Lecture note

Recall: Last time it has been proved:

To factorize $W^{\mu\nu} = \frac{1}{4\pi}\int d^4x e^{iq\cdot x}\langle P|J^\mu(x)J^\nu(0)|P\rangle$,

We use $W_{\mu\nu}(\omega, Q) = 2\,\mathrm{Im}\,T_{\mu\nu}(\omega, Q) = -iT_{\mu\nu}(\omega + i\varepsilon, Q) + iT_{\mu\nu}(\omega - i\varepsilon, Q) = \mathrm{Disc}\,(-iT_{\mu\nu})$,

$Disc\,T^{\mu\nu} = \frac{i}{2}\int dx^- \int \frac{dp^+}{2\pi} e^{-ip^+x^-}\langle P|\bar{\psi}(x)\gamma^\mu(\not{p}+\not{q})\gamma^\nu\psi(0)|P\rangle\delta((p+q)^2)$

$p = p^+{n_-}^\mu + p^-{n_+}^\mu + p^\perp{n_\perp}^\mu$

$p = (p^+, p^-, p^\perp) \sim p^+(1, \lambda^2, \lambda)$

$$Disc\,T^{\mu\nu} = \frac{i}{2}\int dx^- \int \frac{dp^+}{2\pi} e^{-ip^+x^-}\langle P|\bar{\psi}(x)_i{\Gamma^a}_{il}\psi(0)_l|P\rangle\delta((p+q)^2) \times tr(\Gamma_a\gamma^\mu(\not{p}+\not{q})\gamma^\nu)$$

The leading twist requires the large component of ψ

$\psi = (\frac{\gamma^+\gamma^-}{2} + \frac{\gamma^-\gamma^+}{2})\psi$

leading twist component: $\psi_+ = \frac{\gamma^-\gamma^+}{2}\psi$

so we need $\langle P|\bar{\psi}(x)\gamma^+\psi(0)|P\rangle$

$I_{ij}I_{kl} = \frac{1}{4}(\gamma^\mu)_{il}(\gamma\mu)_{kj} + \ldots = \frac{1}{4}(\gamma^0\gamma^0 - \gamma^3\gamma^3) + \ldots = \frac{1}{4}(\frac{1}{2}(\gamma^0+\gamma^3)(\gamma^0-\gamma^3) + \frac{1}{2}(\gamma^0-\gamma^3)(\gamma^0+\gamma^3)) + \ldots = \frac{1}{4}\gamma^+\gamma^- + \frac{1}{4}\gamma^-\gamma^+ + \ldots$

$Disc\,T^{\mu\nu} = \frac{i}{2}\int dx^- \int \frac{dp^+}{2\pi} e^{-ip^+x^-}\langle P|\bar{\psi}(x)\gamma^+\psi(0)|P\rangle\delta((p+q)^2) \times \frac{1}{4}tr(\gamma^-\gamma^\mu(\not{p}+\not{q})\gamma^\nu)$

$x = \frac{Q^2}{2P\cdot q} = \frac{-2q^+q^-}{2p^+q^-} = -\frac{q^+}{P^+}$

$(p+q)^2 = 0 \to 2p^+q^- = -2q^+q^- \to p^+ = -q^+ \to p^+ = xP^+$

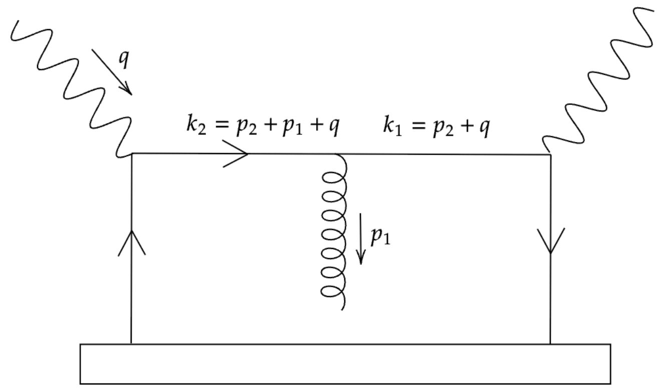
$$W^{\mu\nu} = \frac{1}{2}\int dz^- \int \frac{dp^+}{2\pi} e^{-ixP^+z^-}\langle P|\bar{\psi}(z^-)\gamma^+\psi(0)|P\rangle\delta((p+q)^2) \times \frac{1}{4}tr(\gamma^-\gamma^\mu(\not{p}+\not{q})\gamma^\nu)$$

$$W^{\mu\nu} = \frac{1}{4\pi}\int dz^- e^{-ixP^+z^-}\langle P|\bar{\psi}(z^-)\gamma^+\psi(0)|P\rangle \times \frac{1}{4}\frac{1}{2q^-}tr(\gamma^-\gamma^\mu(\not{p}+\not{q})\gamma^\nu)$$

- The PDF we derived does not contain the Wilson line we need to keep it gauge invariant.

