$\Xi_{cc} \to \Lambda_c$ form factors in QCDSR

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Following the usual QCDSR procedure, we firstly construct a three point correlating function:

$$\Pi_\mu(p_1^2, p_2^2, q^2) = i^2 \int d^4x d^4y e^{ip_2\cdot x - ip_1 \cdot y} \langle 0|T\{j_h(x) j_\mu(0) \bar{j}_H(y)\}|0\rangle$$

Which is a time ordered product of two heavy baryon currents and one V-A current.

$$\begin{aligned} j_h(x) &= \epsilon_{abc}(q_{1a}^T(x) C\gamma^\alpha q_{2b}(x))\gamma_\alpha\gamma_5 Q_c(x) \\ j_\mu(0) &= \bar{q}_{1e}(0)\gamma_\mu(1-\gamma_5)Q_e(0) \\ \bar{j}_H(y) &= \epsilon_{a'b'c'}\bar{q}_{2c'}(y)\gamma_\beta\gamma_5(\bar{Q}_b(y)\gamma^\beta C \bar{Q}_a^T(y)) \end{aligned}$$

$$+\left(\frac{m_c^2\left(q^2-s_1-s_2\right)\left(2\ m_c^2-q^2-s_1+s_2\right)+m_{23}^2\left(-2m_c^2\ \left(q^2+s_1-s_2\right)+q^4-2q^2\ \left(s_1+s_2\right)+\left(s_1-s_2\right)^2\right)}{q^4-2q^2\ \left(s_1+s_2\right)+\left(s_1-s_2\right)^2}\right)r_0$$

$$+\left(\frac{2\left(2m_{23}^2\left(m_c^2+q^2-s_1+s_2\right)+m_c^2\ \left(q^2-s_1-3s_2\right)+s_2\left(q^2+s_1-s_2\right)\right)}{q^4-2q^2\ \left(s_1+s_2\right)+\left(s_1-s_2\right)^2}\right)r_{1,1}$$

$$\frac{m_c^2\left(q^2-s_1-s_2\right)\left(2m_c^2-q^2-s_1+s_2\right)+m_{23}^2\left(-2m_c^2\left(q^2+s_1-s_2\right)+q^4-2q^2\left(s_1+s_2\right)+\left(s_1-s_2\right)^2\right)}{q^4-2q^2\left(s_1+s_2\right)+\left(s_1-s_2\right)^2}$$

$$+\left(\frac{m_c^2\left(q^2-s_1-s_2\right)\left(-q^2+2\ m_c^2-s_1+s_2\right)+m_{23}^2\left(q^4+\left(s_1-s_2\right)^2-2m_c^2\ \left(q^2+s_1-s_2\right)-2q^2\ \left(s_1+s_2\right)\right)}{q^4+\left(s_1-s_2\right)^2-2q^2\ \left(s_1+s_2\right)}\right)r_0$$

$$+\left(\frac{2\left(m_c^2\left(q^2-s_1-3s_2\right)+\left(q^2+s_1-s_2\right)s_2+2\ m_{23}^2\ \left(q^2+m_c^2-s_1+s_2\right)\right)}{q^4+\left(s_1-s_2\right)^2-2q^2\ \left(s_1+s_2\right)}\right)r_{1,1}$$

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$\left(-m_c^2\left(2q^4-7q^2s_2+s_1\left(5s_2-4q^2\right)+2s_1^2+5s_2^2\right)-m_{\mathrm{cq}}^2\left(q^4+q^2s_2+s_1\left(q^2+4s_2\right)-2s_1^2-2s_2^2\right)+2q^6-4q^4s_1-5q^4s_2+2q^2\right.$

Hadronic Calculation

Define the baryons' decay constant:

$$\langle 0|j_h(0)|p_h, s'\rangle = f_h u_h(p_h, s')$$
$$\langle 0|j_H(0)|p_H, s\rangle = f_H u_H(p_H, s)$$

And the form factors:

$$\begin{aligned}\langle p_h, s'|j_\mu(0)|p_H, s\rangle &= \bar{u}_h(p_h, s')[f_1(q^2)\gamma_\mu + f_2(q^2)p_{h\mu} + f_3(q^2)p_{H\mu}]u_H(p_H, s)\\ &- \bar{u}_h(p_h, s')[g_1(q^2)\gamma_\mu + g_2(q^2)p_{h\mu} + g_3(q^2)p_{H\mu}]\gamma_5 u_H(p_H, s)\end{aligned}$$

