

红外发散 KLN 定理

参考文献: T. Muta. Foundations of quantum chromodynamics

李政道. 粒子物理与场论

G. Stermen. An introduction to quantum field theory.

1. 红外发散 黄潜, 王伟 李政道

紫外发散: $k \rightarrow \infty$ 重整化.

红外发散: • 无质量场 e.g. 光子, 胶子, ...

• 无质量场 $\leftrightarrow$ 无质量场相互作用

light quark $\leftrightarrow$ gluon

gluon $\leftrightarrow$ gluon

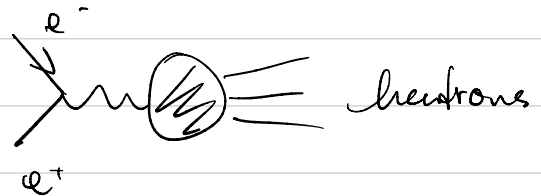
mass singularity / collinear singularity.

Infrared div. $\left\{ \begin{array}{l} \text{infrared div. (soft div.) } \lambda \rightarrow 0 \quad \lambda \text{ (gauge field mass)} \\ \text{mass singularity (coll. div.) } \lambda \rightarrow 0 \& m \rightarrow 0 \end{array} \right.$

2. example. $e^+e^- \rightarrow \text{hadrons}$

2.1 $e^+e^- \rightarrow \text{hadrons}$.

$$\sigma = \frac{4\pi\alpha^2}{3s} \text{ WFS}$$



$$\sigma(e^+e^- \rightarrow \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s}$$

$$R \equiv \frac{\sigma}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} = \omega(s) = N_c \sum_{i=1}^{N_f} Q_i^2 (1 + O(\alpha_s) + \dots)$$

BES II 最重要成果之一.

• Validation of the color:

• determine the strong coupling.

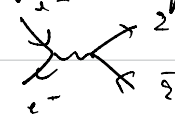
2.2. One-loop QCD corrections

$$\sigma = \underbrace{\sigma_B^2}_{\text{Born}} + \underbrace{\sigma_V}_{\text{Virt}} + \underbrace{\sigma_R}_{\text{Re.}}$$

$$= \sigma_B + \hat{\sigma}_V + \sigma_R$$

$$\hat{\sigma}_V = (2^2 - 1) \sigma_B + \sigma_V$$

diagrams for amplitude:

Born:  $\mathcal{O}(g_s^2)$

tree:  $\mathcal{O}(g_s^2)$ real:  $\mathcal{O}(g_s^4)$  $\mathcal{O}(g_s^4)$

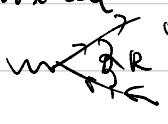
for cross section:

$$| \langle \text{Born} \rangle |^2 + [(\langle \text{tree} \rangle) (\langle \text{Born} \rangle)^* + \text{c.c.}]$$

$$+ | \langle \text{tree} \rangle + \langle \text{tree} \rangle |^2 + \dots$$

$$\alpha_s \equiv \frac{g_s^2}{4\pi}$$

• Virtual corrections:

 $(-ig)^2 \int \frac{d^4 k}{(2\pi)^4} \bar{u}(R_1) \gamma^\mu \frac{i}{k_1 - k - m} \gamma^\rho \frac{i}{k_2 - k - m} \gamma_\rho u(R_2) \frac{-i}{k^2 - k^2 + i\epsilon}$

let $k \rightarrow 0$

integral: $\int \frac{d^4 k}{(2\pi)^4} \frac{N}{k^2 (k^2 - 2R_1 \cdot k) (k^2 + 2R_2 \cdot k)} \frac{1}{k^2 - k^2 + i\epsilon}$

$$N \propto \gamma^\mu (k_1 - k + m) \gamma^\rho (-k_2 - k + m) \gamma_\rho$$

$$\frac{1}{k^2 + i\epsilon} = \frac{1}{2\omega} \left(\frac{1}{k_0 - \omega + i\epsilon} - \frac{1}{k_0 + \omega - i\epsilon} \right) \quad \omega \equiv |\vec{k}|$$

