mu\nu}(p,q) = \frac{1}{4\pi}\int d^4x\, e^{iq\cdot x}\langle p|[J^\mu_{em}(x), J^\nu_{em}(0)]|p\rangle$$

利用态的完备性关系 $\sum_X |X\rangle\langle X|$ 可得

$$W^{\mu\nu}(p,q) = \frac{1}{4\pi}\int d^4x\, e^{iq\cdot x}\sum_X\langle p|J^\mu_{em}(x)|X\rangle\langle X|J^\nu_{em}(0)|p\rangle$$
$$-\frac{1}{4\pi}\int d^4x\, e^{iq\cdot x}\sum_X\langle p|J^\nu_{em}(0)|X\rangle\langle X|J^\mu_{em}(x)|p\rangle$$

利用平移不变性: $\langle p|J^\mu_{em}(x)|X\rangle = \langle p|J^\mu_{em}(0)|X\rangle\, e^{i(p-p_X)\cdot x}$

$$\langle X|J^\mu_{em}(x)|p\rangle = \langle X|J^\mu_{em}(0)|X\rangle\, e^{i(p_X-p)\cdot x}$$

则 $$W^{\mu\nu}(p,q) = \frac{1}{4\pi}\sum_X (2\pi)^4\delta^4(p+q-p_X)\langle p|J^\mu_{em}(0)|X\rangle\langle X|J^\nu_{em}(0)|p\rangle + \underbrace{\text{其它项}}_{\text{运动学不符合要求}}$$

(*) 利用 $W^{\mu\nu}$，散射截面为:

$$\sigma(ep\to eX) = \frac{e^4}{2s}\int\frac{d^3k'}{(2\pi)^3 2E_{k'}}\ \frac{1}{2}\sum_{spin}\bar{u}(k)\gamma_\mu u(k')\,\bar{u}(k')\gamma_\nu u(k)$$
$$\times\ \frac{1}{(Q^2)^2}\ 4\pi W^{\mu\nu}$$

电子旋量部分: $\frac{1}{2}\sum_{spin}\bar{u}(k)\gamma_\mu u(k')\bar{u}(k')\gamma_\nu u(k) = 2(k_\mu k'_\nu - k\cdot k' g_{\mu\nu} + k_\nu k'_\mu)$

(*) 定义 $x = \frac{Q^2}{2p\cdot q} = \frac{2kk'(1-\cos\theta)}{2M(k-k')}$ 初态静止系

$y = \frac{2p\cdot q}{2p\cdot k} = \frac{k-k'}{k}$

$$\frac{\partial(x\ y)}{\partial(k'\cos\theta)} = \frac{2k'}{2M(k-k')} = \frac{2k'}{ys}$$

$$\int\frac{d^3k'}{(2\pi)^3 2k'} = \int\frac{2\pi\ dk'\,k'\,d\cos\theta}{(2\pi)^3\ 2} = \int dx\,dy\ \frac{ys}{(4\pi)^2}$$

(*) 代入截面公式:

$$\frac{d^2\sigma}{dx\,dy} = \frac{4\pi\alpha^2 y}{(Q^2)^2}\ (k_\mu k'_\nu - k\cdot k' g_{\mu\nu} + k_\nu k'_\mu)\ W^{\mu\nu}$$

(*) 利用 $W^{\mu\nu}$ 的协变性

$$W^{\mu\nu} = F_1\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + \frac{F_2}{p\cdot q}\left(p^\mu - \frac{p\cdot q\, q^\mu}{q^2}\right)\left(p^\nu - \frac{p\cdot q\, q^\nu}{q^2}\right)$$

也有人用 $F_L = F_2 - 2xF_1$ $F_T = F_1$ ··· 结构函数.

则
$$\frac{d^2\sigma}{dx\,dy} = \frac{4\pi\alpha^2 y}{Q^4}\left[\frac{2p\cdot k\, p\cdot k'}{p\cdot q}F_2 + 2k\cdot k_1 F_1\right]$$
$$= \frac{4\pi\alpha^2}{Q^4}\left(s(1-y)F_2 + xy^2 s F_1\right)$$

其中已使用 $2k\cdot k' = Q^2 = xys$.