Intersect the single and multiple baryon states:

$$\begin{aligned}\Pi_\mu^{hadron}(p_1^2, p_2^2, q^2) = i^2 \int d^4x d^4y bare^{ip_2\cdot x - ip_1\cdot y} \int \frac{d^4p_h}{(2\pi)^4}\frac{d^4p_H}{(2\pi)^4}\Sigma_{s,s'}\frac{i}{p_h^2 - m_h^2}\frac{i}{p_H^2 - m_H^2}\\ \times \langle 0|j_h(x)|p_h, s'\rangle\langle p_h, s'|j_\mu(0)|p_H, s\rangle\langle p_H, s|\bar{j}_H(y)|0\rangle\\ = \frac{f_h f_H}{(p_2^2 - m_h^2)(p_1^2 - m_H^2)}(\not{p}_2 + m_h)(\Gamma_V - \Gamma_A)(\not{p}_1 + m_H)\end{aligned}$$

$$\begin{aligned}\Pi_\mu^{hadron}(p_1^2, p_2^2, q^2)_V &= \frac{f_h f_H}{(p_2^2 - m_h^2)(p_1^2 - m_H^2)}(\not{p}_2 + m_h)[f(q^2)\gamma_\mu + f_2(q^2)p_{h\mu} + f_3(q^2)p_{H\mu}](\not{p}_1 + m_H)\\ &= \frac{f_h f_H}{(p_2^2 - m_h^2)(p_1^2 - m_H^2)}[m_h m_H f_1\gamma_\mu + (m_h m_H f_2 + 2m_H f_1)p_{2\mu} + m_h m_H f_3 p_{1\mu}\\ &+ m_h f_1\gamma_\mu\not{p}_1 + (m_h f_2 + 2f_1)p_{2\mu}\not{p}_1 + m_h f_3 p_{1\mu}\not{p}_1\\ &- m_H f_1\gamma_\mu\not{p}_2 + m_H f_2 p_{2\mu}\not{p}_2 + m_H f_3 p_{1\mu}\not{p}_2\\ &- f_1\gamma_\mu\not{p}_2\not{p}_1 + f_2 p_{2\mu}\not{p}_2\not{p}_1 + f_3 p_{1\mu}\not{p}_2\not{p}_1]\end{aligned}$$

$$\begin{aligned}\Pi_\mu^{hadron}(p_1^2, p_2^2, q^2)_A &= \frac{f_h f_H}{(p_2^2 - m_h^2)(p_1^2 - m_H^2)}(\not{p}_2 + m_h)[g_1(q^2)\gamma_\mu + g_2(q^2)p_{h\mu} + g_3(q^2)p_{H\mu}]\gamma_5(\not{p}_1 + m_H)\\ &= \frac{f_h f_H}{(p_2^2 - m_h^2)(p_1^2 - m_H^2)}[m_h m_H g_1\gamma_\mu + (m_h m_H g_2 + 2m_H g_1)p_{2\mu} + m_h m_H g_3 p_{1\mu}\\ &- m_h g_1\gamma_\mu\not{p}_1 - (m_h g_2 + 2g_1)p_{2\mu}\not{p}_1 - m_h g_3 p_{1\mu}\not{p}_1\\ &- m_H g_1\gamma_\mu\not{p}_2 + m_H g_2 p_{2\mu}\not{p}_2 + m_H g_3 p_{1\mu}\not{p}_2\\ &+ g_1\gamma_\mu\not{p}_2\not{p}_1 - g_2 p_{2\mu}\not{p}_2\not{p}_1 - g_3 p_{1\mu}\not{p}_2\not{p}_1]\end{aligned}$$

QCD calculation

Perturbative term

With the use of OPE, the correlating function can be expanded into the perturbative term and a series of higher dimensional operator condensates. We firstly calculate the perturbative term.

