红外发散与因子化：

1. 红外发散与KLN定理.　　虚圈顶角修正.与实圈单圈修正.

2. 算符乘积展开：

ⓐ 从4夸克有效算符为例.：　small-x expansion

ⓑ $e^+e^- \to q\bar{q}$ 中的 OPE　formal and phenoma

ⓒ DIS 中的 OPE：　light cone expansion

3. 因子化：共线因子化

ⓐ DIS中的共线因子化：按区域展开：hard-collinear-soft-separation

ⓑ DIS单圈散射核的计算　☆　单圈散射核

4. Drell-Yan与共线因子化.　<u>选讲</u>

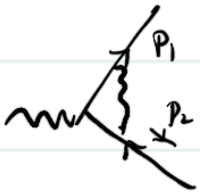
PDF：物理解释：从粒子数密度到规范不变.

0 发散:

- 紫外发散 $\int \frac{d^4k}{(k^2+m^2)^2}$ $k\to\infty$. 对应 $x\to0$
- 红外发散: $k^2\to0$
  - 软发散: 所有分量为0
  - 共线发散 $k^0\sim|\vec{k}|$
- 快度发散: 不是软发散

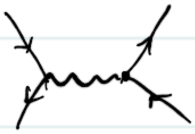
进行上课之前, 先做练习.

计算右图的顶点修正.



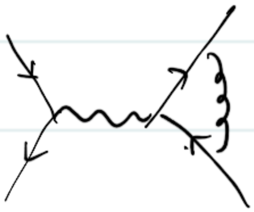
$q = p_1+p_2$

H. 红外发散：以 $e^+e^-\to q\bar{q}$ 为例.

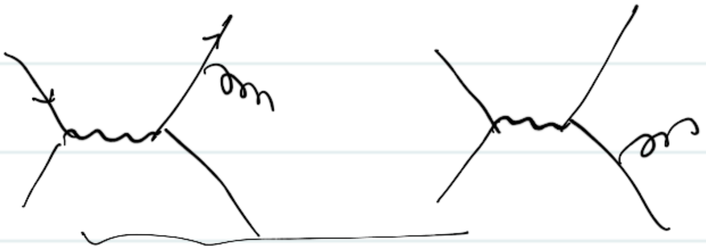
树图  $e^+e^-\to\mu^+\mu^-$ $(-ie\gamma_\mu)$ $e^+e^-\to q\bar{q}$: $(-iee_q\gamma_\mu)$

单圈：

$\alpha_s \sim g_s^2$



虚图



实图.

虚图： 直接计算

$$\Gamma_\mu = \frac{\alpha_s C_F}{4\pi}\left(e^{-\gamma_E}\frac{4\pi}{q^2}\right)^\varepsilon \gamma_\mu\left(-\frac{2}{\varepsilon^2}-\frac{3}{\varepsilon}+\frac{7\pi^2}{6}-8\right)(1+i\varepsilon\pi)$$ ☆ 提前完成

$$\sigma_V = \frac{\alpha_s C_F}{2\pi}\sigma_B\left(e^{-\gamma_E}\frac{4\pi}{q^2}\right)^\varepsilon\left(-\frac{2}{\varepsilon^2}-\frac{3}{\varepsilon}+\frac{7\pi^2}{6}-8\right)$$

$\sigma_B = \frac{4\pi\alpha^2}{3s}$. 树图层次上散射截面.

说明 ⓐ LSZ reduction formula, 需要考虑： 

抽取 Z

ⓑ 自能修正. 用于提取 $\sqrt{Z}$.  自能 $\sim\int\frac{d^4k}{(2\pi)^4}\frac{-i}{k^2}\frac{i}{\not{k}}=0$. 无标度积分

自能修正为0的原因：UV发散与IR发散抵消： $\left(\frac{A}{\varepsilon_{UV}}-\frac{A}{\varepsilon_{IR}}\right)=0$

ⓒ 实修正： $\left.-\frac{2}{\varepsilon^2}-\frac{3}{\varepsilon}\right|_{\substack{UV\\IR}}+\cdots$

实修正中 UV + 自能图中 UV | = 0. 复合算符重整化. 或 矢量流守恒

+

自能图中 IR

||

0

实修正中UV发散 可以认为是来自于 $\sqrt{Z}$ 的红外发散贡献！

正确做法:

$$\begin{tikzcd}
B & A \\
\text{UV.} \arrow[d, leftrightarrow, bend right, "\text{相差负号 ②}"'] \arrow[dr, leftrightarrow, "\text{等价}"] & \underline{\text{IR}} \\
D & C \\
\text{UV.} \arrow[r, leftrightarrow, bend right, "\text{① 相差负号}"'] & \underline{\text{IR}}
\end{tikzcd}$$

$$\rightarrow \underline{A+C} \xrightarrow[C\to B]{} \underline{A+B}$$

正确做法

由①+②

实际做法

红外发散(理论分析)：无质量　$p_1^2=p_2^2=0$

$$i\Gamma^\mu = ig_s^2 C_F \int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2}\frac{1}{(k+p_1)^2}\frac{1}{(k-p_2)^2}\cdot\gamma^\alpha(\not{p}_1+\not{k})\gamma^\mu(\not{k}-\not{p}_2)\gamma_\alpha$$

分子记为 N^μ. 则

$$i\Gamma^\mu = ig_s^2 C_F \int\frac{d^dk}{(2\pi)^d}\frac{N^\mu}{k^2(k^2+2k\cdot p_1)(k^2-2kp_2)}$$

① 将 $\frac{1}{k^2}$ 写为：

$$\frac{1}{k^2}=\frac{1}{2|\vec{k}|}\left[\frac{1}{k_0-|\vec{k}|+i\varepsilon}-\frac{1}{k_0+|\vec{k}|-i\varepsilon}\right]$$

② 采用留数定理方法将 k_0 积分出来

$$\Gamma^\mu \propto \frac{g_s^2C_F}{8(2\pi)^3}\int_0^{2\pi}d\phi\int_{-1}^{1}d\cos\theta\int_0^{\infty}\frac{d|\vec{k}|}{|\vec{k}|}\frac{N^\mu}{[p_{10}-|\vec{p}_1|\cos\theta][p_{20}+|\vec{p}_2|\cos\theta]}$$

θ 为 $\vec{k}$ 与 $\vec{p}_1$ 的夹角.

- $\int\frac{d|\vec{k}|}{|\vec{k}|}$：　当 $|\vec{k}|\to 0$ 时，所有动量分量为0. 软发散
- $\int\frac{d\cos\theta}{[p_{10}-|\vec{p}_1|\cos\theta]}\ \overset{\nearrow}{\underset{\nearrow}{\sim}}\ \int\frac{d\cos\theta}{1-\cos\theta}$

  $p_1^2=0\to p_{10}=|\vec{p}_1|$

  当 $\theta=0$ 即 $\cos\theta=1$ 时 $\vec{k}\parallel\vec{p}_1$ 存在共线发散

类似地 当 $\theta=\pi$ 时 $\cos\theta=-1$. $\vec{k}\parallel\vec{p}_2$ 存在共线发散

另一种分析方法：采用光锥坐标系：

$$k^\mu = (k^+, k^-, k_\perp) \qquad k^\pm = \frac{k^0 \pm k^3}{\sqrt{2}} \qquad \underline{\underline{k_\perp}} = k_T = (k^x, k^y)$$

$$p_1^2 = p_2^2 = 0$$

选择 $p_1^+ \neq 0$ $p_2^- \neq 0$ 即 $p_1^- = p_{1T} = p_{1\perp} = 0$ $p_2^+ = p_{2\perp} = p_{2T} = 0$

$$(k+p_1)^2 = 2(k^+ + p_1^+)k^- - k_\perp^2$$

$$(k-p_2)^2 = k^2 - 2k\cdot p_2 = 2(k^- - p_2^-)k^+ - k_\perp^2$$

$$k^2 = 2k^+k^- - k_\perp^2$$

$$I = \int \frac{dk^+dk^-d^2k_T}{(2\pi)^4} \frac{1}{2(k^++p_1^+)k^- - k_\perp^2} \frac{1}{2k^+(k^- - p_2^-) - k_\perp^2} \frac{1}{2k^+k^- - k_\perp^2}$$

power counting（幂次估计）：

- soft $k^\mu = (k^+, k^-, k_\perp) \sim (\Lambda, \Lambda, \Lambda)$ $p_1^+ \sim p_2^- \sim Q$

$$(k+p_1)^2 = 2(k^++p_1^+)k^- - k_\perp^2 \sim Q\Lambda, \quad (k-p_2)^2 \sim Q\Lambda, \quad k^2 \sim \Lambda^2$$

则积分 I 的幂次为：

$$\frac{\Lambda\,\Lambda\,\Lambda^2}{\Lambda\,\Lambda\,\Lambda^2} \sim O(\Lambda^0)$$

- collinear. $k^\mu = (k^+, k^-, k_\perp) \sim (Q, \frac{\Lambda^2}{Q}, \Lambda)$ $k^2 \sim \Lambda^2$

$$(k+p_1)^2 = 2(k^++p_1^+)k^- - k_\perp^2 \sim \Lambda^2$$

$$(k-p_2)^2 = 2(k^- - p_2^-)k^+ - k_\perp^2 \sim Q^2 \sim \Lambda^0$$

则积分幂次为

$$\frac{\Lambda^0\,\Lambda^2\,\Lambda^2}{\Lambda^2\,\Lambda^0\,\Lambda^2} \sim O(\Lambda^0)$$

都是领头幂次

。围道积分： k^-的极点为：

$$k^- = \frac{k_T^2 - i\varepsilon}{2(k^+ + p_1^+)} \qquad k^- = p_2^- + \frac{k_\perp^2 - i\varepsilon}{2k^+} \qquad k^- = \frac{k_\perp^2 - i\varepsilon}{2k^+}$$

当 $\underline{-p_1^+ < k^+ < 0}$ 时 （对么？） 极点位置分别位于实轴上下方

积分变为： $$\int \frac{dk^+ d^2k_T}{(2\pi)^3} \frac{1}{2(k^+ + p_1^+)\left[2k^+\left(\frac{k_T^2 - i\varepsilon}{2(k^+ + p_1^+)} - p_2^-\right) - k_\perp^2\right]} \frac{1}{2k^+\underline{\left[\frac{k_T^2 - i\varepsilon}{2(k^+ + p_1^+)}\right]} - k_T^2 - i\varepsilon}$$

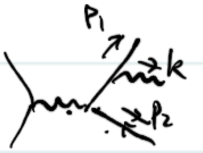
$$= \int \frac{dk^+ d^2k_T}{(2\pi)^3} \frac{2(k^+ + p_1^+)}{2k^+(k_T^2 - 2(k^+ + p_1^+)p_2^-) - 2(k^+ + p_1^+)k_T^2} \frac{1}{(2k^+ - 2k^+ - 2p_1^+)k_T^2}$$

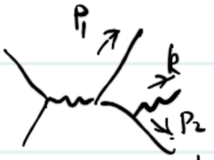
$$\sim \int \frac{dk^+}{\underset{\substack{\uparrow \\ \text{软}}}{k^+}} \frac{d^2k_\perp}{\underset{\substack{\uparrow \\ \text{共线}}}{k_\perp^2}}$$

① 当 $k_\perp = 0$ 时 （k^+可以不为0） $k \parallel \vec{p}_1$(线) 共线发散

② 当 $k^+ = 0$ 时 加上所有分量为0. 软发散

1-2 实图：





$$S_{\mu\nu} = \gamma_\mu \frac{1}{m - \not{p}_1 \not{k}} \gamma_\nu + \gamma_\nu \frac{1}{m + \not{p}_2 + \not{k}} \gamma_\mu$$

加上旋量部分，忽略分子上k贡献.

$$\varepsilon^{*\lambda} \bar{u}(p_1) S_{\lambda\mu} v(p_2) = \left(-\frac{p_{1\lambda}}{p_1 \cdot k} + \frac{p_{2\lambda}}{p_2 \cdot k}\right) \underline{\bar{u}(p_1)\gamma_\mu v(p_2)} \varepsilon^*_\lambda$$

其中. $$\bar{u}(p_1) \not{\varepsilon}^* \frac{\not{k} + \not{p} + m}{(k+p)^2 - m^2} \gamma_\mu v(p_2)$$

$$= \bar{u}(p_1) \not{\varepsilon}^* \frac{\not{p}_1 + m}{2k \cdot p_1} \gamma_\mu v(p_2) = 2 \frac{\varepsilon^* \cdot p_1}{k \cdot p} \bar{u}(p_1) \gamma_\mu v(p_2)$$

$$\not{\varepsilon}^* \not{k} = 2\underset{\substack{\downarrow \\ 0}}{\varepsilon^* \cdot k} - \underset{\substack{\downarrow}}{\not{k}\not{\varepsilon}^*}$$

软 直接压低

共线： $k \parallel p_1$ $\underline{\bar{u}(p_1)\not{k} \sim 0}$

则振幅模方为：

$$F_R = \left(\frac{p_{1\lambda}}{p_1 \cdot k} - \frac{p_{2\lambda}}{p_2 \cdot k}\right)^2 \mathrm{tr}\left[(\not{p}_1 + m)\gamma_\mu (\not{p}_2 - m)\gamma_\nu\right] \times$$ 另一半.