(*) OPE in DIS

定义编时乘积算符

$$t_{\mu\nu} = i\int d^4x\, e^{iq\cdot x}\, T[J^{\mu}_{em}(x)\, J^{\nu}_{em}(0)]$$

算符矩阵元为: $T_{\mu\nu} = \langle p(\varphi)|\, t_{\mu\nu}\, |p(\varphi)\rangle$

$$T_{\mu\nu} = T_1\left[-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right] + \frac{T_2}{p\cdot q}\left[p_\mu - \frac{p\cdot q}{q^2}q_\mu\right]\left[p_\nu - \frac{p\cdot q}{q^2}q_\nu\right]$$

可以证明: $\mathrm{Im}\, T_{1,2}(w+i\varepsilon, Q^2) = 2\pi\, f_{1,2}(w, Q^2)$

对于算符乘积: $T[O_a(z)\, O_b(0)]$ 一般可展开为

$$T[O_a(z) O_b(0)] = \sum C_{abk}(z)\, O_k(0)$$

动量空间形式为:

$$\int d^4z\, e^{iq\cdot z}\, T[O_a(z) O_b(0)] = \sum C_{abk}(q)\, O_k(0)$$

从 T_1 与 T_2 可以看到, 存在两种指标收缩情况, μ与ν收缩, μ与μ_i, ν与ν_i收缩

算符 O中的指标要么是μ,ν 要么与 q 收缩, 现考虑全部指标与q收缩情况:

$C_{\mu_1\mu_2\cdots\mu_n}\, O^{\mu_1\cdots\mu_n}_{d,n}$ 其中 d为量纲, n为对称指标数

- $\underbrace{\langle O^{\mu_1\cdots\mu_n}_{d,n}\rangle}_{d-2} = \underbrace{p^{\mu_1} p^{\mu_2}\cdots p^{\mu_n}}_{n} \cdot \underbrace{m_p^{d-n-2}}_{d-n-2}$

- $C_{\mu_1\mu_2\cdots\mu_n} \sim \underbrace{q^{\mu_1}\cdots q^{\mu_n}}_{n} \times Q^{2-d-n}$

- $\boxed{C_{\mu_1\mu_2\cdots\mu_n}\langle O\rangle \text{ 的总体量纲为 } 0}$

— $C_{\mu_1\mu_2\cdots\mu_n}\langle O_{d,n}^{\mu_1\cdots\mu_n}\rangle \Rightarrow \frac{q_{\mu_1}q_{\mu_2}\cdots q_{\mu_n}}{Q^n}\cdot Q^{2-d} \times m_p^{d-n-2}\,p^{\mu_1}p^{\mu_2}\cdots p^{\mu_n}$

$$\Rightarrow \frac{(p\cdot q)^n}{Q^n}\,Q^{2-d}\,m_p^{d-n-2}$$

$$\Rightarrow w^n\left(\frac{Q}{m_p}\right)^{2+n-d}$$

$w=\frac{2p\cdot q}{Q^2}$ $\Rightarrow w^n\left(\frac{Q}{m_p}\right)^{2-t}$

$t=d-n$ = dimension − spin

算符的重要性可以按照 t 的增大而减小.