The next lecture will show: Collinear gluon generate the Wilson line we need.

DIS factorization (1-Loop)



Goal: To show that the collinear part of 1-loop diagram generates the Wilson line we need to keep it gauge invariant.

$$T^{\mu\nu} = \frac{i}{4\pi}\int d^4x \int d^4y\, e^{iq\cdot x}\langle P|J^\mu(x)(ig)\bar\psi(y)\not{A}\psi(y)J^\nu(0)|P\rangle$$

$$= \frac{i}{4\pi}\int d^4x d^4y \int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4} e^{iqx}(-ig)\langle P|\bar\psi(x)\gamma^\mu \frac{\not{k}_1}{k_1^2+i\epsilon}\not{A}\frac{\not{k}_2}{k_2^2+i\epsilon}\gamma^\nu\psi(0)|P\rangle e^{-ik_1(x-y)-ik_2 y}$$

Change variable: $k_2 = p_2 + p_1 + q, k_1 = p_2 + q$

$= \frac{g}{4\pi}\int d^4xd^4y\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}e^{-ixp_2}e^{-iyp_1}\langle P|\bar\psi(x)\gamma^\mu\frac{\not{p}_2+\not{q}}{(p_2+q)^2+i\epsilon}\not{A}\frac{\not{p}_2+\not{p}_1+\not{q}}{(p_1+p_2+q)^2+i\epsilon}\gamma^\nu\psi(0)|P\rangle$

- The imaginary part only comes from the propagator

The imaginary part of a propagator:

$\mathrm{Im}\frac{1}{p^2+i\epsilon} = \frac{1}{2i}(\frac{1}{p^2+i\epsilon} - \frac{1}{p^2-i\epsilon}) = \lim_{\epsilon\to 0^+}\frac{-\epsilon}{(p^2)^2+\epsilon}$

Theorem (Delta sequences): Take any nonnegative normalized function f. Define $f_j(x) = j^n f(jx)$.

$$\lim_{j\to\infty} f_j \to \delta$$

Set $\epsilon \to \frac{1}{j}$, so $\mathrm{Im}\frac{1}{p^2+i\epsilon} = -\pi\delta(p^2)$

In our case:

$$\begin{aligned}
&\frac{1}{2i}(\frac{1}{p_1^2+i\epsilon}\frac{1}{p_2^2+i\epsilon} - \frac{1}{p_1^2-i\epsilon}\frac{1}{p_2^2-i\epsilon}) && (1)\\
=&\frac{1}{2i}(\frac{1}{p_1^2+i\epsilon}\frac{1}{p_2^2+i\epsilon} - \frac{1}{p_1^2+i\epsilon}\frac{1}{p_2^2-i\epsilon} + \frac{1}{p_1^2+i\epsilon}\frac{1}{p_2^2+i\epsilon} - \frac{1}{p_1^2-i\epsilon}\frac{1}{p_2^2-i\epsilon}) && (2)\\
=&\frac{1}{p_1^2+i\epsilon}(-\pi\delta(p_2^2)) + (\pi\delta(p_1^2))\frac{1}{p_2^2-i\epsilon} && (3)
\end{aligned}$$

- One may be confused that we start from a symmetric expression but the final result is not. Check the symmetry property.

Consider the first part:

$$ImT^{\mu\nu} = \frac{g}{4\pi}\int d^4xd^4y\int\frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}e^{-ixp_2}e^{-iyp_1}\langle P|\bar\psi(x)\gamma^\mu(\not{p}_2+\not{q})\not{A}\frac{\not{p}_2+\not{p}_1+\not{q}}{(p_1+p_2+q)^2+i\epsilon}\gamma^\nu\psi(0)|P\rangle(-\pi)\delta((p_2+q$$

- Collinear quark $p_2 = p_2^+(1,\lambda^2,\lambda)$
- Now we consider the momentum of gluon p_1 is collinear: $p_1 = p_1^+(1,\lambda^2,\lambda)$.