散射截面. $$d\sigma_R \sim \sigma_B \cdot \frac{d^3k}{(2\pi)^3 2E} \sum_{\lambda=1,2} \left| \frac{p_1 \cdot \varepsilon^*}{p_1 \cdot k} - \frac{p_2 \cdot \varepsilon^*}{p_2 \cdot k} \right|^2$$

将 $\sum \varepsilon^*_\mu \varepsilon_\nu \to -g_{\mu\nu}$，则积分部分为：

$$\sim \int \frac{d^3k}{(2\pi)^3} (-g_{\mu\nu}) \left(\frac{p_1^\mu}{k\cdot p_1} - \frac{p_2^\mu}{k\cdot p_2}\right)\cdot\left(\frac{p_1^\nu}{k\cdot p_1} - \frac{p_2^\nu}{k\cdot p_2}\right)$$

$$\sim \int \frac{d^3k}{(2\pi)^3} (+1) \quad \frac{2p_1\cdot p_2}{k\cdot p_1\, k\cdot p_2}$$

选择： $k^\mu = (k^0, \vec{k})$ $\quad p_1^\mu = E(1, \vec{v})$ $\quad p_2^\mu = E(1, \vec{v}')$ $\quad$ ($|\vec{v}| = |\vec{v}'| = 1$ 当 $m=0$)

$p_1\cdot p_2 = 1 - \vec{v}\cdot\vec{v}'$， $k\cdot p_1 = k^0 E(1 - \vec{v}\cdot\hat{\vec{k}})$ $\quad k\cdot p_2 = k^0 E(1 - \vec{v}'\cdot\hat{\vec{k}})$

对k积分分解为大小与角度部分：

$$\int d\sigma_R \sim \sigma_B \frac{\alpha}{\pi} \int_0^{|\vec{q}|} \frac{dk^0}{k^0} I(v, v') \qquad \frac{d^3k}{(2\pi)^3 2f} \to \frac{d^4k}{(2\pi)^4} (2\pi)\delta c \quad)$$

$$I(v, v') = \int \frac{d\Omega_k}{4\pi} \frac{1 - \vec{v}\cdot\vec{v}'}{[1 - \hat{k}\cdot\vec{v}][1 - \hat{k}\cdot\vec{v}']}$$

大小积分有发散（红外）。角度部分仍可拆分为 $(1-\cos\theta)(1+\cos\theta)$

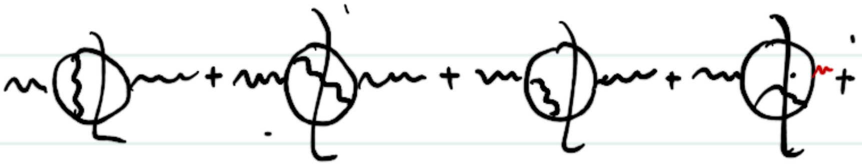
(*) 在DR下完整计算结果为：

$$\sigma_R = \frac{\alpha_s C_F}{2\pi} \sigma_B \left(e^{-\gamma_E} \frac{4\pi}{q^2}\right)^\varepsilon \cdot \left(\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + \frac{19}{2} - \frac{7\pi^2}{6}\right)$$ （直接计算） ☆ 提前完成任务.

注意：① 先计算出 DR下 σ_B

(*) $\sigma_B + \sigma_V + \sigma_R = \sigma_B \left(1 + \frac{3\alpha_s C_F}{4\pi}\right)$

(*) 可以用 KLN 定理描述：下面所有图贡献相加后无红外发散！


交叉图

2、算符乘积展开 (operator product expansion)

2-1 四夸克有效算符

粒子物理电弱标准模型中 $b\to c\ell\tilde{\nu}$

$$iM = \bar{u}(p_c)\frac{ig_2}{2\sqrt{2}}\gamma_\mu(1-\gamma_5)V_{cb}\,u(p_b)$$

$$\times\ \bar{u}(p_{e^-})\frac{ig_2}{2\sqrt{2}}\gamma_\nu(1-\gamma_5)\,v(p_{\tilde{\nu}_e})\ \times\ \frac{-ig^{\mu\nu}}{k^2-m_W^2}$$

其中 $k^\mu = (p_b-p_c)^\mu$

(*) m_b : 4.8 GeV

m_c : 1.5GeV $\qquad\ll\quad m_W = 81.4$ GeV

$m_e \sim m_{\tilde{\nu}_e} \sim 0$

b夸克静止系

$$k^2 = (p_b-p_c)^2 = p_b^2 - 2p_b\cdot p_c + p_c^2 \ \overset{\downarrow}{\le}\ p_b^2 = m_b^2 \ll m_W^2$$

$$iM \approx \frac{-ig_2^2}{8m_W^2}V_{cb}\,\bar{u}(p_c)\gamma_\mu(1-\gamma_5)u(p_b)\ \ \bar{u}(p_e)\gamma^\mu(1-\gamma_5)v(p_{\tilde{\nu}_e})$$

$$\frac{G_F}{\sqrt{2}} = \frac{g_2^2}{8m_W^2}$$

$\underline{iM} \sim \langle c e \tilde{\nu}|\exp[i\int d^4x\, H_{eff}]|b\rangle$

① 有效理论: $H_{eff} = \frac{G_F}{\sqrt{2}}V_{cb}\ \bar{c}\gamma^\mu(1-\gamma_5)b\ \ \bar{e}\gamma^\mu(1-\gamma_5)\tilde{\nu}$ △

c.b.e.ν 均为费米子场算符.

S矩阵元:

$$\langle c e \tilde{\nu}_e|-i\int d^4x\, H_{eff}(x)|b\rangle = (2\pi)^4\delta^4(\ )\cdot \underline{iM}$$

如何从完整理论得到了该有效理论？

(*) 算符分析：

$$S = T\exp\left[i\int d^4x\, \mathcal{L}_I\right] \qquad \mathcal{H}_I = -\mathcal{L}_I$$

$$= T\Big\{ \underline{\frac{ig_2}{2\sqrt{2}} V_{cb} \int d^4x\, \bar{c}\gamma_\mu(1-\gamma_5) b\, W^{+\mu}} \qquad i\int d^4x\, \mathcal{L}_I$$

$$\times \frac{ig_2}{2\sqrt{2}} \int d^4y\, \bar{e}\gamma_\nu(1-\gamma_5)\tilde{\nu}_e\, W^{-\nu}(y) \Big\} \qquad i\int d^4y\, \mathcal{L}_I$$

$$T\{ W^{+\mu}(x)\, W^{-\nu}(y) \} = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \underline{\frac{-ig^{\mu\nu}}{k^2-m_W^2+i\varepsilon}}$$

$$\overset{k\ll m_W}{=\!=} \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{-ig^{\mu\nu}}{-m_W^2+i\varepsilon}$$

$$= \frac{-ig^{\mu\nu}}{-m_W^2} \delta^4(x-y)$$

则 $$S \cong \frac{-ig_2^2}{8m_W^2} T\Big\{ \int d^4x\, \bar{c}\gamma^\mu(1-\gamma_5) b \int d^4y\, \bar{e}\gamma^\nu(1-\gamma_5)\tilde{\nu}_e(y) \cdot \delta^4(x-y) \Big\}$$

$$= \frac{-iG_F}{\sqrt{2}} T\Big\{ \underline{\int d^4x\, \bar{c}\gamma^\mu(1-\gamma_5) b\, \bar{e}\gamma_\mu(1-\gamma_5)\tilde{\nu}_e} \Big\}$$

有效理论的形成. 相当于

$$\underbrace{O_1(x)\, O_2(y)}_{y\sim x} = O_1(x)\, O_2(x+y-x) = \underbrace{O_1(x)\, O_2(x)}_{\hat{O}_1(x)} + (y-x)\, O_1(x)\, O_2'(x) + \cdots$$

$$O_1(x)\, O_2(y) = \hat{O}_1(x) + \cdots$$

把算符的乘积变成了新的定域算符，这就是算符乘积展开. 也是有效理论前提.

有漏洞么？

ⓐ $y-x$ 很大呢？　　$\underline{\text{幂次压低}}$

ⓑ 复合算符重整化呢？

$$O_1(x)\, O_2(y) = \sum_i \underset{\uparrow}{C_i}\, \hat{O}_i(x)$$

弥补 UV 之间的差别，称之为短程系数. 或 Wilson 系数

㈣ 路径积分方法:

$$Z_W = \int [dW^+][dW^-] \exp[i\int d^4x \mathcal{L}_I]$$

$$\mathcal{L}_W = -\frac{1}{2}(\partial_\mu W^+_\nu - \partial_\nu W^+_\mu)(\partial^\mu W^{-\nu} - \partial^\nu W^{-\mu}) + m_W^2 W^+_\mu W^-_\nu$$

$$+ \underbrace{\frac{g_2}{\sqrt{2}}(J^+_\mu W^{+\mu} + J^-_\mu W^{-\mu})}_{\text{相互作用}}$$

利用路径积分中方法，可将生成泛函写为:

$$Z_W = \int [dW^+][dW^-] \exp\{[i\int d^4x\, d^4y\, W^+_\mu(x)\, K^{\mu\nu}(x,y)\, W^-_\nu(y)] + \frac{ig_2}{2\sqrt{2}} \int d^4x\, [J^{+\mu} W^+_\mu + J^{-\mu} W^-_\mu]\}$$

$$K_{\mu\nu}(x,y) = \delta^4(x-y)[g_{\mu\nu}(\partial^2 + m_W^2) - \partial_\mu \partial_\nu]$$

它的逆为 $\Delta_{\mu\nu}(x,y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{-i}{k^2 - m_W^2} [g_{\mu\nu} - \frac{k_\mu k_\nu}{m_W^2}]$

② 积分出 $W^\pm$，则生成泛函为

$$Z_W \sim \exp[-\frac{ig_2^2}{8} \int d^4x\, d^4y\, J^-_\mu(x)\, \Delta^{\mu\nu}(x,y)\, J^+_\nu(y)\}$$

③ 对 $\Delta_{\mu\nu}(x,y)$ 取截断，即

$$\Delta_{\mu\nu}(x,y) = \frac{g_{\mu\nu}}{M_W^2} \delta^4(x-y)$$

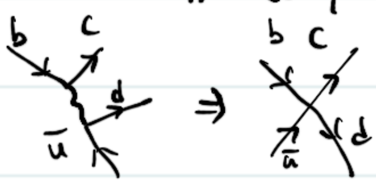
④ 代入后得

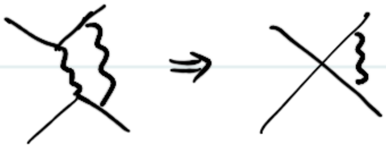
$$Z_W \approx \exp[-i\frac{G_F}{\sqrt{2}} \int d^4x\, J^+_\mu(x)\, J^{-\mu}(x)]$$

这与算符分析方法一致:

⑤ 一般地，需要考虑 QCD 修正:

$$\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}} \{V_{ub} V^*_{ud} (C_1 \underline{\underline{\hat{O}_1}} + C_2 \underline{\underline{\hat{O}_2}}) - V_{tb} V^*_{td} \sum_{i=3}^{10} C_i O_i\}$$

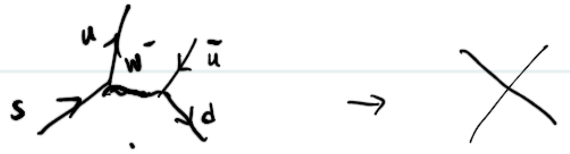




① 对于半轻衰，复合算符重整化系数为1.

② 对于非轻衰，则需计算演化，见 Peskin P607.

非轻：



$s \to u\bar{u}d$

树图：$\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}} \underbrace{(\bar{d}\gamma^\mu(1-\gamma_5)u)(\bar{u}\gamma_\mu(1-\gamma_5)s)}_{O_1}$ $\qquad \underline{k^2 \ll m_W^2}$

加入QCD修正后：

$$\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}}(\underline{C_1}O_1 + \underline{C_2 O_2}) \cdot V_{ud}^* V_{us}$$

$$O_1 = (\bar{d}\gamma^\mu(1-\gamma_5)u)(\bar{u}\gamma_\mu(1-\gamma_5)s)$$

$$O_2 = (\bar{u}\gamma^\mu(1-\gamma_5)u)(\bar{d}\gamma_\mu(1-\gamma_5)s) = (\bar{d}^\alpha\gamma^\mu(1-\gamma_5)u^\beta)(\bar{u}^\beta\gamma_\mu(1-\gamma_5)s^\alpha)$$

颜色不一样

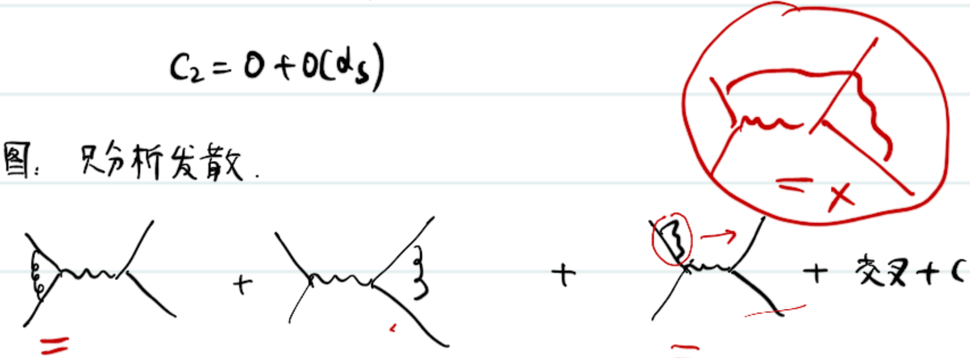
C_1 与 C_2 为短程系数

$$C_1 = 1 + O(\alpha_s)$$

$$C_2 = 0 + O(\alpha_s)$$

一单圈图：只分析发散.