— 算符指标为 μ,ν 也是类似的

算符的一般形式为:

$$O_{q,V}^{\mu_1\cdots\mu_n}=\frac{1}{2}\left(\frac{i}{2}\right)^{n-1}S\left\{\bar{q}\gamma^{\mu_1}\overleftrightarrow{D}^{\mu_2}\overleftrightarrow{D}^{\mu_3}\cdots\overleftrightarrow{D}^{\mu_n}q\right\}$$

$$O_{q,A}^{\mu_1\cdots\mu_n}=\frac{1}{2}\left(\frac{i}{2}\right)^{n-1}S\left\{\bar{q}\gamma^{\mu_1}\overleftrightarrow{D}^{\mu_2}\cdots\overleftrightarrow{D}^{\mu_n}\gamma_5 q\right\}$$

$$\bar{A}\overleftrightarrow{D}^{\mu}B=\bar{A}\overrightarrow{D}^{\mu}B-\bar{A}\overleftarrow{D}^{\mu}B$$

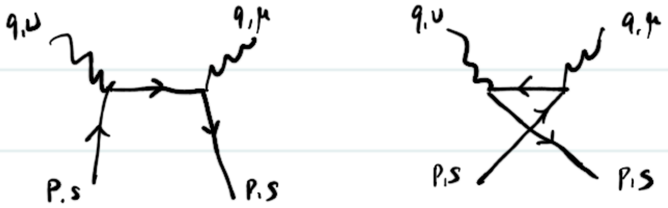
$$O_{g,V}^{\mu_1\cdots\mu_n}=-\frac{1}{2}\left(\frac{i}{2}\right)^{n-2}S\left\{G_A^{\mu_1\alpha}\overleftrightarrow{D}^{\mu_2}\cdots\overleftrightarrow{D}^{\mu_{n-1}}G_{A\alpha}{}^{\mu_n}\right\}$$

— 所以考虑到矢量流守恒, $q\Leftrightarrow -q$ 的对称性.

$$t_{\mu\nu}=\sum_{n=2,4}^{\infty}\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)\frac{2^n q_{\mu_1}\cdots q_{\mu_n}}{(-q^2)^n}\sum_{j=q,g}2\,C_{j,n}^{(1)}\,O_{j,V}^{\mu_1\cdots\mu_n}$$
$$+\sum_{n=2,4}^{\infty}\left(g_{\mu\mu_1}-\frac{q_\mu q_{\mu_1}}{q^2}\right)\left(g_{\nu\mu_2}-\frac{q_\nu q_{\mu_2}}{q^2}\right)\frac{2^n q_{\mu_3}\cdots q_{\mu_n}}{(-q^2)^{n-1}}\sum_{j=g,q}2\,C_{j,n}^{(2)}\,O_{j,V}^{\mu_1\cdots\mu_n}$$

$C_j^{(1)}$ 与 $C_j^{(2)}$ 与相应矩程系数.

- OPE for DIS tree-level



$$iM^{\mu\nu} = i\left\{\bar{u}(p,s)\gamma^\mu \frac{i(\not p+\not q)}{(p+q)^2}\gamma^\nu u(p,s) + \bar{u}(p,s)\gamma^\nu \frac{i(\not p-\not q)}{(p-q)^2}\gamma^\mu u(p,s)\right\} \quad (i为约定)$$

$$(p+q)^2 = 2p\cdot q+q^2 = q^2\left[1+\frac{2p\cdot q}{q^2}\right] = q^2(1-\omega)$$

$$\bar{u}(p,s)\gamma^\mu(\not p+\not q)\gamma^\nu u(p,s) = \bar{u}(p,s)\left\{(p+q)^\mu\gamma^\nu + (p+q)^\nu\gamma^\mu - g^{\mu\nu}(\not p+\not q) + i\varepsilon^{\mu\nu\kappa\lambda}(p+q)_\kappa\gamma_\lambda\gamma_5\right\}u(p,s)$$

利用 $\not p\, u(p,s)=0$ $\quad \bar{u}(p,s)\gamma_\lambda u(p,s) = 2p_\lambda$ $\quad \bar{u}(p,s)\gamma_\lambda\gamma_5 u(p,s) = 2h\,p_\lambda$ (h: helicity)

可得. $\bar{u}(p,s)\gamma^\mu(\not p+\not q)\gamma^\mu u(p,s) = 2(p+q)^\mu p^\nu + 2(p+q)^\nu p^\mu - 2g^{\mu\nu}p\cdot q$