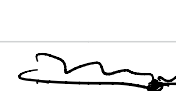
$$\int \frac{d^4 k}{(2\pi)^4} \frac{1}{2\omega} \left(\frac{1}{R_0 - \omega + i\epsilon} - \frac{1}{k_0 + \omega - i\epsilon} \right) \frac{N}{(2k_1 \cdot k)(2R_2 \cdot k)}$$

$$\propto \frac{1}{(k_1^0 k_0 - \vec{k}_1 \cdot \vec{k}) (R_2^0 k_0 - \vec{R}_2 \cdot \vec{k})} \quad \text{center of mass frame.}$$

$$\propto \int \frac{d\omega d\cos\theta d\phi}{\omega (R_1^0 - |\vec{R}_1| \cos\theta) (R_2^0 + |\vec{R}_2| \cos\theta)}$$

$$k_1^0 = \sqrt{|\vec{k}_1|^2 + m^2} \quad k_2^0 = \sqrt{|\vec{k}_2|^2 + m^2}$$

two singularities:

- $\alpha \rightarrow 0$ Soft
- $m \rightarrow 0$ $\theta \rightarrow 0, \pi$  $\vec{k}$ collinear

2.2 Cancellation of the IR div.

dim. reg.

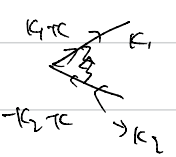
$$\sigma_V = \frac{1}{8s} \int \frac{d^3 k_1}{(2\pi)^3 k_{10}} \frac{d^3 k_2}{(2\pi)^3 k_{20}} (2\pi)^4 (k_1 + k_2 - p_1 - p_2) F_V$$

$$F_V = \sum_{\text{pol.}} |iM|^2 = \sum_{\text{pol.}} \left[\bar{u}(p_1) (-ie\gamma_\mu) u(p_1) \frac{-i}{s} \bar{u}(k_1) \gamma^\mu v(k_2) \right]^2$$

$$= \frac{e^2 e^2}{s^2} \text{Tr}[k_2 \gamma^\mu k_1 \gamma^\mu] \text{Tr}[k_1 \gamma^\mu k_2 \gamma^\mu] + \text{c.c.}$$



$$= \mu^{2\epsilon} \int \frac{d^4 k}{(2\pi)^4} \bar{u}(k_1) (-ig_s \gamma^\mu) \frac{i}{k-k_1} \gamma^\mu \frac{i}{-k-k_2} (-ig_s \gamma^\nu) u(k_2) \frac{-i}{k^2}$$

$$= i g_s^2 C_F \mu^{2\epsilon} \int \frac{d^4 k}{(2\pi)^4} \bar{u}(k_1) \gamma^\mu \frac{k-k_1}{(k-k_1)^2} \gamma^\mu \frac{k+k_2}{(k+k_2)^2} u(k_2) \frac{1}{k^2}$$


$$\mu^{2\epsilon} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 (k_1-k)^2 (k+k_2)^2}$$

$$= \mu^{2\epsilon} \int \frac{d^4 k}{(2\pi)^4} \Gamma(3) \int dx_1 dx_2 \delta(1-x_1-x_2) \frac{1}{(k^2 - 2x_1 k \cdot p_1 + 2x_2 k \cdot p_2)^3}$$

$$= \mu^{2\epsilon} \Gamma(3) \int \frac{d^4 k}{(2\pi)^4} \int dx_1 dx_2 \delta(1-x_1-x_2) \frac{1}{(k^2 - 2x_1 k \cdot p_1 + 2x_2 k \cdot p_2)^3}$$

$$= \mu^{2\epsilon} \int dx_1 dx_2 \delta(1-x_1-x_2) \frac{-i}{(4\pi)^2} \Gamma(3-\epsilon) \frac{1}{[(x_1 p_1 + x_2 p_2)^2]^{3-\epsilon}}$$

$$= \frac{-i}{(4\pi)^2} (\mu^2)^{\epsilon} \int dx_1 dx_2 \delta(1-x_1-x_2) \Gamma(1+\epsilon) \frac{1}{(-2x_1 x_2 p_1 \cdot p_2)^{1+\epsilon}}$$

$$= \frac{-i}{(4\pi)^2} (\mu^2)^{\epsilon} \frac{1}{(-s)^{1+\epsilon}} \int_0^1 dx \frac{1}{x^{1+\epsilon} (1-x)^{1+\epsilon}}$$

$$= \frac{-i}{(4\pi)^2} (\mu^2)^{\epsilon} \frac{1}{(-s)^{1+\epsilon}} \frac{\Gamma(-\epsilon)}{\Gamma(-2\epsilon)}$$