完整：



\+ 交叉 + ct

(*) 其实不用加 ct. 因为矢量流守恒，所有 uv 发散都抵消

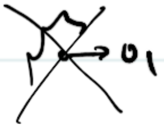
有效

\+ 交叉 + + + 复合算符重整化.

自动抵消

(*) 

$$= -\, g_s^2 \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^2}\left[\frac{1}{2}\bar{u}\gamma^\mu(1-\gamma_5)u\,\bar{d}\gamma_\mu(1-\gamma_5)s - \frac{1}{6}\bar{d}\gamma^\mu(1-\gamma_5)u\,\bar{u}\gamma_\mu(1-\gamma_5)s\right]$$



$$= +\, g_s^2 \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^2}\,\frac{1}{4}\left[\frac{1}{2}\bar{u}\gamma^\mu(1-\gamma_5)u\,\bar{d}\gamma_\mu(1-\gamma_5)s - \frac{1}{6}\bar{d}\gamma^\mu(1-\gamma_5)u\,\bar{u}\gamma_\mu(1-\gamma_5)s\right]$$

四张图相加 $i\mathcal{M} = -\frac{3}{4}g_s^2\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^2}\left[\bar{u}\gamma^\mu(1-\gamma_5)u\,\bar{d}\gamma_\mu(1-\gamma_5)s - \frac{1}{3}\bar{d}\gamma^\mu(1-\gamma_5)u\,\bar{u}\gamma_\mu(1-\gamma_5)s\right]$

(*) 对于 O_2 也是类似地，需交换 $\langle O_1\rangle$ 与 $\langle O_2\rangle$

可以类似于复合算符重整化：

$$O_1^0 = \underline{Z_{11}O_1 + Z_{12}O_2} \qquad O_2^0 = Z_{21}O_1 + Z_{22}O_2$$

$$\begin{pmatrix}O_1^0\\ O_2^0\end{pmatrix} = \begin{pmatrix}Z_{11} & Z_{12}\\ Z_{21} & Z_{22}\end{pmatrix}\begin{pmatrix}O_1\\ O_2\end{pmatrix} \qquad \begin{pmatrix}O_1\\ O_2\end{pmatrix} = \begin{pmatrix}Z_{11} & Z_{12}\\ Z_{21} & Z_{22}\end{pmatrix}^{-1}\begin{pmatrix}O_1^0\\ O_2^0\end{pmatrix}$$

$$= \begin{pmatrix}1+\delta_{11} & \delta_{12}\\ \delta_{21} & 1+\delta_{22}\end{pmatrix}\begin{pmatrix}O_1\\ O_2\end{pmatrix}$$

↓

逆矩阵为：$\begin{pmatrix}1-\delta_{11} & -\delta_{12}\\ -\delta_{21} & 1-\delta_{22}\end{pmatrix}$

$\langle O_1\rangle$ 的矩阵元为：$\langle O_1^t\rangle + \langle O_1^{1\text{loop}}\rangle + \text{c.t.}$ → 有限

$$-\frac{3}{4}g_s^2\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^2}\left[\langle O_2^t\rangle - \frac{1}{3}\langle O_1^t\rangle\right] - \delta_{11}\langle O_1^t\rangle - \delta_{12}\langle O_2^t\rangle = \text{有限}$$

$$\delta_{11} = \frac{1}{4}g_s^2\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^2} \qquad \delta_{12} = -\frac{3}{4}g_s^2\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^2}$$

类似地 $\delta_{21} = \delta_{12}$ $\quad \delta_{22} = \delta_{11}$

$$\gamma = \frac{1}{2}\frac{\partial Z}{\partial \ln\mu} = \mu\frac{\partial}{\partial\mu}\begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22}\end{bmatrix}$$

$$= \frac{g_s^2}{(4\pi)^2}\begin{pmatrix} -2 & 6 \\ 6 & -2 \end{pmatrix}$$

(4) 对角化算符:

$$O^{1/2} = \frac{1}{2}\Big(\overset{O_1}{\bar{d}\gamma^\mu(1-\gamma_5)u\,\bar{u}\gamma_\mu(1-\gamma_5)s} - \overset{O_2}{\bar{u}\gamma^\mu(1-\gamma_5)u\,\bar{d}\gamma_\mu(1-\gamma_5)s}\Big)$$

$$O^{3/2} = \frac{1}{2}\Big(\bar{d}\gamma^\mu(1-\gamma_5)u\,\bar{u}\gamma_\mu(1-\gamma_5)s + \bar{u}\gamma^\mu(1-\gamma_5)u\,\bar{d}\gamma_\mu(1-\gamma_5)s\Big]$$

$$\gamma_{1/2} = -8\frac{g_s^2}{(4\pi)^2} \qquad \gamma_{3/2} = 4\frac{g_s^2}{(4\pi)^2}$$

$$\underline{\langle O_1\rangle\big|_{\mu=m_W}} = \langle O^{1/2}\rangle_{\mu=m_W} + \langle O^{3/2}\rangle_{\mu=m_W}$$

$$= \left(\frac{\log m_W^2/\Lambda^2}{\log m_K^2/\Lambda^2}\right)^{4/b_0}\langle O^{1/2}\rangle\Big|_{\mu=m_K} \qquad b_0 = 11-\frac{2}{3}n_f$$

$$+ \left(\frac{\log m_W^2/\Lambda^2}{\log m_K^2/\Lambda^2}\right)^{-2/b_0}\langle O^{3/2}\rangle\Big|_{\mu=m_K}$$

$$= 2.1\,\langle O^{1/2}\rangle\Big|_{\mu=m_K} + 0.7\,\langle O^{3/2}\rangle\Big|_{\mu=m_K}$$

$$= 2.1\left(\frac{1}{2}\langle O_1\rangle + \frac{1}{2}\langle O_2\rangle\right)\Big|_{\mu=m_K} + 0.7\left(\frac{1}{2}\langle O_1\rangle - \frac{1}{2}\langle O_2\rangle\right)\Big|_{\mu=m_K}$$

$$= 1.4\,\underline{\underline{\langle O_1\rangle}}\Big|_{\mu=m_K} + \underline{\underline{0.7}}\,\langle O_2\rangle\Big|_{\mu=m_K}$$

$$\mathcal{H} = \frac{G_F}{\sqrt{2}}\big(C_1(\mu)\,O_1(\mu) + C_2(\mu)\,O_2(\mu)\big)$$

$$\rightarrow \quad C_1(m_W) = 1 \qquad C_2(m_W) = 0$$

$$C_1(m_K) = 1.4 \qquad C_2(m_K) = 0.7.$$

↓ Wilson 系数的演化.

正负电子湮灭与算符乘积展开：

① 运动学

② 无相互作用 OPE

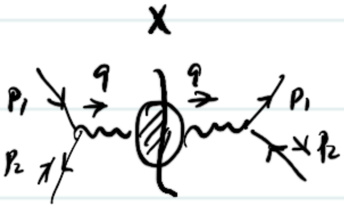
③ 领头项近似

④ OPE 的一般形式

⑤ OPE 适用范围与解析延拓

1. $e^+e^- \to X$ 的散射截面：

a. 图像： $s=(p_1+p_2)^2=q^2$



光学定理：

$\sigma(e^+e^-\to X) = \frac{1}{2s}\,\mathrm{Im}\,M(e^+e^-\to e^+e^-)$ 。 $e^-(p_1)\,e^+(p_2) \to e^-(p_1)\,e^+(p_2)$ 向前散射

$$iM(e^+e^-\to e^+e^-) = (-ie)^2\,\bar{u}(p_1)\,\gamma_\mu\,v(p_2)\cdot\frac{-i}{s}\cdot i\Pi_h^{\mu\nu}(q^2)\ \ \frac{-i}{s}\ \ \bar{v}(p_2)\,\gamma_\nu\,u(p_1)$$

$$\Pi_h^{\mu\nu}(q^2) = (q^2g^{\mu\nu}-q^\mu q^\nu)\,\Pi_h(q^2)$$

其中 $q^\mu q^\nu$ 项不产生贡献。（EOM 或矢量流守恒）

$$\frac{1}{4}\sum_{spin}\bar{u}(p_1)\,\gamma_\mu\,v(p_2)\ \ \bar{v}(p_2)\,\gamma^\mu\,u(p_1) = \frac{1}{4}\,\mathrm{tr}\left[\not{p}_1\,\gamma_\mu\,\not{p}_2\,\gamma^\mu\right] = \frac{1}{4}(-2)\,4(p_1\cdot p_2) = -s$$

所以 $\sigma(e^+e^-\to X) = -\frac{4\pi\alpha}{s}\,\mathrm{Im}\,\Pi_h(q^2)$

b. 算符分析：

不变振幅为： $iM(e^+e^-\to X) = \bar{v}(p_2)\,(-ie\gamma_\mu)\,u(p_1)\cdot\frac{-i}{s}\cdot\underline{\langle X|\,(-ieJ^\mu_{em})\,|0\rangle}$

散射截面为：

$$\sigma = \frac{1}{2s}\ \frac{1}{4}\sum_{spin}\ \sum_X\,(2\pi)^4\,\delta^4(p_X-q)\ |iM|^2$$

注意此处的求和也包含动量空间积分 $\frac{d^3p_1}{(2\pi)^3 2E_1}\cdots\frac{d^3p_X}{(2\pi)^3 2E_X}$

化简振幅模方:

$$\frac{1}{4}\sum_{spin}|iM|^2 = \frac{1}{4}\frac{e^4}{s^2}\,\mathrm{tr}[\not p_2\gamma_\mu \not p_1\gamma_\nu]\cdot\langle 0|J^\mu_{em}|X\rangle\langle X|J^\nu_{em}|0\rangle$$

$$=\frac{e^4}{s^2}\left(p_{2\mu}p_{1\nu}-g_{\mu\nu}\,p_1\cdot p_2+p_{1\mu}p_{2\nu}\right)\langle 0|J^\mu_{em}|X\rangle\langle X|J^\nu_{em}|0\rangle$$

则 $\quad \sigma=\frac{e^4}{2s^3}\,l^{\mu\nu}\,W_{\mu\nu}$

其中 $\quad l^{\mu\nu}=p_1^\mu p_2^\nu+p_1^\nu p_2^\mu-\frac{q^2}{2}g^{\mu\nu}$

$$\underline{W^{\mu\nu}=\sum_X(2\pi)^4\delta^4(p_X-q)\,\langle 0|\underline{J^\mu_{em}}(x)|X\rangle\langle X|J^\nu_{em}|0\rangle}$$

可以将張量矩阵之重新表述为 $\longrightarrow$ $\langle 0|J^\mu_{em}(x)|X\rangle=\langle 0|J^\mu_{em}(0)|X\rangle\cdot e^{-ip_X\cdot x}$

$$W^{\mu\nu}=\int d^4x\; e^{iq\cdot x}\langle 0|J^\mu_{em}(x)\,J^\nu_{em}(0)|0\rangle$$

$I=\sum_X |X\rangle\langle X|$ 这里求和包含 $\frac{d^3p_1}{(2\pi)^3 2E_1}\cdots\frac{d^3p_X}{(2\pi)^3 2E_X}$

由于 $q^0>0$, 则可以改写为 $\quad J^\mu_{em}(x)J^\nu_{em}(0)-J^\nu_{em}(0)J^\mu_{em}(x)$

$$\underline{W^{\mu\nu}=\int d^4x\; e^{iq\cdot x}\langle 0|[J^\mu_{em}(x),J^\nu_{em}(0)]|0\rangle}\,,\qquad \underline{q^0\sim Q}\quad \underline{x^0}\sim\frac{1}{Q}$$