- Also the power counting of gluon field in Lorentz gauge gives: $A^\mu \sim A^+$

Reduce $(\not{p}_2 + \not{q})\not{A}\frac{\not{p}_2+\not{p}_1+\not{q}}{(p_1+p_2+q)^2+i\epsilon}$:

$(p_2 + p_1 + q)^2 = (p_2 + q)^2 + p_1^2 + 2(p_2 + q)\cdot p_1 = 2q^- p_1^+ + i\epsilon$

$(\not{p}_2 + \not{q})\not{A}(\not{p}_2 + \not{p}_1 + \not{q}) = 2(\not{p}_2 + \not{q})A^+q^- - (\not{p}_2 + \not{q})(\not{p}_2 + \not{p}_1 + \not{q})\not{A} = 2(\not{p}_2 + \not{q})A^+q^-$

So,

$(\not{p}_2 + \not{q})\not{A}\frac{\not{p}_2+\not{p}_1+\not{q}}{(p_1+p_2+q)^2+i\epsilon} = (\not{p}_2 + \not{q})\frac{2q^-A^+}{2q^-p_1^+ + i\epsilon}$

$\frac{2p\cdot q}{Q^2} > 0 \to 2p\cdot q > 0 \to p^+q^- > 0 \to q^- > 0$

$x = \frac{Q^2}{2P\cdot q} = \frac{-2q^+q^-}{2p^+q^-} = -\frac{q^+}{P^+}$

$(p_2+q)^2 = 0 \to 2p_2^+q^- = -2q^+q^- \to p_2^+ = -q^+ \to p_2^+ = xP^+$

$\delta((p_2+q)^2) = \delta(2p_2^+q^- - 2xP^+q^-)$

Set $p_2^+ = yP^+$

$\delta((p_2+q)^2) = \frac{1}{2q^-P^+}\delta(y-x)$

$(\not{p}_2 + \not{q})\not{A}\frac{\not{p}_2+\not{p}_1+\not{q}}{(p_1+p_2+q)^2+i\epsilon} = (\not{p}_2 + \not{q})\frac{A^+}{p_1^+ + i\epsilon}$

$$2ImT^{\mu\nu} = \frac{g}{4\pi}\int dz^-dy^-\int\frac{dp_1^+}{2\pi}\frac{dp_2^+}{2\pi}e^{-iz^-yP^+}e^{-iy^-p_1^+}\langle P|\bar{\psi}(z^-)\gamma^\mu(\not{p}_2+\not{q})\gamma^\nu\psi(0)|P\rangle(-2\pi)\delta((p_2+q)^2)\frac{A^+(y^-)}{p_1^+ + i\epsilon}$$

- Color index …

$$2ImT^{\mu\nu} = -\frac{g}{2}\int dz^-dy^-\int\frac{dp_1^+}{2\pi}\frac{dp_2^+}{2\pi}e^{-iz^-xP^+}e^{-iy^-p_1^+}\langle P|\bar{\psi}(z^-)\gamma^\mu(\not{p}_2+\not{q})\gamma^\nu\psi(0)|P\rangle\delta((p_2+q)^2)\frac{A^+(y^-)}{p_1^+ + i\epsilon}$$

$W^{\mu\nu} = 2ImT^{\mu\nu} = -\frac{g}{2}\int dz^-dy^-\int\frac{dp_1^+}{2\pi}\frac{dp_2^+}{2\pi}e^{-iz^-xP^+}e^{-iy^-p_1^+}\langle P|\bar{\psi}(z^-)\gamma^+\psi(0)|P\rangle\delta((p+q)^2)\frac{A^+(y^-)}{p_1+i\epsilon}\frac{1}{4}tr(\gamma^\mu(\not{p}_2 + \not{q})\gamma^\nu\gamma^-)$

$= -\frac{g}{2}\int dz^-\frac{dp_2^+}{2\pi}e^{-iz^-xP^+}\int\frac{dp_1^+}{2\pi}dy^-e^{-iy^-p_1^+}\langle P|\bar{\psi}(x)\gamma^+\psi(0)|P\rangle\delta((p+q)^2)\frac{A^+(y^-)}{p_1^+ + i\epsilon}\frac{1}{4}tr(\gamma^\mu(\not{p}_2+\not{q})\gamma^\nu\gamma^-)$