The total result reads $\Gamma^\mu = g^\mu \frac{g_s^2}{8\pi^2} (\frac{4\pi\mu^2}{-s})^{\epsilon} \Gamma(1+\epsilon) B(\epsilon, 2-\epsilon) (\frac{1}{\epsilon_{UV}} - \frac{2}{\epsilon_{IR}} - 2)$

$$B(p, q) = \frac{\Gamma(p) \Gamma(q)}{\Gamma(p+q)} \quad \text{Euler's Beta function}$$

The gluon self energy:



$$-i\Sigma(p) = \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \gamma^\mu \frac{i}{\not{p}-\not{k}-i\epsilon} \gamma_\mu \frac{-i}{n^2} (-ig_s)^2 C_F$$

$$= -\mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{\gamma^\mu (\not{p}-\not{k}) \gamma_\mu}{(p-k)^2 k^2} g_s^2 C_F$$

$$= -(2-d) \mu^{2\epsilon} g_s^2 C_F \int \frac{d^d k}{(2\pi)^d} (\not{p}-\not{k}) \frac{1}{k^2 (p-k)^2}$$

$$= (d-2) \mu^{2\epsilon} g_s^2 C_F \int \frac{d^d k}{(2\pi)^d} \int_0^1 dx \frac{\not{p}-\not{k}}{(k^2 - 2xp \cdot k)^2}$$

$$= (d-2) \mu^{2\epsilon} g_s^2 C_F \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{\not{p}(1-x)}{k^4}$$

$$\frac{\partial \Sigma}{\partial \not{p}} \Big|_{p^2=0} = i(d-2) \mu^{2\epsilon} g_s^2 C_F \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} (1-x) \frac{1}{k^4}$$



$$-i\Sigma(p) = \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \gamma^\mu \frac{i}{\not{p}-\not{k}} \gamma_\mu \frac{-i}{(p-k)^2} (-ig_s)^2 C_F$$

$$= -\mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \gamma^\mu \not{k} \gamma_\mu \frac{1}{k^2 (p-k)^2} g_s^2 C_F$$

$$= -(2-d) \mu^{2\epsilon} g_s^2 C_F \int \frac{d^d k}{(2\pi)^d} \not{k} \frac{1}{k^2 (p-k)^2}$$

$$= (d-2) \mu^{2\epsilon} g_s^2 C_F \int \frac{d^d k}{(2\pi)^d} \int_0^1 dx \frac{\not{k}}{(k^2 - 2xp \cdot k)^2}$$

$$= (d-2) \mu^{2\epsilon} g_s^2 C_F \int \frac{d^d k}{(2\pi)^d} \int_0^1 dx \frac{\not{p}}{k^4}$$

$$\frac{\partial \Sigma(p)}{\partial \not{p}} \Big|_{p^2=0} = i(d-2) \mu^{2\epsilon} g_s^2 C_F \int \frac{d^d k}{(2\pi)^d} \int_0^1 dx \frac{\not{p}}{k^4}$$

$$\mathcal{Z}_1 = 1 - \frac{\partial \Sigma}{\partial \not{p}} \Big|_{p^2=0} = i(2-d) \mu^{2\epsilon} g_s^2 C_F \frac{1}{2} \cdot \frac{i}{(4\pi)^2} \left(\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} \right)$$

$$\mathcal{Z}_2 = 1 + \frac{\partial \Sigma}{\partial \not{p}} \Big|_{p^2=0} = 1 - \frac{g_s^2 C_F}{(4\pi)^2} \left(\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} \right)$$