参数化: $\quad W^{\mu\nu}=(q^\mu q^\nu-q^2g^{\mu\nu})\,\frac{1}{6\pi}\,W(s)$

则 $\quad \sigma=\frac{e^4}{2s^3}\left(p_1^\mu p_2^\nu+p_1^\nu p_2^\mu-\frac{q^2}{2}g^{\mu\nu}\right)\cdot(-q^2g_{\mu\nu})\,\frac{1}{6\pi}\,W(s)$

$$=\frac{(4\pi\alpha)^2}{2s^3}(-q^2)\left(\overset{q^2}{2p_1\cdot p_2}-\frac{q^2}{2}\cdot 4\right)\frac{1}{6\pi}W(s)$$

$$=\frac{4\pi\alpha^2}{3s}W(s)$$

$$\int d^4x\, e^{iq\cdot x}\langle 0|J^\nu_{em}(0)\,J^\mu_{em}(x)|0\rangle$$

$$=\int d^4x\, e^{iq\cdot x}\sum_X\langle 0|J^\nu_{em}(0)|X\rangle\langle X|J^\mu_{em}(x)|0\rangle$$

$$=\int d^4x\, e^{iq\cdot x}\sum_X\langle 0|J^\nu_{em}(0)|X\rangle\langle X|J^\mu_{em}(0)|0\rangle\, e^{ip_X\cdot x}$$

$$=\sum_X(2\pi)^4\underset{X}{\delta^4(p_X+q)}$$

(4) 无相互作用分析：

假设忽略 QCD 相互作用，分析 下面组合

$$T[J^\mu(x) J^\nu(0)] = -\mathrm{Tr}\left[\langle 0|T[\psi(0)\bar{\psi}(x)]|0\rangle \gamma^\mu_x \langle 0|T[\psi(x)\bar{\psi}(0)]|0\rangle \gamma^\nu \right]$$

$$+ N\left[\bar{\psi}(x)\gamma^\mu [\langle 0|T[\psi(x)\bar{\psi}(0)]|0\rangle] \gamma^\nu \psi(0)\right]$$

$$+ N\left[\bar{\psi}(0)\gamma_\nu [\langle 0|T[\psi(0)\bar{\psi}(x)]|0\rangle] \gamma^\mu \psi(x)\right]$$

$$+ N\left[\bar{\psi}(x)\gamma^\mu \psi(x)\bar{\psi}(0)\gamma^\nu \psi(0)\right]$$

利用坐标空间传播子的表达式

$$\Delta(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{i}{p^2-m^2+i\varepsilon}$$

$$= \frac{-1}{4\pi^2}\frac{1}{x^2-i\varepsilon} + \text{less singular term}$$

$$S_F(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon}$$

$$= (i\not{\partial}+m) \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{i}{p^2-m^2+i\varepsilon}$$

$$= (i\not{\partial}+m)\left(\frac{-1}{4\pi^2}\frac{1}{x^2-i\varepsilon}\right)$$

$$= \cdots$$

$$x^{-6} \qquad x^{-3}$$

代入可得： $$T[J_\mu(x) J_\nu(0)] = \frac{x^2 g_{\mu\nu} - 2x_\mu x_\nu}{\pi^4 (x^2-i\varepsilon)^4} + \frac{i x^\lambda}{2\pi^2(x^2-i\varepsilon)^2} a_{\mu\lambda\nu\rho} O_V^\rho(x,0)$$

$$+ \frac{x^\lambda}{2\pi^2(x^2-i\varepsilon)^2}\varepsilon_{\mu\lambda\nu\rho} O_A^\rho(x,0) + O_{\mu\nu}(x,0)$$

$$O_V^\mu(x,y) = \bar{\psi}(x)\gamma^\mu\psi(y) - \bar{\psi}(y)\gamma^\mu\psi(x)$$

$$O_A^\mu(x,y) = \bar{\psi}(x)\gamma^\mu\gamma_5\psi(y) - \bar{\psi}(y)\gamma^\mu\gamma_5\psi(x)$$

$$O_{\mu\nu}(x,y) = \bar{\psi}\gamma^\mu\psi(x)\ \bar{\psi}\gamma^\nu\psi(y)$$

$$a_{\mu\lambda\nu\rho} = g_{\mu\lambda}g_{\nu\rho} + g_{\mu\rho}g_{\nu\lambda} - g_{\mu\nu}g_{\lambda\rho}$$

$\varepsilon_{0123} = +1$

利用编时乘积组合： $\Theta(x^0)$

$$T[J_\mu(x)J_\nu(0)] - T[J_\mu(x)J_\nu(0)]^\dagger = \underset{\uparrow}{\varepsilon(x_0)}\,[J_\mu(x), J_\nu(0)]$$

$$\varepsilon(x_0) = \frac{x_0}{|x_0|}$$

$\underline{且}$
$$\underline{\frac{1}{x^2 - i\varepsilon}} = \frac{P}{x^2} + i\pi\,\delta(x^2)$$

$$\frac{1}{(x^2 - i\varepsilon)^n} = \frac{P}{(x^2)^n} + i\pi\,\frac{(-1)^{n-1}}{(n-1)!}\,\delta^{(n-1)}(x^2)$$

可以得到：

$$\begin{aligned}
\varepsilon(x_0)\underline{[J_\mu(x), J_\nu(0)]} = &\frac{i}{3\pi^3}\,(2x_\mu x_\nu - x^2 g_{\mu\nu})\,\underline{\delta^{(3)}(x^2)} &&✓\\
&+ \frac{1}{\pi}\,x^\lambda a_{\mu\lambda\nu\rho}\,\delta^{(1)}(x^2)\,O_V^\rho(x,0) &&✓\\
&- \frac{i}{\pi}\,x^\lambda\,\delta^{(1)}(x^2)\,\varepsilon_{\mu\lambda\nu\rho}\,O_A^\rho(x,0) &&✓\\
&+ O_{\mu\nu}(x,0) - O_{\nu\mu}(0,x) &&✓
\end{aligned}$$

(*). 无相互作用. 领头阶近似 即 parton model.，只取 $\delta^{(3)}(x^2)$ 项：

$$W_{\mu\nu} = \Sigma e_q^2\,\frac{i}{3\pi^2}\left(g_{\mu\nu}\frac{\partial}{\partial q}\cdot\frac{\partial}{\partial q} - 2\frac{\partial}{\partial q^\mu}\frac{\partial}{\partial q^\nu}\right)I_3$$

$$\underline{I_n = \int d^4x\, e^{iq\cdot x}\,\varepsilon(x_0)\,\delta^{(n)}(x^2)}$$

可在动量空间化简得：

$$I_n = \frac{i\pi^2}{4^{n-1}(n-1)!}\,(q^2)^{n-1}\,\underline{\underline{\varepsilon(q_0)}}\,\theta(q^2)$$

代入矩阵元： $W_{\mu\nu} = \sum e_q^2 \frac{1}{6\pi} (q_\mu q_\nu - q^2 g_{\mu\nu}) \varepsilon(q_0) \theta(q^2)$

则 $\sigma = \frac{4\pi\alpha^2}{3s} N_c \sum e_q^2$.

$$\sigma(e^+e^- \to X) = \sum_q \sigma(e^+e^- \to q\bar{q})$$

(六) 一般性理论：包含相互作用之后：

$\underline{x很小}$. $\underset{(+6)}{\underline{\underline{J_\mu(x) J_\nu(0)}}} \sim C^1_{\mu\nu}(x)\cdot 1 + C^{\bar{q}q}_{\mu\nu}(x)\, \bar{q}q(0) + C^{F^2}_{\mu\nu}(x)\cdot \underset{+4}{(F^a_{\alpha\beta})^2(0)} + \cdots \underset{+4}{\underline{\bar{q}\overleftrightarrow{D}q}}$

$\bar{q}\overleftrightarrow{D}q$

量纲分析：$C^1_{\mu\nu}(x) \sim x^{-6}$

$C^{\bar{q}q}_{\mu\nu} \sim m x^{-2}$

$C^{F^2}_{\mu\nu} \sim x^{-2}$

做付立叶变换

$$-e^2 \int d^4x\, e^{iq\cdot x} J_\mu(x) J_\nu(0)$$

$$= -ie^2 (q^2 g_{\mu\nu} - q_\mu q_\nu) \left[C^1(q^2)\cdot 1 + C^{\bar{q}q}(q^2)\, m\bar{q}q + C^{F^2}(q^2) (F^a_{\alpha\beta})^2 + \cdots \right]$$

$C^1 \sim q^0 \qquad C^{\bar{q}q} \sim (q^2)^{-2} \qquad (C^{F^2}) \sim (q^2)^{-2}$

在领头阶

$$C^1(q^2) = -\left(3\sum_f Q_f^2\right) \frac{\alpha}{3\pi} \log(-q^2 + i\varepsilon)$$

$$\sigma(e^+e^- \to \text{hadrons}) = \frac{4\pi\alpha^2}{s} \left\{ \mathrm{Im}\, C^1(q^2) + \mathrm{Im}\, C^{\bar{q}q}(q^2) \langle 0|\bar{q}q|0\rangle + \mathrm{Im}\, C^{F^2}(q^2) \langle 0|(F^a_{\alpha\beta})^2|0\rangle + \cdots \right\}$$

领头阶： $\sigma(e^+e^- \to \text{hadrons}) = \frac{4\pi\alpha^2}{s} \sum_q e_q^2$.

(*) 适用区域问题：

在OPE使用中需要保证$\underline{小x}$为主$\overset{\text{类空}}{\wedge}$. 但对于类时区域，中间产生了大量强子，不一定能保障适用性！

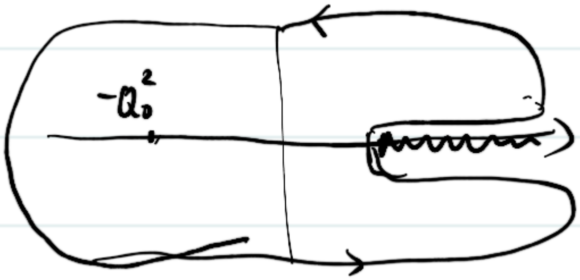
不过人们可以通过色散积分做延拓进行联系.

定义n阶矩

$$I_n = -4\pi\alpha \oint \frac{dq^2}{2\pi i} \frac{1}{(q^2+Q_0^2)^{n+1}} \Pi_h(q^2)$$

如果 contract the contour on the pole. we find

$$I_n = -4\pi\alpha \frac{1}{n!} \frac{d^n}{d(q^2)^n} \Pi_h(q^2)\Big|_{q^2=-Q_0^2} \qquad ①$$



选择如上围道 可得

$$I_n = -4\pi\alpha \int \frac{dq^2}{2\pi i} \frac{1}{(q^2+Q_0^2)^{n+1}} \operatorname{Disc} \Pi_h(q^2)$$

$$= -4\pi\alpha \int \frac{dq^2}{2\pi} \frac{1}{(q^2+Q_0^2)^{n+1}} \frac{1}{i} 2i \operatorname{Im} \Pi_h(q^2)$$

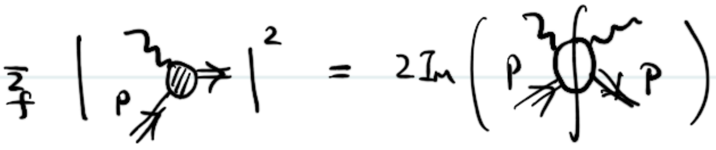
$$= \frac{1}{\pi} \int_0^\infty ds \frac{s}{(s+Q_0^2)^{n+1}} \sigma(s)$$

由公式①可得：

$$\int_0^\infty ds \frac{s}{(s+Q_0^2)^{n+1}} \sigma(s) = \frac{4\pi\alpha^2}{n(Q_0^2)^n} \sum_f Q_f^2 + \mathcal{O}(\alpha_s(Q_0^2)) + \mathcal{O}((Q_0^2)^{-2})$$

在$q^2=-Q_0^2$的OPE，与散射截面的积分有关.