$= -\frac{g}{2}\int dz^-\frac{dy}{2\pi}e^{-iz^-yP^+}\int\frac{dp_1^+}{2\pi}dy^-e^{-iy^-p_1^+}\langle P|\bar{\psi}(x)\gamma^+\psi(0)|P\rangle\delta(x-y)\frac{1}{2q^-}\frac{A^+(y^-)}{p_1^+ + i\epsilon}\frac{1}{4}tr(\gamma^\mu(\not{p}_2+\not{q})\gamma^\nu\gamma^-)$

PDF part:

$f(x) = -g\int dz^-e^{-iz^-xP^+}\int\frac{dp_1^+}{2\pi}dy^-e^{-iy^-p_1^+}\langle P|\bar{\psi}(z^-)\gamma^+\frac{A^+(y^-)}{p_1^+ + i\epsilon}\psi(0)|P\rangle$

Integrate p_1^+

If $y^- < 0$, close the contour on the upper half plane, no pole.

$\int\frac{dp_1^+}{2\pi}e^{-iy^-p_1^+}\frac{1}{p_1^+ + i\epsilon} = \frac{1}{2\pi} * (-2\pi i) * \theta(y^-) = -i\theta(y^-)$

First part: $f(x) = \int dz^-e^{-iz^-xP^+}\langle P|\bar{\psi}(z^-)\gamma^+(ig\int_0^\infty A^+(y^-)dy^-)\psi(0)|P\rangle$

- The second part gives $(ig\int_{z^-}^\infty A^+(y^-)dy^-)^\dagger$

Total 1-loop $f(x) = \int dz^-e^{-iz^-xP^+}\langle P|\bar{\psi}(z^-)\gamma^+(ig\int_o^{z^-}A^+(y^-)dy^-)\psi(0)|P\rangle$

Conclusion: 1-loop collinear gluon gives the 1-loop expression for Wilson line. An all order proof (Collins: Foundation of perturbative QCD) shows:

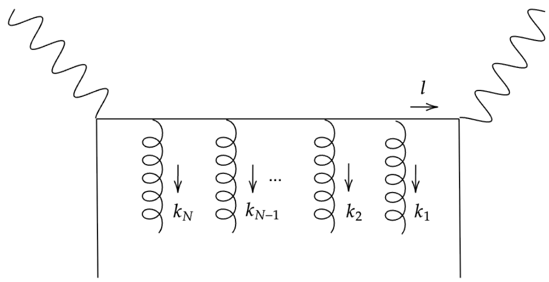
$$f(x) = \int dx^-e^{-iz^-xP^+}\langle P|\bar{\psi}(z^-)\gamma^+e^{ig\int_o^{z^-}A^+(y^-)dy^-}\psi(0)|P\rangle$$

- You can shift the role of p_1, p_2 in Eq.(3). See what you arrived at.

Higher loop

Here is a proof from Collins *Foundation of perturbative QCD* to show that:

Collinear gluon from inserting n interaction $\bar{\psi}\not{A}\psi$ generates the expansion of Wilson line at n loop.



$W_N = \frac{i}{\not l - m}\gamma^- \frac{i}{\not l+\not k_1 - m}\cdots\gamma^- \frac{i}{\not l+\not k_1+...+\not k_N - m}$

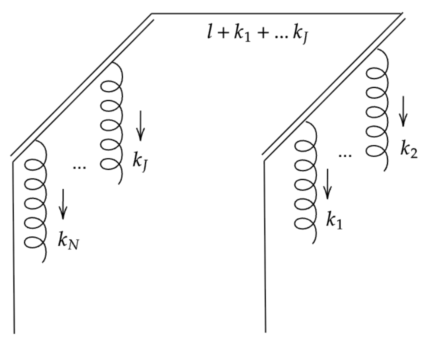
Claim: $W_N = \sum_{J=0}^{N} R_J M_J L_{N,J}$, where:

$L_{N,J} = \frac{i}{k^+_{J+1}+i\epsilon}\cdots\frac{i}{k^+_{J+1}+...+k^+_N+i\epsilon}$

$M_J = \frac{i}{\not l+\not k_1+...+\not k_j - m+i\epsilon}$

$R_J = \frac{i}{k^+_J+i\epsilon}\cdots\frac{i}{k^+_1+...+k^+_J+i\epsilon}$

$$T^{\mu\nu} = \sum_{J=0}^{N}$$



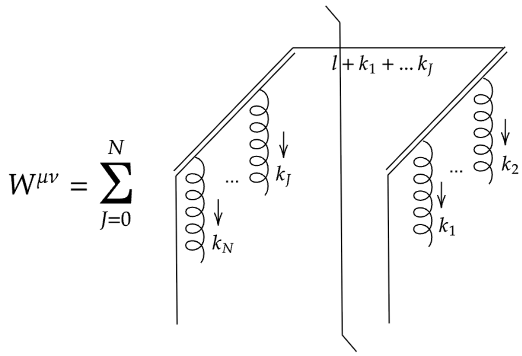
Proof: By induction

Now we want to determine the imaginary part of the sum.