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2)^2}$$

$$\frac{(i)^2 i}{(4\pi)^2} \Gamma(2-\epsilon) \frac{1}{(m^2)^\epsilon}$$

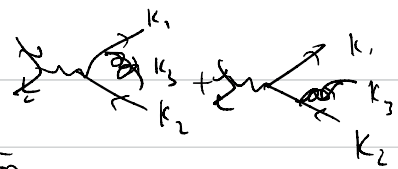
$$\frac{i}{(4\pi)^2} \Gamma(\epsilon)$$

Note: the UV & IR cancel out

Total result: $\widetilde{G}_\mu = A_\mu \otimes B$ with

$$A_\mu = \frac{g_s}{\pi} C_F \left(\frac{\phi \pi \mu^2}{s} \right)^2 \frac{\cos \pi \epsilon}{\Gamma(1-\epsilon)} \left(-\frac{1}{\epsilon^2} - \frac{3}{2\epsilon} - \gamma_E + \dots \right)$$

• real corrections:



$$\sigma_R = \frac{1}{8s} \int \frac{d^4 k_i}{(2\pi)^4} (2\pi)^4 \delta^4 \left(\sum_{i=1}^3 k_i - p_1 - p_2 \right) \bar{F}_R$$

$$\bar{F}_R = \hbar^2 M^2$$

$$\begin{aligned} iM &= \bar{u}(p_2) (-i \not{e} \gamma^\rho) u(p_1) \frac{-i}{s} \bar{u}(k_1) (-i \not{e} \not{q}) \left[\not{e}^*(k_3) \frac{1}{k_1 + k_2} \gamma^\rho \right. \\ &\quad \left. + \gamma^\rho \frac{1}{-k_2 - k_3} \not{e}^*(k_3) \right] u(k_2) \\ &= \frac{1}{s} e_1 e_2 e^2 g_s \bar{u}(p_2) \gamma^\rho u(p_1) \bar{u}(k_1) \left[\gamma^\mu \frac{1}{k_1 + k_2} \gamma^\rho \right. \\ &\quad \left. + \gamma^\rho \frac{1}{-k_2 - k_3} \gamma^\mu \right] u(k_2) \end{aligned}$$

$$|iM|^2 = - \frac{e_1^2 e_2^2 e^4 g_s^2 C_F}{s^2} \text{Tr}[\not{p}_2 \not{p}_1 \not{p}_1 \not{p}_2] \frac{s^{\mu\rho} \epsilon_\mu(k_3)}{s^{\mu\rho} \epsilon_\mu(k_3)} + \text{Tr} \left[\frac{L^{\rho\rho'}}{K_1} S_\rho K_2 S_{\rho'} \right] (g_{\mu}{}^{\mu'})$$

$$L^{\rho\rho'} = \text{Tr}[\not{K}_2 \gamma^\rho \not{K}_1 \gamma^{\rho'}] = 4 \left(K_1^\rho K_2^{\rho'} + K_2^\rho K_1^{\rho'} - g^{\rho\rho'} \frac{s}{2} \right)$$

$$\int d\pi_3 G^{\rho\rho'} \equiv I^{\rho\rho'}$$

Ward identity: $g_\rho I^{\rho\rho'} = 0 \Rightarrow I^{\rho\rho'} = I(g^2) \left(\frac{2^\rho 2^{\rho'}}{g^2} - g^{\rho\rho'} \right)$

$$I(g^2) = \frac{1}{1-d} g_{\rho\rho'} I^{\rho\rho'}$$

$$L^{\rho\rho'} I_{\rho\rho} = 2I(g^2) g^2 (d-2)$$

$$= \frac{2(d-2)g^2}{1-d} g_{\rho\rho'} I^{\rho\rho'} = \frac{2(d-2)g^2}{1-d} \int d\pi_3 g_{\rho\rho'} G^{\rho\rho'}$$

$$g_{\rho\rho'} G^{\rho\rho'} = \frac{8(1-\epsilon)(x_1^2 + x_2^2 - \epsilon x_3^2)}{(1-x_1)(1-x_2)}$$

$$x_i = \frac{2k_i \cdot q}{g^2}$$

$$x_1 + x_2 + x_3 = 2$$

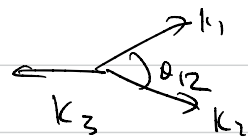
the phase space:

$$\int d\bar{u}_3 = \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \frac{d^d k_3}{(2\pi)^d} (2\pi) \delta^d(k_1 + k_2 + k_3 - q)$$