DIS:



运动学:

(*) DIS 为电子与核子深度非弹性散射:

$$e(k)+p(p)\to e(k')+X(P_X) \qquad q=k-k'$$

$$iM(ep\to ef) = -ie\,\bar{u}(k')\gamma_\mu u(k)\,\frac{-i}{q^2}\,(ie)\langle X|J^\mu(0)|p\rangle$$

散射截面与下面物理量有关:

$$\sum_X (2\pi)^4\delta^4(q+p-P_X)\langle p|J^\mu_{em}(0)|X\rangle\langle X|J^\nu_{em}(0)|p\rangle$$ 这里 $\sum_X$ 包括动量求和

(*) 引入矩阵元张量:

$$W^{\mu\nu}(p,q) = \frac{1}{4\pi}\int d^4x\, e^{iq\cdot x}\langle p|[J^\mu_{em}(x), J^\nu_{em}(0)]|p\rangle$$

利用态的完备性关系 $\sum_X|X\rangle\langle X$ 可得

$$W^{\mu\nu}(p,q) = \frac{1}{4\pi}\int d^4x\, e^{iq\cdot x}\sum_X\langle p|J^\mu_{em}(x)|X\rangle\langle X|J^\nu_{em}(0)|p\rangle$$
$$-\frac{1}{4\pi}\int d^4x\, e^{iq\cdot x}\sum_X\langle p|J^\nu_{em}(0)|X\rangle\langle X|J^\mu_{em}(x)|p\rangle$$

利用平移不变性: $\langle p|J^\mu_{em}(x)|X\rangle = \langle p|J^\mu_{em}(0)|X\rangle\cdot e^{i(P-P_X)\cdot x}$

$$\langle X|J^\mu_{em}(x)|p\rangle = \langle X|J^\mu_{em}(0)|X\rangle e^{i(P_X-P)\cdot x}$$

则 $$W^{\mu\nu}(p,q) = \frac{1}{4\pi}\sum_X(2\pi)^4\delta^4(p+q-P_X)\langle p|J^\mu_{em}(0)|X\rangle\langle X|J^\nu_{em}(0)|p\rangle + \underbrace{\text{其它项}}_{\text{运动学不符合要求}}$$

(*) 利用 $W^{\mu\nu}$，散射截面为：

$$\sigma(ep\to eX) = \frac{e^4}{2s}\int\frac{d^3k'}{(2\pi)^3 2E_{k'}}\ \frac{1}{2}\sum_{spin}\bar{u}(k)\gamma_\mu u(k')\ \bar{u}(k')\gamma_\nu u(k)$$

$$\times\ \frac{1}{(Q^2)^2}\ 4\pi W^{\mu\nu}$$

电子旋量部分： $\frac{1}{2}\sum_{spin}\bar{u}(k)\gamma_\mu u(k')\bar{u}(k')\gamma_\nu u(k) = 2(k_\mu k'_\nu - k\cdot k' g_{\mu\nu} + k_\nu k'_\mu)$

(*) 定义 $x = \frac{Q^2}{2p\cdot q} = \frac{2kk'(1-\cos\theta)}{2M(k-k')}$ 初态静止系

$$y = \frac{2p\cdot q}{2p\cdot k} = \frac{k-k'}{k}$$

$$\frac{\partial(x\ y)}{\partial(k'\cos\theta)} = \frac{2k'}{2M(k-k')} = \frac{2k'}{ys}$$

$$\int\frac{d^3k'}{(2\pi)^3 2k'} = \int\frac{2\pi\ dk'\ k'\ d\cos\theta}{(2\pi)^3\ 2} = \int dx\,dy\ \frac{ys}{(4\pi)^2}$$

(*) 代入截面公式：

$$\frac{d^2\sigma}{dx\,dy} = \frac{4\pi\alpha^2 y}{(Q^2)^2}\ (k_\mu k'_\nu - k\cdot k' g_{\mu\nu} + k_\nu k'_\mu)\ W^{\mu\nu}$$

(*) 利用 $W^{\mu\nu}$ 的协变性

$$W^{\mu\nu} = F_1\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + \frac{F_2}{pq}\left(p^\mu - \frac{p.q\,q^\mu}{q^2}\right)\left(p^\nu - \frac{p.q\,q^\nu}{q^2}\right)$$

也有人用 $F_L = F_2 - 2xF_1$ $F_T = F_1$ ··· 结构函数.

则 $$\frac{d^2\sigma}{dx\,dy} = \frac{4\pi\alpha^2 y}{Q^4}\left[\frac{2p\cdot k\,p\cdot k'}{p\cdot q}F_2 + 2k\,k_1 F_1\right]$$

$$= \frac{4\pi\alpha^2}{Q^4}\left(s(1-y)F_2 + xy^2 s F_1\right)$$

其中已使用 $2k\cdot k' = Q^2 = xys$.

(*) OPE in DIS

定义编时乘积算符　　　　$\bar{\psi}\gamma^{\mu}\psi \quad \bar{\psi}\gamma^{\nu}\psi$

$$t_{\mu\nu} = i\int d^4x\, e^{iq\cdot x}\, T[\underline{J^{\mu}_{em}(x)\ J^{\nu}_{en}(0)}]$$

算符矩阵元为：　$T_{\mu\nu} = \langle p(\varphi)|t_{\mu\nu}|p(\varphi)\rangle$

$$T_{\mu\nu} = T_1\left[-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right] + \frac{T_2}{p\,q}\left[p_\mu - \frac{p\cdot q}{q^2}q_\mu\right]\left[p_\nu - \frac{p\,q}{q^2}q_\nu\right]$$

可以证明：$\mathrm{Im}\, T_{1,2}(w+i\varepsilon, Q^2) = 2\pi f_{1,2}(w, Q^2)$

对于算符乘积：　$T[O_a(z)\,O_b(0)]$ 一般可展开为

$$T[O_a(z)O_b(0)] = \sum C_{abk}(z)\, O_k(0)$$

动量空间形式为：

$$\int d^4z\, e^{iq\cdot z}\, T[O_a(z)O_b(0)] = \sum C_{abk}(q)\, O_k(0)$$

从 T_1 与 T_2 可以看到，存在两种指标收缩情况，μ与ν收缩，μ与μ_1，ν与ν_1收缩

算符 O 中的指标要么是 μ,ν 要么与 q 收缩，现考虑全部指标与 q 收缩情况

可能的算符：$\underline{\bar{\psi}\gamma^{\mu_1}\psi}$，$\bar{\psi}D^{\mu_1}\psi$，$\bar{\psi}\gamma^{\mu_1}D^{\mu_2}\psi$　$\bar{\psi}D^{\mu_1}D^{\mu_2}\psi$ ……　量纲不同，对称指标数不同、

考虑一般情况：$C_{\mu_1\mu_2\cdots\mu_n}\, O^{\mu_1\cdots\mu_n}_{d,n}$　　其中 d 为量纲，n 为对称指标数

- $$\langle \underbrace{O^{\mu_1\cdots\mu_n}_{d,n}}_{d-2}\rangle = \underbrace{p^{\mu_1}p^{\mu_2}\cdots}_{n}p^{\mu_n}\cdot \underbrace{m_p^{d-n-2}}_{d-n-2}$$

- $$C_{\mu_1\mu_2\cdots\mu_n} \approx \overset{n}{\underline{q^{\mu_1}\cdots}}\, q^{\mu_n} \times Q^{2-d-n}$$

- $\boxed{C_{\mu_1\mu_2\cdots\mu_n}\ \langle O\rangle \text{ 的总体量纲为 } 0}$

- $C_{\mu_1\mu_2\cdots\mu_n}\langle O_{d,n}^{\mu_1\cdots\mu_n}\rangle \Rightarrow \frac{q_{\mu_1}q_{\mu_2}\cdots q_{\mu_n}}{Q^n}\cdot Q^{2-d} \times m_p^{d-n-2}\, p^{\mu_1}p^{\mu_2}\cdots p^{\mu_n}$

$$\Rightarrow \frac{(p\cdot q)^n}{Q^n}\, Q^{2-d}\, m_p^{d-n-2}$$

$$\Rightarrow w^n \left(\frac{Q}{m_p}\right)^{2+n-d}$$

$\underline{w=\frac{2p\cdot q}{Q^2}\sim 1}$ $\quad \Rightarrow w^n\left(\frac{Q}{m_p}\right)^{2-t}$

$t = d-n$ = dimension − spin

算符的重要性可以按照t的增大而减小

- 算符指标加μ, ν也是类似的 · ·

例：$\bar{\psi}\gamma^\mu\psi$, $t=2$ $\quad \bar{\psi}D^\mu\psi$: $t=3$. $\quad \bar{\psi}\psi$: $t=3$. $\quad$ $\underline{最小\ t=2}$

说明：这里对应 μ 指标取大分量 $\quad \underline{\bar{\psi}\gamma_\perp^\mu\psi,\ t=3}$

(*) 领头扭度算符 ($t=2$) 的一般形式为：

$$O_{q,V}^{\mu_1\cdots\mu_n} = \frac{1}{2}\left(\frac{i}{2}\right)^{n-1} S\left\{\bar{q}\gamma^{\mu_1}\overleftrightarrow{D}^{\mu_2}\overleftrightarrow{D}^{\mu_3}\cdots\overleftrightarrow{D}^{\mu_n}q\right\}$$

$$O_{q,A}^{\mu_1\cdots\mu_n} = \frac{1}{2}\left(\frac{i}{2}\right)^{n-1} S\left\{\bar{q}\gamma^{\mu_1}\overleftrightarrow{D}^{\mu_2}\cdots\overleftrightarrow{D}^{\mu_n}\gamma_5 q\right\}$$

$$\bar{A}\overleftrightarrow{D}^\mu B = \bar{A}\overrightarrow{D}^\mu B - \bar{A}\overleftarrow{D}^\mu B$$

$$O_{g,V}^{\mu_1\cdots\mu_n} = -\frac{1}{2}\left(\frac{i}{2}\right)^{n-2} S\left\{G_A^{\mu_1\alpha}\overleftrightarrow{D}^{\mu_2}\cdots\overleftrightarrow{D}^{\mu_{n-1}}G_{A\alpha}{}^{\mu_n}\right\}$$

- 考虑到矢量流守恒，$q\leftrightarrow -q$ 的对称性.

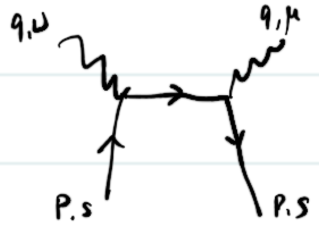
$$t_{\mu\nu} = \sum_{n=2,4}^{\infty}\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)\frac{2^n q_{\mu_1}\cdots q_{\mu_n}}{(-q^2)^n}\sum_{j=q,g} 2\, C_{j,n}^{(1)}\, O_{j,V}^{\mu_1\cdots\mu_n}$$
$$+\sum_{n=2,4}^{\infty}\left(g_{\mu\mu_1}-\frac{q_\mu q_{\mu_1}}{q^2}\right)\left(g_{\nu\mu_2}-\frac{q_\nu q_{\mu_2}}{q^2}\right)\frac{2^n q_{\mu_3}\cdots q_{\mu_n}}{(-q^2)^{n-1}}\sum_{j=g,q} 2\, C_{j,n}^{(2)}\, O_{j,V}^{\mu_1\cdots\mu_n}$$

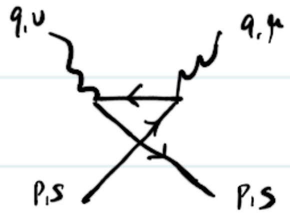
$\underline{C_{j,n}^{(1)}与\ C_{j,n}^{(2)}}$ 为相应矩程系数，它们的确定可以通过计算夸克矩阵元得到.

$\langle t_{\mu\nu}\rangle$：nonlocal 算符矩阵元

$\langle O_V\rangle$：local 算符矩阵元

- OPE for DIS tree-level (夸克矩阵元)





$$iM^{\mu\nu} = i\left\{\bar u(p,s)\,\gamma^\mu\,\frac{i(\not p+\not q)}{(p+q)^2}\,\gamma^\nu u(p,S) + \bar u(p,s)\gamma^\nu\,\frac{i(\not p-\not q)}{(p-q)^2}\,\gamma^\mu u(p,s)\right\} \quad (i\text{为约定})$$

$$(p+q)^2 = 2p\cdot q+q^2 = q^2\left[1+\frac{2p\cdot q}{q^2}\right] = q^2(1-\omega)$$

$$\bar u(p,s)\gamma^\mu(\not p+\not q)\gamma^\nu u(p,s) = \bar u(p,s)\left\{(p+q)^\mu\gamma^\nu + (p+q)^\nu\gamma^\mu - g^{\mu\nu}(\not p+\not q) + i\Sigma^{\mu\nu\alpha\lambda}(p+q)_\alpha\gamma_\lambda\gamma_5\right\}u(p,s)$$

利用 $\not p\, u(p,s)=0$ $\quad \bar u(p,s)\,\gamma_\lambda u(p,s) = 2p_\lambda$ $\quad \bar u(p,s)\gamma_\lambda\gamma_5 u(p,s) = 2h\,p_\lambda$ (h: helicity)

可得. $\bar u(p,s)\gamma^\mu(\not p+\not q)\gamma^\mu u(p,s) = 2(p+q)^\mu p^\nu + 2(p+q)^\nu p^\mu - 2g^{\mu\nu}p\cdot q$

代入并考虑 q 对称性:

$$iM^{\mu\nu} = -\frac{2}{q^2}\sum_{n=0}^{\infty}\omega^n\left[(p+q)^\mu p^\nu + (p+q)^\nu p^\mu - g^{\mu\nu}p\cdot q\right] + (\mu\leftrightarrow\nu,\, q\to -q,\, \omega\to-\omega)$$

$$= -\frac{4}{q^2}\sum_{n=0,2,4}^{\infty}\omega^n\, 2p^\mu p^\nu - \frac{4}{q^2}\sum_{n=1,3,5}^{\infty}\omega^n(q^\mu p^\nu + q^\nu p^\mu - g^{\mu\nu}p\cdot q)$$

$$= -\frac{8}{q^2}\sum_{n=0,2,4}^{\infty}\frac{2^n(p\cdot q)^n}{(-q^2)^n}\left(p^\mu - \frac{p\cdot q\, q^\mu}{q^2}\right)\left(p^\nu - \frac{p\cdot q\, q^\nu}{q^2}\right)$$

$$-\frac{4}{q^2}\sum_{n=1,3,5}^{\infty}\frac{2^n(p\cdot q)^{n+1}}{(-q^2)^n}\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right)$$