If we take the imaginary part of $\frac{i}{k_i^+ + i\epsilon}$, it gives $Im\frac{i}{k_i^+ + i\epsilon} \sim \delta(k_i^+)$.

However, it tells you the gluon is not collinear.

So we can only take the imaginary part of $\frac{i}{\not l+\not k_1+...+\not k_j - m+i\epsilon} \to \delta(l+k_1+...k_j)^2$



Aside: Why we can apply optical theorem to $W^{\mu\nu}$ and $T^{\mu\nu}$

Schwinger Dyson equation

We start with the path integral definition of a one-point green function

$$\langle\hat{\phi}(x)\rangle = \frac{1}{Z}\int D\phi\, e^{i\int d^4y(-\frac{1}{2}\phi\Box\phi+\mathcal{L}_{int}[\phi])}\phi(x),$$

where we know the on-shell EOM is:

$$\Box\phi = \partial_\phi\mathcal{L}_{int}[\phi] = \mathcal{L}'_{int}[\phi].$$

- Prove the EOM. It is nontrivial since you need a generalized Euler-Lagrange equation from this definition

Let's implement a redefinition of integration variable:

$\phi'(x) = \phi(x) + \epsilon(x)$.

$$\langle\hat{\phi}'(x)\rangle = \frac{1}{Z}\int D\phi'\, e^{i\int d^4y(-\frac{1}{2}\phi'\Box\phi'+\mathcal{L}_{int}[\phi'])}\phi'(x),$$
$$\langle\hat{\phi}(x)\rangle = \frac{1}{Z}\int D\phi\, e^{iS}\left(\phi(x)+\epsilon(x)-i\phi(x)\int d^4y\,\epsilon(y)\Box_y\phi(y)+i\epsilon(y)\mathcal{L}'_{int}(y)\right),$$

where I used $\mathcal{L}_{int}[\phi+\epsilon] \sim \mathcal{L}_{int}[\phi] + \epsilon\mathcal{L}'_{int}[\phi']$.

So we get an equation:

$$\frac{1}{Z}\int D\phi\, e^{iS}\left(\epsilon(x)-i\phi(x)\int d^4y\,\epsilon(y)\Box_y\phi(y)+i\epsilon(y)\mathcal{L}'_{int}(y)\right) = 0$$

Let's write: $\epsilon(x) = \int d^4y\delta^4(x-y)\epsilon(y)$

$$\frac{1}{Z}\int d^4y\epsilon(y)\int D\phi\, e^{iS}\left(\epsilon(y)-i\phi(x)\Box_y\phi(y)+i\mathcal{L}'_{int}(y)\right) = 0$$

Since the equation is valid for any $\epsilon(x)$,

$$\int D\phi\, e^{iS}\left(\delta^4(x-y)-i\phi(x)\Box_y\phi(y)+i\mathcal{L}'_{int}(y)\right) = 0$$
$$\Box_y\langle\phi(x)\phi(y)\rangle = \langle\mathcal{L}'_{int}(\phi(y))\phi(x)\rangle - i\delta^4(y-x)$$

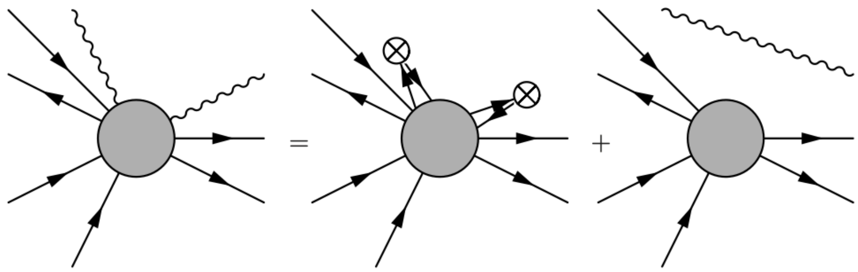
- Schwinger-Dyson equation applies to every classic conservation law and gives a quantum conservation equation.