代入并考虑 q 对称性:

$$iM^{\mu\nu} = -\frac{2}{q^2}\sum_{n=0}^{\infty}\omega^n\left[(p+q)^\mu p^\nu + (p+q)^\nu p^\mu - g^{\mu\nu}p\cdot q\right] + (\mu\leftrightarrow\nu,\ q\to -q,\ \omega\to-\omega)$$

$$= -\frac{4}{q^2}\sum_{n=0,2,4}^{\infty}\omega^n\, 2p^\mu p^\nu - \frac{4}{q^2}\sum_{n=1,3,5}^{\infty}\omega^n\left(q^\mu p^\nu + q^\nu p^\mu - g^{\mu\nu}p\cdot q\right)$$

$$= -\frac{8}{q^2}\sum_{n=0,2,4}^{\infty}\frac{2^n(p\cdot q)^n}{(-q^2)^n}\left(p^\mu - \frac{p\cdot q\, q^\mu}{q^2}\right)\left(p^\nu - \frac{p\cdot q\, q^\nu}{q^2}\right)$$

$$- \frac{4}{q^2}\sum_{n=1,3,5}^{\infty}\frac{2^n(p\cdot q)^{n+1}}{(-q^2)^n}\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right)$$

将 $p\cdot q$ 拆分开可得:

$$iM^{\mu\nu} = -\frac{8}{q^2}\sum_{n=0,2,4}^{\infty}\frac{2^n q^{\mu_3}\cdots q^{\mu_{n+2}}}{(-q^2)^n}\left(g^{\mu\mu_1} - \frac{q^\mu q^{\mu_1}}{q^2}\right)\left(g^{\nu\mu_2} - \frac{q^\nu q^{\mu_2}}{q^2}\right)\cdot p_{\mu_1}\cdots p_{\mu_{n+2}}$$

$$- \frac{4}{q^2}\sum_{n=1,3,5}^{\infty}\frac{2^n q^{\mu_1}\cdots q^{\mu_{n+1}}}{(-q^2)^n}\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right)(p_{\mu_1}\cdots p_{\mu_{n+1}})$$

利用矩阵元形式:

$$iM^{\mu\nu} = -\frac{8}{q^2}\sum_{n=0,2,4}^{\infty}\frac{2^n q^{\mu_3}\cdots q^{\mu_{n+2}}}{(-q^2)^n}\left(g^{\mu\mu_1} - \frac{q^\mu q^{\mu_1}}{q^2}\right)\left(g^{\nu\mu_2} - \frac{q^\nu q^{\mu_2}}{q^2}\right)\langle p|O_{q,\nu\mu_1\cdots\mu_{n+2}}|p\rangle$$

$$- \frac{4}{q^2}\sum_{n=1,3,5}^{\infty}\frac{2^n q^{\mu_1}\cdots q^{\mu_{n+1}}}{(-q^2)^n}\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right)\langle p|O_{q,\nu\mu_1\cdots\mu_{n+1}}|p\rangle$$

平移 n 的求和可得:

$$i M^{\mu\nu} = 2 \sum_{n=2,4,6}^{\infty} 2^n \frac{q^{\mu_3} \cdots q^{\mu_n}}{(-q^2)^{n-1}} \left(g^{\mu\mu_1} - \frac{q^\mu q^{\mu_1}}{q^2}\right)\left(g^{\nu\mu_2} - \frac{q^\nu q^{\mu_2}}{q^2}\right) O_{q,\nu\mu_1\cdots\mu_n}$$

$$+ 2 \sum_{n=2,4,6}^{\infty} 2^n \frac{q^{\mu_1} \cdots q^{\mu_n}}{(-q^2)^n} \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) O_{q,\nu,\mu_1\cdots\mu_n}$$

对比可得:

$$C^{(1,2)}_{q,\nu,n} = 1 + O(\alpha_s)$$

$$C^{(1,2)}_{g,\nu,n} = 0 + O(\alpha_s)$$