将 P·q 拆分开可得:

$$iM^{\mu\nu} = -\frac{8}{q^2}\sum_{n=0,2,4}^{\infty}2^n\frac{q^{\mu_3}\cdots q^{\mu_{n+2}}}{(-q^2)^n}\left(g^{\mu\mu_1} - \frac{q^\mu q^{\mu_1}}{q^2}\right)\left(g^{\nu\mu_2} - \frac{q^\nu q^{\mu_2}}{q^2}\right)\cdot\underline{p_{\mu_1}\cdots p_{\mu_{n+2}}}$$

$$-\frac{4}{q^2}\sum_{n=1,3,5}^{\infty}2^n\frac{q^{\mu_1}\cdots q^{\mu_{n+1}}}{(-q^2)^n}\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right)\left(\underline{p_{\mu_1}\cdots p_{\mu_{n+1}}}\right)$$

* 夸克 local 算符矩阵元: $\frac{1}{2}\langle q(p)|\bar\psi\gamma^\mu\psi|q(p)\rangle = \frac{1}{2}\bar u\gamma^\mu u = \frac{1}{4}\mathrm{tr}[\not p\gamma^\mu] = .\,p^\mu$

$$\frac{1}{2}\,\frac{i}{2}\langle q(p)|\bar\psi\gamma^\mu\overleftrightarrow{D}^\nu\psi|q(p)\rangle = \frac{1}{2}p^\nu\,\bar u\gamma^\mu u = p^\mu p^\nu$$

(*) 利用矩阵元形式:

$$
\begin{aligned}
iM^{\mu\nu} = & -\frac{8}{q^2}\sum_{n=0,2,4}^{\infty}\frac{2^n q^{\mu_3}\cdots q^{\mu_{n+2}}}{(-q^2)^n}\left(g^{\mu\mu_1}-\frac{q^\mu q^{\mu_1}}{q^2}\right)\left(g^{\nu\mu_2}-\frac{q^\nu q^{\mu_2}}{q^2}\right)\langle P|O_{q,\nu\mu_1\cdots\mu_{n+2}}|P\rangle \\
& -\frac{4}{q^2}\sum_{n=1,3,5}^{\infty}\frac{2^n q^{\mu_1}\cdots q^{\mu_{n+1}}}{(-q^2)^n}\left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)\langle P|O_{q,\nu\mu_1\cdots\mu_{n+1}}|P\rangle
\end{aligned}
$$

平移n的求和可得:

$$
\begin{aligned}
iM^{\mu\nu} = & \ 2\sum_{n=2,4,6}^{\infty}\frac{2^n q^{\mu_3}\cdots q^{\mu_n}}{(-q^2)^{n-1}}\left(g^{\mu\mu_1}-\frac{q^\mu q^{\mu_1}}{q^2}\right)\left(g^{\nu\mu_2}-\frac{q^\nu q^{\mu_2}}{q^2}\right) O_{q,\nu\mu_1\cdots\mu_n} \\
& +2\sum_{n=2,4,6}^{\infty}\frac{2^n q^{\mu_1}\cdots q^{\mu_n}}{(-q^2)^n}\left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right) O_{q,\nu,\mu_1\cdots\mu_n}
\end{aligned}
$$

对比可得:

$$C^{(1,2)}_{q,\nu,n} = 1+O(\alpha_s)$$

$$C^{(1,2)}_{g,\nu,n} = 0+O(\alpha_s)$$

Twist:

(1) From OPE

(2) From EFT

(3) From conformal theory

(1) From OPE:

in 4-fermion

$$T[W^+_\mu(x)\,W^-_\nu(y)] = \int\frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\,\frac{-i}{k^2-M_W^2} \curvearrowright \frac{i}{M_W^2-k^2} \times g_{\mu\nu}$$

$$= \int\frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\,\frac{i}{M_W^2}\left[1+\frac{k^2}{M_W^2}+\cdots\right]\times g_{\mu\nu}$$

$$= \frac{i}{M_W^2}\,\delta^4(x-y) + \frac{i}{M_W^2}(2\pi)^4\,\frac{\partial_x^2}{M_W^2}\,\delta^4(x-y)\times g_{\mu\nu}$$

$$= \frac{i}{M_W^2}g_{\mu\nu}\left[1+\frac{\partial^2}{M_W^2}+\cdots\right]\delta^4(x-y)$$

$$S\sim T\left[\frac{ig}{2\sqrt{2}}\int d^4x\;\bar\psi\gamma^\mu(1-\gamma_5)\psi W^+_\mu(x)\;\;\frac{ig}{2\sqrt{2}}\int d^4y\;\bar\psi\gamma^\nu(1-\gamma_5)\psi W^+_\nu(y)\right]$$

$$= T\Big[\frac{-ig^2}{8M_W^2}\int d^4x\,d^4y\;\bar\psi\gamma^\mu(1-\gamma_5)\psi(x)\;\bar\psi\gamma_\mu(1-\gamma_5)\psi(y)$$

$$\times\left\{1+\frac{\partial^2}{M_W^2}+\cdots\right\}\delta^4(x-y)\Big]$$

$$= \frac{-ig^2}{8M_W^2}\Big\{\int d^4x\;\bar\psi\gamma^\mu\underline{(1-\gamma_5)\psi(x)\times\bar\psi\gamma_\mu(1-\gamma_5)\psi(x)}$$

$$+\frac{1}{M_W^2}\int d^4x\;\partial^2\left(\bar\psi\gamma^\mu(1-\gamma_5)\psi(x)\right)\bar\psi\gamma_\mu(1-\gamma_5)\psi\Big\}$$

$$O(x) = \bar\psi\gamma^\mu(1-\gamma_5)\psi\;\bar\psi\gamma_\mu(1-\gamma_5)\psi$$

$$O'(x) = \partial^2(\bar\psi\gamma^\mu(1-\gamma_5)\psi)\;\bar\psi\gamma_\mu(1-\gamma_5)\psi$$

$O'(x)$ 相对于 $O(x)$ 量纲增加了 2，$\langle O'(x)\rangle / O(x) \sim k^2$.

考虑到系数 $\frac{1}{M_W^2}$，$\frac{\langle C'O'\rangle}{\langle CO\rangle} = \frac{k^2}{M_W^2} \ll 1$. 高量纲算符的贡献被压低

定域展开的重要性按照量纲即可判断.

(*) <u>DIS呢?</u> 按照量纲?

$\underline{\bar\psi\gamma^\mu\psi}$ $\quad$ $\underline{\bar\psi\gamma^\mu D^\nu\psi}$

$\langle\bar\psi\gamma^\mu\psi\rangle\sim p^\mu$ $\qquad$ $\underline{\langle\bar\psi\gamma^\mu D^\nu\psi\rangle\sim p^\mu p^\nu}$

不要忘记 p^μ 的分量 (p^+) 可以很大，<u>在Q量级</u>

所以即使考虑可能的系数压低，高量纲算符也不压低！那怎么办？

(*) 固定量纲：

例如 $\underline{\bar\psi\gamma^\mu D^\nu\psi}$ $\quad$ $\underline{\bar\psi D^\mu\psi}$

相应矩阵元 $\overset{-1}{\langle p(p)|}\,\overset{4}{\underline{\bar\psi\gamma^\mu D^\nu\psi}}\,\overset{-1}{|p(p)\rangle}\sim p^\mu p^\nu$

$\underset{-1}{\langle p(p)|}\,\underset{4}{\underline{\bar\psi D^\mu\psi}}\,\underset{-1}{|p(p)\rangle}\sim m p^\mu$

取 p^+: $\langle\bar\psi D^\mu\psi\rangle\ll\langle\bar\psi\gamma^\mu D^\nu\psi\rangle$

量纲相同，<u>指标越多</u> 贡献越大

(*) 固定指标：

例如： $\underset{4}{\underline{\bar\psi\gamma^\mu D^\nu\psi}}$ $\quad$ $\underset{5}{\underline{\bar\psi D^\mu D^\nu\psi}}$

$\langle\bar\psi\gamma^\mu D^\nu\psi\rangle\sim p^\mu p^\nu$

$\langle\bar\psi D^\mu D^\nu\psi\rangle\sim m p^\mu p^\nu$ $\qquad$ 系数需要 $\frac{1}{Q}$

$\langle\bar\psi D^\mu D^\nu\psi\rangle\ll\langle\bar\psi\gamma^\mu D^\nu\psi\rangle$

指标相同，量纲越小，贡献越大

统一：定义 $\tau = d-n$.

From EFT:

p^+ 大分量

$\sim$

(*) 夸克场大小分量： $p^\mu \sim (p^+, p^-, p_\perp) \sim (Q, Q\lambda^2, \lambda Q)$ $\quad \not{p} = \frac{n_- p}{2}\not{n}_+ + \frac{n_+ p}{2}\not{n}_- + \not{p}_\perp$

$$T[\psi(x)\bar\psi(y)] = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{i\not{p}}{p^2}$$

$$\psi(x) = \left(\frac{\not{n}_+\not{n}_-}{4} + \frac{\not{n}_-\not{n}_+}{4}\right)\psi(x) = \psi_c(x) + \eta_c(x)$$

$$T\Big[\underbrace{\frac{\not{n}_+\not{n}_-}{4}\psi(x)}_{\psi_c}\,\underbrace{\bar\psi(y)\frac{\not{n}_-\not{n}_+}{4}}_{\bar\psi_c}\Big] = \int\frac{d^4p}{(2\pi)^4} e^{-ip(x-y)}\frac{i}{p^2}\underbrace{\frac{\not{n}_+\not{n}_-}{4}\not{p}\frac{\not{n}_-\not{n}_+}{4}}$$

$$\frac{\not{n}_+\not{n}_-}{4}\Big(\underset{\text{只有这项非零}}{\underline{\underline{\frac{n_-\cdot p}{2}}}}\not{n}_+ + \frac{n_+\cdot p}{2}\not{n}_- + \not{p}_\perp\Big)\frac{\not{n}_-\not{n}_+}{4}$$

$$\sim \frac{Q\,Q\lambda^2(Q\lambda)^2}{Q^2\lambda^2}\cdot\times 1 \Rightarrow \lambda^2$$

$\Rightarrow \psi_c \sim \lambda$ $\quad$ 大分量

$$T\Big[\frac{\not{n}_-\not{n}_+}{4}\psi(x)\bar\psi(y)\frac{\not{n}_+\not{n}_-}{4}\Big] = \int\frac{d^4p}{(2\pi)^4} e^{-ip(x-y)}\frac{i}{p^2}\frac{\not{n}_-\not{n}_+}{4}\underset{\frac{n_+\cdot p}{2}\not{n}_-}{\underset{=}{\not{p}}}\frac{\not{n}_- - \not{n}_-}{4}$$

$$\sim \frac{Q\,Q\lambda^2(Q\lambda)^2}{Q^2\lambda^2}\cdot\lambda^2 \sim \lambda^4$$

$\Rightarrow \eta_c \sim \lambda^2$ $\quad$ 小分量

也有人写γ^+

(*) $\bar\psi\frac{\not{n}_-}{2}\psi = (\bar\psi_c + \bar\eta_c)\frac{\not{n}_-}{2}(\psi_c + \eta_c) = \bar\psi_c\frac{\not{n}_-}{2}\psi_c$ $\quad$ 都是大分量. leading twist $\quad$ 2.

$\bar\psi \times 1\,\psi = (\bar\psi_c + \bar\eta_c)(\psi_c + \eta_c) = \bar\psi_c\eta_c + \bar\eta_c\psi_c$ $\quad$ 一个小分量 next to leading twist $\quad$ 3

$\bar\psi\frac{\not{n}_+}{2}\psi = \quad - \quad - \quad = \bar\eta_c\frac{\not{n}_+}{2}\eta_c$ $\quad$ 两个小分量 N^2L twist $\quad$ 4.

$\rightarrow$ $\bar\psi\gamma^\mu\psi$ 的领头项

胶子:

$$T[A_\mu(x)A_\nu(y)] = \int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\frac{-i}{p^2}\left(g_{\mu\nu}-(1-\alpha)\frac{p_\mu p_\nu}{p^2}\right)$$

$$T[n_-\cdot A(x)\, n_-\cdot A(y)] = \int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\frac{-i}{p^2}\left(0-(1-\alpha)\frac{n_-\cdot p\, n_-\cdot p}{p^2}\right)\sim\frac{\lambda^4}{\lambda^2}\cdot\frac{1}{\lambda^2}\sim\lambda^0$$

$n_-\cdot A(x)\sim\lambda^0\sim1$ leading

$$T[A_{\perp\mu}(x)\,A_{\perp\nu}(y)] = \int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\frac{-i}{p^2}\left(g_{\perp\mu\nu}+\cdots\right)\sim\frac{\lambda^4}{\lambda^2}\cdot1\sim\lambda^2$$

$A_\perp(x)\sim\lambda$ next-to-leading

$$T[n_+\cdot A(x)\, n_+\cdot A(y)] = \quad\cdots\quad\sim\frac{\lambda^4}{\lambda^2}\cdot\frac{\lambda^4}{\lambda^2}\sim\lambda^4$$

$n_+\cdot A\sim\lambda^2$ N²L

∴ 幂次与动量一样！ $\underline{\underline{n_-\cdot D \quad 领头项}}$

leading twist (t=2) 都是大分量

(*) local: $\bar\psi_c\frac{\gamma^-}{2}n_-\cdot D\,\psi_c = \underbrace{\frac{n_{-\mu}n_{-\nu}}{2}}_{\text{对称}}\overset{\text{对称}}{\underline{\underline{\bar\psi\gamma^\mu D^\nu\psi}}}$ leading twist $\underline{\underline{\text{moments}}}$

(*) nonlocal : $\bar\psi\frac{\gamma^-}{2}P\exp[\underline{ig_s\int ds\, n_-\cdot A(sn_-)}]\psi$ --- leading twist PDF

(*) 纯胶子: $F^{\mu\nu}$ $F_{\mu\nu}$ 最低量纲

$F^{++}=0$

$F^{+\perp}$ · 最大贡献

$F^{+\perp}$. $F^{+}{}_{\perp}$ leading twist.