C.O.M : $q = (q^0, 0)$ $x_i = \frac{2k_i^0}{q^0}$, $dk_i^0 = \frac{q^0}{2} dx_i$

$$\int d\bar{u}_3 = \left(\frac{q^0}{4\pi}\right)^{2d-3} \frac{1}{(q^0)^3} \int dx_1 d\Omega_{d-1} \int dx_2 d\Omega_{d-1} \frac{(x_1 x_2)^{d-2}}{x_1 x_2 x_3} \delta(x_1 + x_2 + x_3 - 2)$$

$$= \left(\frac{q^0}{4\pi}\right)^{2d-3} \frac{\Omega_{d-1} \Omega_{d-2}}{(q^0)^3} \int dx_1 \int dx_2 (x_1 x_2)^{d-3}$$



$$\int_{-1}^1 d\cos\theta \left(\sin^2\theta\right)^{\frac{d-4}{2}} \frac{1}{x_3} \delta(x_1 + x_2 + x_3 - 2)$$

$$\int_{-1}^1 d\cos\theta \delta\left(x_1 + x_2 + \sqrt{x_1^2 + x_2^2 + 2x_1 x_2 \cos\theta} - 2\right)$$

$$= \frac{1}{2x_1 x_2} = \frac{x_3}{x_1 x_2}$$

$$2\sqrt{x_1^2 + x_2^2 + 2x_1 x_2 \cos\theta}$$

$$= \left(\frac{q^0}{4\pi}\right)^{2d-3} \frac{\Omega_{d-1} \Omega_{d-2}}{(q^0)^3} \int dx_1 \int dx_2 (x_1 x_2)^{d-4}$$

$$\int_{-1}^1 d\cos\theta = \frac{x_3^2 - x_1^2 - x_2^2}{2x_1 x_2} = \frac{2 + x_1(x_2 - 2) - 2x_2}{x_1 x_2} \quad \perp$$

$$= \left(\frac{q^0}{4\pi}\right)^{2d-3} \frac{\Omega_{d-1} \Omega_{d-2}}{(q^0)^3} \int_0^1 dx_1 \int_{1-x_1}^1 dx_2$$

$$\left[4(1-x_1)(1-x_2)(x_1+x_2-1) \right]^{\frac{d-4}{2}}$$

$$= \frac{S\left(\frac{q^0}{4\pi}\right)^{-2\epsilon}}{(2\pi)^3 \Gamma(2-2\epsilon)} \int_0^1 dx_1 \int_{1-x_1}^1 dx_2 \frac{1}{[(1-x_1)(1-x_2)(x_1+x_2-1)]^\epsilon}$$

$$\sigma_R = (\sum_i Q_i^2) \alpha_s^2 \Gamma \frac{2}{5} \left(\frac{4\pi\hbar^2}{s} \right)^{2\epsilon} \frac{(1-\epsilon)^2}{(3-2\epsilon)\Gamma(1-2\epsilon)} \quad \leftarrow \text{(from Master)}$$

$$K = \left(\frac{p}{\epsilon^2} - \frac{12}{\epsilon} + 10 - 4\epsilon \right) B(1-\epsilon, 2-2\epsilon) B(1-\epsilon, 1-\epsilon) + \dots$$

$$\sigma_R = A_R \sigma_B, \quad \sigma_R = \frac{4\pi\alpha^2}{3s} (\sum Q_i^2) \left(\frac{4\pi}{s} \right)^\epsilon \frac{3(1-\epsilon)\Gamma(2-\epsilon)}{(3-2\epsilon)\Gamma(1-2\epsilon)}$$

$$A_R = \frac{d_s}{\pi} \Gamma \left(\frac{4\pi\hbar^2}{s} \right)^\epsilon \frac{\cos\pi\epsilon}{\Gamma(1-\epsilon)} \left(\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} + \frac{19}{4} + \dots \right)$$

$$\sigma = (1 + A_v + A_R) \sigma_B = \left(1 + \frac{3}{4} \Gamma \frac{\alpha_s}{\pi} \right) \sigma_B$$

IR div. cancel out!