For photon

EOM reads $\Box A^\mu = J^\mu$

$$\Box^k_{\alpha\beta}\Box_{\mu\nu}\langle A_\nu(x)\cdots A_\beta(x_k)\cdots\rangle = \Box^k_{\alpha\beta}\Big[\langle j_\mu(x)\cdots A_\beta(x_k)\cdots\rangle - i\delta^4(x-x_k)g_{\mu\beta}\langle\cdots\rangle\Big]$$
$$= \langle j_\mu(x)\cdots j_\alpha(x_k)\cdots\rangle + \Box^k_{\mu\alpha}\Box D_F(x,x_k)\langle\cdots\rangle,$$

- If done in Lorentz gauge, discard the index for $\Box$



Graphically, the D_F terms correspond to a non-interacting photon. Since we only care about connected diagrams due to superselection rule, we have

$$\Box^k_{\alpha\beta}\Box_{\mu\nu}\langle A_\nu(x)\cdots A_\beta(x_k)\cdots\rangle = \langle j_\mu(x)\cdots j_\alpha(x_k)\cdots\rangle.$$

Amplitudes in DIS

In DIS process, we care about two amplitudes

$$\langle\gamma(\epsilon,q)\,p|S|\gamma(\epsilon,q)\,p\rangle\,\&\langle\gamma(\epsilon,q)\,p|S|X\rangle\,,$$

which are related by optical theorem.

The LSZ reduction formula for $\langle\gamma(\epsilon,q)\,p|S|\gamma(\epsilon,q)\,p\rangle$ reads：

$$\langle\gamma(\epsilon,q)\,p|S|\gamma(\epsilon,q)\,p\rangle = \Big[i^4\int d^4x\,e^{iqx}\int d^4x_1\,e^{-iqx_1}\int d^4y_i\,e^{ipy}\big(i\partial\!\!\!/_y + m_1\big)\cdots\Big]$$
$$\times\epsilon_\mu\epsilon_\nu\langle j_\mu(x)j_\alpha(x_k)\cdots\psi(y)\cdots\rangle$$

Where we used the Schwinger Dyson Equation for photon.

Since the proof of LSZ reduction formula is done iteratively, it suffices to write

$\langle\gamma(\epsilon,q)\,p|S|\gamma(\epsilon,q)\,p\rangle = i^2\int d^4x\,e^{iqx}\int d^4x_1\,e^{-iqx_1}\epsilon_\mu\epsilon_\nu\langle p|j_\mu(x)j_\nu(x_1)|p\rangle$
$i\mathcal{M}\delta^4(p+q-p-q) = i^2\int d^4x\,e^{iqx}\int d^4x_1\,e^{-iqx_1}\epsilon_\mu\epsilon_\nu\langle p|j_\mu(x-x_1)j_\nu(0)|p\rangle$

Change variables: $\{x,x_1\}\to\{x'=x-x_1,x_1\}$

$$i\mathcal{M}\delta^4(0) = i^2\int d^4x'\,e^{iqx'}\int d^4x_1\,\epsilon_\mu\epsilon_\nu\langle p|j_\mu(x')j_\nu(0)|p\rangle$$
$$\mathcal{M}(\gamma\,p\to\gamma\,p) = i\int d^4x'\,e^{iqx'}\epsilon_\mu\epsilon_\nu\langle p|j_\mu(x')j_\nu(0)|p\rangle$$

The result is not new however:

- It does not depend on perturbative theory: In perturbative derivation, there is always interchanging between. Free operators and Heisenberg operators. In this derivation, there is no such ambiguity. Heisenberg operators are used the whole time.
- The derivation for $M(\gamma\,p\to X)$ resembles, by Optical theorem:

$$2\,\mathrm{Im}\,M(A \to A) = \sum_X \int d\Pi_X (2\pi)^4 \delta^4(p_A - p_X) |M(A \to X)|^2.$$

we know

$$W_{\mu\nu}(\omega, Q) = 2\,\mathrm{Im}\,T_{\mu\nu}(\omega, Q) = -iT_{\mu\nu}(\omega + i\varepsilon, Q) + iT_{\mu\nu}(\omega - i\varepsilon, Q) = \mathrm{Disc}\,(-iT_{\mu\nu}),$$