Fierz transformation:

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$$(\Gamma_A)_{ij}\ (\Gamma_B)_{kl} = \sum_{CD} C_{ABCD}\ (\Gamma_C)_{il}\ (\Gamma_D)_{kj} \qquad (1)$$

Γ: I, γ_μ, $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$, $\gamma_\mu\gamma_5$, γ_5

可以证明：这些γ矩阵满足 $(\Gamma)^2 = \pm I$，且在求迹下相互正交

则 $tr[\Gamma_A \Gamma_B] = \underline{\pm 4}\,\delta_{AB}$

Fierz变换：在(1)式左右同乘上 $(\Gamma_C')_{li}$ $(\Gamma_D')_{jk}$，遍历不同γ即可得到。

下面以 $\Gamma_A = I$，$\Gamma_B = I$ 为例介绍：

$$\underline{(I)_{ij}\,(I)_{kl} = \sum_{CD} C_{CD}\,(\Gamma_C)_{il}\,(\Gamma_D)_{kj}}$$

① 左右同乘 $\underline{(I)_{li}\,(I)_{jk}}$:

$$左 = I_{li}\,I_{ij}\,I_{jk}\,I_{kl} = tr[I] = 4$$

$$右 = \sum_{CD} C_{CD}\ tr[\Gamma_C]\ tr[\Gamma_D] \overset{C,D=I}{=} C_{II} \times 16$$

$$\rightarrow\ C_{II} = \frac{1}{4}$$

② 左右同乘 $(\gamma_0)_{li}$ $(\gamma_0)_{jk}$:

$$左 = (\gamma_0)_{li}\,(I)_{ij}\,(\gamma_0)_{jk}\,(I)_{kl} = tr[\gamma_0\gamma_0] = 4$$

$$右 = \sum_{CD} C_{CD}\ tr[\Gamma_C\gamma_0]\ tr[\Gamma_D\gamma_0] = C_{\gamma_0\gamma_0} \times 16$$

$$C_{\gamma_0\gamma_0} = \frac{1}{4}$$

③ 左右同乘 γ_i γ^i. (注意. 无求和)

$$左 = (\gamma_i)_{li}\,(I)_{ij}\,(\gamma^i)_{jk}\,(I)_{kl} = tr[\gamma_i\gamma^i] = 4$$

$$右 = \sum_{CD} C_{CD}\ tr[\Gamma_C\gamma_i]\ tr[\Gamma_D\gamma^i] = C_{\gamma_i\gamma^i} \times 16$$

$$C_{\gamma_i \gamma^i} = \frac{1}{4}$$

⊗左右同乘 $\gamma_5 \quad \gamma_5$

左 $= (\gamma_5)_{li} (I)_{ij} (\gamma_5)_{jk} (I)_{kl} = tr[\gamma_5 \gamma_5] = 4$

右 $= \sum_{CD} C_{CD} \; tr[\Gamma_C \gamma_5] \, tr[\Gamma_D \gamma_5] = C_{\gamma_5 \gamma_5} \times 16$

$$C_{\gamma_5 \gamma_5} = \frac{1}{4}$$

⊗左右同乘 $(\gamma_0 \gamma_5)_{li} \quad (\gamma_0 \gamma_5)_{jk}$

左 $= (\gamma_0 \gamma_5)_{li} (I)_{ij} (\gamma_0 \gamma_5)_{jk} (I)_{kl} = tr[\gamma_0 \gamma_5 \gamma_0 \gamma_5] = -4$

右 $= \sum_{CD} C_{CD} \; tr[\Gamma_C \gamma_0 \gamma_5] \, tr[\Gamma_D \gamma_0 \gamma_5] = C_{\gamma_0 \gamma_5 ,\, \gamma_0 \gamma_5} \times 16$

$$C_{\gamma_0 \gamma_5 \; \gamma_0 \gamma_5} = -\frac{1}{4}$$

⊗左右同乘 $(\gamma_i \gamma_5)_{li} \quad (\gamma_0 \gamma_5)_{jk}$ （无求和）

左 $= (\gamma_i \gamma_5)_{li} (I)_{ij} (\gamma^i \gamma_5)_{jk} (I)_{kl} = tr[\gamma_i \gamma_5 \gamma^i \gamma_5] = -4$

右 $= \sum_{CD} C_{CD} \; tr[\Gamma_C \gamma_i \gamma_5] \, tr[\Gamma_D \gamma^i \gamma_5] = C_{\gamma_i \gamma_5 \; \gamma^i \gamma_5} \times 16$

$$C_{\gamma_i \gamma_5 \gamma^i \gamma_5} = -\frac{1}{4}$$

⊗左右乘 (σ_{ij}) - - -

所以 $(I)_{ij} (I)_{kl} = \frac{1}{4} I_{il} I_{kj} + \underline{\underline{\frac{1}{4} (\gamma_\mu)_{il} (\gamma^\mu)}}_{kj}$
$+ \frac{1}{4} (\gamma_5)_{il} (\gamma_5)_{kj} - \frac{1}{4} (\gamma_\mu \gamma_5)_{il} (\gamma^\mu \gamma_5)_{kj}$
$+ \frac{1}{8} (\sigma_{\mu\nu})_{il} (\sigma^{\mu\nu})_{kj}$ - - - ?

(*) 什么时候用 Fierz 变换？

$$\underline{\bar{\psi}\,\gamma_\mu \frac{i(\not p+\not q)}{(p+q)^2}\,\gamma_\nu}\,\psi \xRightarrow{Q(1)} \bar{\psi}\,(\quad)\,\psi$$

$$\bar{\psi}_i\, I_{ij}\, (\qquad)_{jk}\, (I)_{kl}\, \psi_l$$

$$\Rightarrow \frac{1}{4}\,\underline{\bar{\psi}_i (I)_{il} \psi_l}\ \vdots\ I_{kj}\,(\qquad)_{jk}$$

$$+\frac{1}{4}\,\bar{\psi}_i (\gamma_\mu)_{il} \psi_l\ \vdots\ (\gamma^\mu)_{kj}\,(\quad)_{jk}$$

$$+\frac{1}{4}\ \cdots$$

低能算符　　　　hard kernel

只需考虑 leading twist

(*) Factorization for DIS:

$$T^{\mu\nu}(q,p) = \frac{1}{4\pi}\int d^4x\, e^{iq\cdot x} \langle p|T[J^{\mu}(x)\, J^{\nu}(0)]|p\rangle$$

$$\underline{W^{\mu\nu}(q,p)} = 2\,\mathrm{Im}\, T^{\mu\nu}(q,p)$$

① leading order in perturbation theory:

$$T^{\mu\nu}(q,p) = \frac{1}{4\pi}\int d^4x\, e^{iq\cdot x}\langle p|T[\underline{\bar q(x)\gamma^\mu q(x)}\ \underline{\bar q(0)\gamma^\nu q(0)}]|p\rangle + \bar q(x)\text{-----}q(0)$$

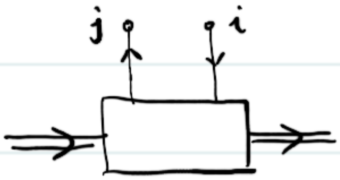
忽略 反夸克贡献

$$= \frac{1}{4\pi}\int d^4x\, e^{iq\cdot x}\underline{\int\frac{d^4 l}{(2\pi)^4}\, e^{-il\cdot(x-0)}}\ \langle p|T[\bar q(x)\underset{\underset{(I)_{ij}}{\equiv}}{\gamma^\mu}\ \underset{=}{\frac{i\not{l}}{l^2}}\ \underset{\underset{(I)_{kl}}{\equiv}}{\gamma^\nu} q(0)]|p\rangle.$$

① 直接做 Fierz 变换

② 或分析: 算符部分(动量空间): $\Gamma_{ji}(k,p) \equiv \int d^4x\, e^{-ik\cdot x}\langle p|T[\bar q_i(x)\, q_j(0)]|p\rangle$

对应:



$$\langle p|T[\bar q_i(x)\, q_j(0)]|p\rangle = \int\frac{d^4k}{(2\pi)^4}\, e^{ik\cdot x}\ \Gamma_{ji}(k,p)$$

② i, j 为旋量指标

先代入到 $W^{\mu\nu}$ 中:

$$W^{\mu\nu} = \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\left(\gamma^\mu(\not{k}+\not{q})\gamma^\nu\right)_{ij}\ \delta((k+q)^2)\ \ \Gamma_{ji}(k,p)$$

4维动量积分.

② $\Gamma_{ji}(k,p)$ 类似于 波函数. 其中 k 的量纲为:

$$k_\perp \sim \Lambda, \quad k^- \sim \frac{\Lambda^2}{p^+}. \quad k^+ \sim p^+ \qquad \Lambda \sim \Lambda_{QCD}$$

no hard interaction inside. 引入 $\hat{k}^\mu = (k^+, 0, 0, 0)$ $\quad k^\mu = \hat{k}^\mu (1 + O(\Lambda))$

③ 如果仅保留领头项. 相当于对 $\langle p|\bar{q}_i(x)\, q_j(0)|p\rangle$ 做 $\vec{x}_\perp$ 和 x^+ 的展开.

④ 定义: $\langle p|\bar{q}_i(x)\, q_j(0)|p\rangle = \langle p|\bar{q}_i(x^- n)\, q_j(0)|p\rangle + O(x_\perp, x^+)$

$$\int \frac{dx^-}{2\pi}\, e^{-ik^+x^-} \langle p|T[\bar{q}_i(x^-n)\, q_j(0)]|p\rangle = \frac{1}{2}(\gamma^-)_{ji}\, f_{q/p}(z). \qquad z = k^+/p^+$$

$$\frac{1}{2}\int \frac{dx^-}{(2\pi)}\, e^{-ik^+x^-} \langle p|T[\bar{q}(x^-n)\gamma^+ q(0)]|p\rangle = f_{q/p}(z)$$

$$\langle p|T[\bar{q}_i(x^-n)\, q_j(0)]|p\rangle = \frac{1}{2}\int dk^+\, e^{ik^+x^-} (\gamma^-)_{ji}\, f_{q/p}(z) \qquad z = k^+/p^+$$

重新代入至 $W^{\mu\nu}$

$$W^{\mu\nu} = \frac{1}{4}\int_0^{p^+} dk^+\, \underline{f_{q/p}(z)}\, \delta((\hat{k}+q)^2)\, \mathrm{Tr}[\underline{\gamma^\mu \overset{q^-\gamma^+}{\underline{(\hat{\not k}+\not q)}} \gamma^\nu} \gamma^-]$$

到这里和做Fierz变换一样.

$$(\hat{k}+q)^2 = 2(\hat{k}+q)^+ q^- = 0 \rightarrow \hat{k}^+ + \hat{q}^+ = 0 . \rightarrow (\hat{k}+q)^\mu = (0, q^-, 0, 0)$$

可以展开为:

$$F_1(x, Q^2) = \frac{1}{2}\int dz\, \delta(x-z)\, f_{q/p}(z) = \frac{1}{2} f_{q/p}(x)$$

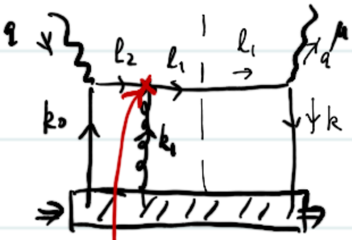
$$F_2(x, Q^2) = 2x F_1(x, Q^2) = x\, f_{q/p}(x)$$

部分子模型.

- DIS Factorization at 1-loop

交换 collinear gluon

等效于 expansion by region



$$\boxed{\langle p|\bar q(0)\gamma^+ W_c(\infty,0)\, q(0)|P\rangle}$$

$$iM^{(1)} = \frac{1}{4\pi}\int d^4x\, e^{iq\cdot x}\langle p|T[J^\mu(x)\, J^\nu(0)]|P\rangle$$

$$= \frac{1}{4\pi}\int d^4x\, e^{iq\cdot x}\int d^4z_1\langle p|T[\bar q\gamma^\mu q(x)\ \ \underline{\bar q\, i g_s \not{A}\, q(z_1)}\ \ \bar q\gamma^\nu q(0)]|P\rangle$$

$$= \frac{1}{4\pi}\int d^4x\int d^4z_1\int\frac{d^4l_1}{(2\pi)^4}\int\frac{d^4l_2}{(2\pi)^4}\, e^{iq\cdot x}\, e^{-il_1(x-z)}\, e^{-il_2(z-0)}$$

$$\langle p|\bar q(x)\gamma^\mu\frac{i\not{l}_1}{l_1^2}\, i g_s t^a\gamma_\mu A^{a\mu}(z_1)\frac{i\not{l}_2}{l_2^2}\gamma^\nu q(0)|P\rangle$$

矩阵元中 $\langle p|\bar q_i(x^- n)\ \ t^a A^{a\mu}(z_1^- n)\ \ q(0)|P\rangle$ 只有 x^- 分量，则 x^+ 与 $x_\perp$ 分量可以积分出来

相应的 l_1 与 l_2 也可以积分出 l^- 与 $l_\perp$ 分量.