Bloch-Nordsieck: soft divergences cancel out after summation

3. *Kinoshita - Lee - Nauenberg theorem.* *final state*
 transition rates are free of IR (soft & collinear)
 i.e. if the summation over the initial and final degenerate states is carried out.

proof: old fashioned (time ordered) P.T.

- S-matrix.
 $A = H_0 + g\hat{H}$
 $i\frac{\partial}{\partial t} |\psi(t)\rangle = g\hat{H}_1(t) |\psi(t)\rangle$
 $\hat{H}_1(t) = e^{iH_0 t} \hat{H}_1 e^{-iH_0 t}$
 time evolution operator: $U(t, t') = \mathcal{T} \exp \left[-i \int_{t'}^t \hat{H}_1(t'') dt'' \right]$
 S-matrix operator: $S = \lim_{\substack{t' \rightarrow -\infty \\ t \rightarrow +\infty}} U(t, t')$

$$S = U(0, \infty)^\dagger U(0, -\infty)$$

- transition rate for $a \rightarrow b$

$$\begin{aligned} |\langle b | S | a \rangle|^2 &= \sum_{ij} \langle b | U(0, \infty)^\dagger | i \rangle \langle i | U(0, -\infty) | a \rangle \\ &\quad (\langle b | U(0, \infty)^\dagger | j \rangle)^* (\langle j | U(0, -\infty) | a \rangle)^* \\ &= \sum_{ij} (R_{b,i}^\dagger)^* R_{a,i,j} \end{aligned}$$

$$\text{with } R_{a,i,j}^\dagger = \langle i | U(0, \pm\infty) | a \rangle^* \langle j | U(0, \pm\infty) | a \rangle$$

• perturbation theory:

$$\langle i | U(0, \pm\infty) | j \rangle = \delta_{ij} + \langle i | -i \int_{\pm\infty}^0 d\lambda g \hat{H}_1(\lambda) | j \rangle$$

$$= \delta_{ij} + \langle i | -i \int_{\pm\infty}^0 d\lambda e^{iH_0\lambda} H_1 e^{-iH_0\lambda} | j \rangle$$

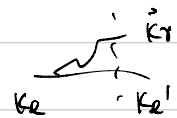
$$= \delta_{ij} - i \int_{\pm\infty}^0 d\lambda e^{i(E_i - E_j)\lambda} g H_{1,ij}$$

$$= \delta_{ij} - \frac{g H_{1,ij}}{E_i - E_j \pm i\epsilon} + \dots \quad H_{1,ij} = \langle i | H_1 | j \rangle$$

$$R_{aij}^{\pm} = \delta_{ia} \delta_{ja} - \frac{g H_{1,ia}}{E_i - E_a \pm i\epsilon} \delta_{ja} - \frac{g H_{1,ja}}{E_j - E_a \pm i\epsilon} \delta_{ia} + O(g^2)$$

$H_{1,ia}$: transition from $|a\rangle$ to $|i\rangle$

e.g. in QED: $|e\rangle \rightarrow |e\gamma\rangle$



div. emerge when $E_i = E_a$ i.e. energy degenerate.

Case 1: γ is soft $E_e \approx E_e + E_\gamma$ when $E_\gamma \rightarrow 0$

Case 2: e/γ collinear $E_a = |\vec{k}_e|$

$$E_i = |\vec{k}_e| + |\vec{k}_{e'}|$$

$$E_a - E_i = \sqrt{(\vec{k}_e + \vec{k}_{e'})^2} - (|\vec{k}_e| + |\vec{k}_{e'}|)$$

$$= \sqrt{k_e^2 + k_{e'}^2 + 2k_e k_{e'} \cos\theta} - (k_e + k_{e'})$$

≥ 0 when $\cos\theta = 1$

• Summation over the degenerate states:

$D(E)$: the set of states with energy E , (E')

$$\sum_{a \in D(E)} \sum_{b \in D(E')} |\langle b | S | a \rangle|^2$$

$$\bullet a \in D(E)$$

$$R_{ij}^{\pm}(E) = \sum_{a \in D(E)} R_{aij}^{\pm}$$

$$R_{ij}^{\pm}(E) = \begin{cases} 0, & \text{if } i, j \notin D(E) \\ -\frac{g H_{1,ij}}{E_i - E_j \pm i\epsilon} & \text{if } i \notin D(E) \text{ but } j \in D(E) \\ -\frac{g H_{1,ji}}{E_j - E_i \pm i\epsilon} & \text{if } j \notin D(E) \text{ but } i \in D(E) \\ \delta_{ij} & \text{if } i, j \in D(E) \end{cases}$$

No. div. at $O(g^0)$!

- All orders: induction

H can be diagonalized by $U(0, \pm\infty)$.

$$U^\dagger H U = \hat{H}_0$$

unitarity: $U H_0 = H U$

See S.S. Schweber: An introduction to relativistic quantum field theory, pp. 320

$$[U, \hat{H}_0] = (H - \hat{H}_0) U = (g H_1 + \Delta) U$$

$$\Delta = H_0 - \hat{H}_0 \quad \text{diagonal operator}$$

$$R.T.: \quad \Delta = \sum_n g^n \Delta_n \quad U = \sum_n g^n U_n \quad R = \sum_n g^n R_n$$

$$R_{ij}^\pm(E) = \sum_{rs} g^{r+s} \sum_{\alpha \in D(E)} \langle i | U_r(0, \pm\infty) | \alpha \rangle^* \langle j | U_s(0, \pm\infty) | \alpha \rangle$$

$$R_{n,ij}^\pm(E) = \sum_r \sum_{\alpha \in D(E)} \langle i | U_r(0, \pm\infty) | \alpha \rangle^* \langle j | U_{n+r}(0, \pm\infty) | \alpha \rangle$$

Lemma: if $\langle i | \Delta_n | i \rangle$ is finite of IR for $n \leq N$, then $R_{n,ij}^\pm$ is IR finite for $n \leq N+1$. when?

Proof: induction.

Case 1: $i \notin D(E)$

$$\langle i | U H_0 - H_0 U | \alpha \rangle = (E_\alpha - E_i) U_{i\alpha}$$

$$\langle i | (g H_1 + \Delta) U | \alpha \rangle = g \sum_k H_{ik} U_{k\alpha} + \Delta_i U_{i\alpha}$$

$$U_{i\alpha} = \frac{1}{E_\alpha - E_i} \left(g \sum_k H_{ik} U_{k\alpha} + \Delta_i U_{i\alpha} \right)$$

$$U_{r,i\alpha} = \frac{1}{E_\alpha - E_i} \left(\sum_k H_{ik} U_{r-1,k\alpha} + \sum_{s=1}^r \Delta_{s,i} U_{r-s,i\alpha} \right)$$

$$\Rightarrow R_{n,ij}^\pm(E) = \frac{1}{E - E_i} \left[\sum_k H_{ik} R_{n-1,kj}^\pm(E) + \sum_{s=1}^n \Delta_{s,i} R_{n-s,i,j}^\pm(E) \right]$$

$$R_n \rightarrow R_m (m < n), \Delta_m (m < n) \quad \text{IR finite.}$$

Case 2: $j \notin D(E)$ Same with Case 1.

Case 3: $i, j \in D(E)$

$$\text{Unitarity: } U^\dagger U = 1$$

$$\Rightarrow \sum_r U_r^\dagger U_{n+r} = 0 \quad (n \neq 0)$$

$$\sum_r \sum_{\alpha \in D(E)} U_{r,i\alpha}^* U_{n+r,j\alpha} + \sum_r \sum_{\alpha \notin D(E)} U_{r,i\alpha}^* U_{n+r,j\alpha} = 0$$

$$R_{n+1,ij}^\pm(E) = - \sum_{r=0}^\infty \sum_{\alpha \notin D(E)} U_{r,i\alpha}^* U_{n+1+r,j\alpha}$$

Unitarity $\rightarrow$ KLN.

not degenerate.

• Uniform cut

$$+ + + + \dots$$

= finite,