* change of momentum: $l_1 - q \rightarrow k$ $\quad l_1 - l_2 \rightarrow k_1$

$l_1 = q + k$ $\quad l_2 = l_1 - k_1$

$$iM^{(1)} = \frac{1}{4\pi}\int d^4x\, d^4z_1\int\frac{d^4k}{(2\pi)^4}\frac{d^4k_1}{(2\pi)^4}\, e^{-ik\cdot x}\, e^{ik_1 z_1}\times$$

$$\langle p|\bar q(x^- n)\gamma^\mu\frac{i(\not{k}+\not{q})}{(k+q)^2}\, i g_s t^a\gamma_\mu A^{a\mu}(z_1 n)\frac{i(\not{k}+\not{q}-\not{k}_1)}{(k+q+k_1)^2}\times\gamma^\nu q(0)|P\rangle$$

$$= \frac{1}{4\pi}\int dx^- dz_1^-\int\frac{dk^+}{2\pi}\int\frac{dk_1^+}{2\pi}\, e^{-ik^+x^-}\, e^{ik_1^+z_1^-}\times$$

$$\times\langle p|\underset{(I)_{ij}}{\bar q(x^- n)}\gamma^\mu\frac{i(\hat{\not{k}}+\not{q})}{(\hat k+q)^2}\, i g_s t^a\gamma_\alpha A^{a\alpha}(z_1 n)\frac{i(\hat{\not{k}}+\not{q}-\hat{\not{k}}_1)}{(\hat k+q-\hat k_1)^2}\gamma^\nu \underset{(I)_{kl}}{q(0)}|P\rangle$$

Fierz 变换: $I_{ij}I_{kl} = \frac{1}{4}(\gamma_\beta)_{il}(\gamma^\beta)_{kj} + \frac{1}{4}\cdots -$

$$iM^{(1)} = \frac{1}{4\pi}\int dx^- dz_1^-\int\frac{dk^+}{2\pi}\int\frac{dk_1^+}{2\pi}\, e^{-ik^+x^-}\, e^{ik_1^+z_1^-}$$

$$\times\frac{1}{4}\langle p|\bar q(x^- n)\gamma_\beta\times t^a A^{a\alpha}(z_1^- n)\ q(0)|P(p)\rangle$$

注意颜色指标

$$\times\ \mathrm{tr}\left[\gamma^\beta\gamma^\mu\frac{i(\hat{\not{k}}+\not{q})}{(\hat k+q)^2}\, i g_s\gamma_\alpha\frac{i(\hat{\not{k}}+\not{q}-\hat{\not{k}}_1)}{(\hat k+q-\hat k_1)^2}\gamma^\nu\right]$$

$$(\hat{k}+q)^2 = 2(\hat{k}^+ + q^+)q^- = 0$$

$$\hat{k}^\mu = (k^+,0,0,0) \quad \hat{k}_1^\mu = (k_1^+,0,0,0)$$

取在壳

$$iM^{(1)} = \frac{1}{4\pi}\int dx^- dz_1^- \int\frac{dk^+}{(2\pi)}\int\frac{dk_1^+}{2\pi}\, e^{-ik^+x^-}\, e^{ik_1^+z_1^-} \qquad (2\pi)\delta((\hat{k}+q)^2)$$

$$\times \frac{1}{4}\langle p(p)|\bar{q}(x^-n)\,\gamma^+\, t^a A^{a+}(z_1^-n)\, q(0)|p(p)\rangle \qquad ⊗$$

$$\times\, tr[\gamma^-\gamma^\mu\ iq^-\gamma^+\ ig_s\underbrace{\gamma^-\ \frac{iq^-\gamma^+}{-2q^-\cdot\hat{k}_1^+ + i\varepsilon}}\ \gamma^\nu] \qquad ●$$

(*) $\rightarrow \underset{\uparrow}{\frac{\gamma^-\gamma^+}{2}} \times \frac{i}{-\hat{k}_1^+ + i\varepsilon}$

投影算符 可以拿走

(*) $$i\int\frac{dk_1^+}{2\pi}\frac{1}{-\hat{k}_1^+ + i\varepsilon}e^{ik_1^+z_1^-} = i\int\frac{dk_1^+}{2\pi}\frac{1}{\hat{k}_1^+ + i\varepsilon}e^{-ik_1^+z_1^-} = \underline{\theta(z_1^-)}$$

(*) $$iM^{(1)} = \frac{1}{4\pi}\times\frac{1}{4} \qquad \int\frac{dk^+}{2\pi}\, e^{-ik^+x^-}\ (2\pi)\delta((\hat{k}+q)^2) \qquad \rightarrow$$

$$\times\int dx^- e^{-ik^+x^-}\int d\underline{z_1^-}\,\langle p(p)|\bar{q}(x^-n)\ \not{n} \times ig_s t^a\, n\cdot A^a(z_1^-n)\cdot q(0)|p(p)\rangle \quad \theta(z_1^-) \qquad \rightarrow$$

$\times\, tr[\gamma^-\gamma^\mu\ iq^-\gamma^+\ \gamma^\nu]$ --- tree-level kernel

中间行可写为：

$$\int dx^- e^{-ik^+x^-} \times \langle p(p)|\bar{q}(x^-n)\ \not{n}\ W_c(\infty,0)\ q(0)|p(p)\rangle$$ 展开至一阶 ⊙

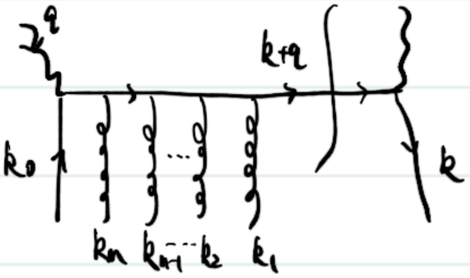
$$W_c = P\exp[ig_s\int_0^\infty dz_1^-\ n\cdot A^a(z_1^-n)\,t^a]$$

$\hookrightarrow$ 此为PDF定义：即 $f_{q/p}(z)$ - $z = \frac{k^+}{p^+}$

(*) 则+ $iM^{(1)} = \frac{1}{4\pi} \times \frac{1}{4} \int \frac{dk^+}{2\pi} e^{-ik^+x^-} (2\pi)\,\delta((\hat{k}+q)^2)\ f_{q/p}(z) \cdot tr[\gamma^-\gamma^\mu i\not{q}^-\gamma^+\gamma^\nu]$

这就是单圈阶因子化. 只考虑了一个图.

(4) N gluon



$$W^{\mu\nu} = \frac{1}{4}\int \frac{dk^+}{2\pi} Tr[\gamma^\mu \underline{i\gamma\cdot(\hat{k}+q)} \gamma^\nu \gamma^-]\ \delta((\hat{k}+q)^2)$$

$$\times \frac{1}{2}\int \prod_{i=1}^{n} \frac{dk_i}{2\pi} \frac{ig_s}{-k_1^+ + i\varepsilon} \frac{ig_s}{-k_1^+ - k_2^+ + i\varepsilon} \cdots \frac{ig_s}{-k_1^+ - k_2^+ \cdots - k_m^+ + i\varepsilon}$$

$$\times \int dx^- e^{-ix^-k^+} \prod_{i=1}^{n} dz_i^- \underline{e^{iz_ik_i^+}} \qquad ?$$

$$\times \langle p|\bar{q}(x^-n)\gamma^+ A^+(z_1n)A^+(z_2n)\cdots A^+(z_nn)\ q(0)|p(p)\rangle$$

可以定义 $\tilde{k}_i^+ = k_1^+ + k_2^+ + \cdots + k_i^+$

$$W_c(\infty, x) = P\exp[ig_s\int_0^\infty dz^- A^+(z^-n+x)$$

$$= 1 + \sum_i (ig_s)^i \int \prod_{j=1}^{i} dz_j^- A^+(z_1n+x) A^+(z_2n+x) \cdots A^+(z_jn+x)$$

$$\times \theta(z_1^- - z_2) \theta(z_2^- - z_3^-) \cdots \theta(z_{j-1}^- - z_j) \theta(z_j)$$

利用 $\theta(\lambda) = i\int\frac{dw}{2\pi} e^{-iw\lambda} \frac{1}{w+i\varepsilon}$

代入得:

$$W_c(\infty, x) = 1 + \sum_{i=1}^{\infty} (ig_s)^i \int \prod_{j=1}^{i} \frac{dk_j^+}{2\pi}$$

$$\times \frac{i}{-k_1^+ + i\varepsilon} \frac{i}{-\tilde{k}_2^+ + i\varepsilon} \cdots \frac{i}{-k_j^+ + i\varepsilon}$$

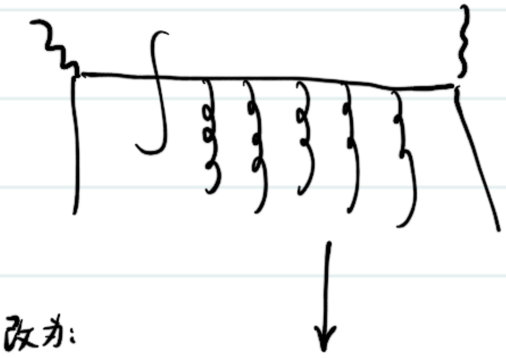
$$\times \int \prod_{j=1}^{i} dz_j^- \, e^{ik_j^+ z_j^+} A^+(z_1^- n + x) A^+(z_2^- n + x) \cdots A^+(z_j^- n + x)$$

代入 $W^{\mu\nu}$ 得

$$W^{\mu\nu} = \frac{1}{4} \int \frac{dk^+}{2\pi} \operatorname{Tr}\left[\gamma^\mu \gamma(\hat{k}+q) \gamma^\nu \gamma^-\right] \delta((\hat{k}+q)^2)$$

$$\times \frac{1}{2} \int dx^- e^{-ik^+x^-} \langle P(p)| \bar{q} \gamma^+ W_c(\infty, \overset{\Lsh}{0}) q(0) |P(p)\rangle$$

加入另一半:



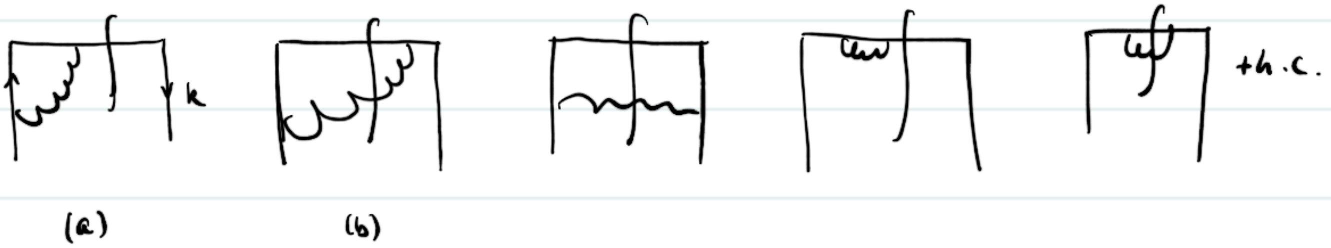
其中的PDF改为:

$$\int dx^- e^{-ik^+x^-} \langle P(p)| \underline{\underline{\bar{q}(x^- n) W_c^+(\infty, x^- n)}} \gamma^+ W_c(\infty, 0) q(0) | P(p)\rangle$$

$$W_c(\infty, x) = P\exp\left[ig_s \int_0^{\infty} ds \, n \cdot A(x+sn)\right]$$

Factorization for DIS in momentum space:

(*) integration by region



Consider Fig (a): 取 $u\bar{u} = \gamma^-$ <u>非极化</u>

$$W_a^{\mu\nu} = \frac{1}{4}\int \frac{d\hat{k}^+}{2\pi}\,\delta((\hat{k}+q)^2|\int\frac{d^4k_1}{(2\pi)^4}\,\frac{-i}{k_1^2+i\varepsilon}\,\mathrm{Tr}\Big[\gamma^\mu\gamma\cdot(\hat{k}+q)\,(ig_s\gamma^\rho t^a)\,\frac{i(\hat{\not k}+\not q-\not k_1)}{(\hat{k}+q-k_1)^2+i\varepsilon}\gamma^\nu$$

$$\times\frac{i(\hat{\not k}-\not k)}{(\hat{k}-q)^2+i\varepsilon}\,(ig_s\gamma_\rho t^a)\,\underset{\longrightarrow\text{来自于 } u\bar{u}}{\gamma^-}\Big]$$

Consider the region: $k_1^\mu \sim (1, \lambda^2, \lambda, \lambda)\cdot Q$ $Q \sim \hat{k}^+$.

expand the amplitude in λ

$W_a^{\mu\nu} = \cdots$