$$
\begin{aligned}
\Pi_\mu^{pert}(p_1^2,p_2^2,q^2) &= i^2\int d^4xd^4ye^{ip_2\cdot x-ip_1\cdot y}\langle 0|T\{j_h(x)j_\mu(0)\bar{j}_H(y)\}|0\rangle \\
&= i^2\epsilon_{abc}\epsilon_{a'b'c'}\int d^4xd^4ye^{ip_2\cdot x-ip_1\cdot y} \\
&\times (q_{1a}^T(x)C\gamma^\alpha q_{2b}(x))\gamma_\alpha\gamma_5 Q_c(x)\bar{q}_{1e}(0)\gamma_\mu(1-\gamma_5)Q_e(0)\bar{q}_{2c'}(y)\gamma_\beta\gamma_5(\bar{Q}_b(y)\gamma^\beta C\bar{Q}_a^T(y)) \\
&= 2i^2\epsilon_{abc}\epsilon_{a'b'c'}\delta_{ab'}\delta_{bc'}\delta_{ca'}\int d^4xd^4ye^{ip_2\cdot x-ip_1\cdot y} \\
&\times [\gamma_\alpha\gamma_5 S_Q(x-y)C^T\gamma^{\beta T}S_Q^T(-y)(1-\gamma_5)\gamma_\mu^T S_{q_1}^T(x)C\gamma^\alpha S_{q_2}(x-y)\gamma_\beta\gamma_5] \\
&= 12i^2\int d^4xd^4ye^{ip_2\cdot x-ip_1\cdot y}[\gamma_\alpha\gamma_5 S_Q(x-y)\gamma^\beta CS_Q^T(-y)CC(1-\gamma_5)\gamma_\mu^T CCS_{q_1}^T(x)C\gamma^\alpha S_{q_2}(x-y)\gamma_\beta\gamma_5] \\
&= 12i^2\int d^4xd^4ye^{ip_2\cdot x-ip_1\cdot y}[\gamma_\alpha\gamma_5 S_Q(x-y)\gamma^\beta CS_Q^T(-y)C(1-\gamma_5)\gamma_\mu CS_{q_1}^T(x)C\gamma^\alpha S_{q_2}(x-y)\gamma_\beta\gamma_5]
\end{aligned}
$$

Where we have used some relations:

$$
\begin{aligned}
C^T\gamma^{\beta T} &= \gamma^\beta C \\
C(1-\gamma_5)\gamma_\mu^T C &= (1-\gamma_5)\gamma_\mu
\end{aligned}
$$

And with the following result:

$$
CS^T(x)C = \int \frac{d^4k}{(2\pi)^4}\frac{i(\not{k}-m)}{k^2-m^2+i\epsilon}e^{-ik\cdot x}
$$

We have

$$
\begin{aligned}
\Pi_\mu^{pert}(p_1^2,p_2^2,q^2) &= 12i^2\int d^4xd^4ye^{ip_2\cdot x-ip_1\cdot y}\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}\frac{d^4k_4}{(2\pi)^4} \\
&\times [\gamma_\alpha\gamma_5\frac{i}{\not{k}_3-m_Q}\gamma^\beta\frac{i(\not{k}_1-m_Q)}{k_1^2-m_Q^2+i\epsilon}(1-\gamma_5)\gamma_\mu\frac{i(\not{k}_2-m_1)}{k_2^2-m_1^2+i\epsilon}\gamma^\alpha\frac{i}{\not{k}_4-m_2}\gamma_\beta\gamma_5] \\
&\times e^{-ik_3\cdot(x-y)}e^{-ik_1\cdot(-y)}e^{-ik_2\cdot x}e^{-ik_4\cdot(x-y)} \\
&= -12\int\frac{d^4k_3}{(2\pi)^4}\frac{d^4k_4}{(2\pi)^4}\frac{N_\mu}{(k_1^2-m_Q^2)(k_2^2-m_1^2)(k_3^2-m_Q^2)(k_4^2-m_2^2)}
\end{aligned}
$$

Where:

$$
\begin{aligned}
N_\mu &= \gamma_\alpha\gamma_5(\not{k}_3+m_Q)\gamma^\beta(\not{k}_1-m_Q)(1-\gamma_5)\gamma_\mu(\not{k}_2-m_1)\gamma^\alpha(\not{k}_4+m_2)\gamma_\beta\gamma_5 \\
k_1 &= p_1-k_3-k_4, \qquad k_2 = p_2-k_3-k_4
\end{aligned}
$$

Now we need to write the correlator function as a double dispersion relation, which has the form:

$$
\Pi_\mu^{pert}(p_1^2,p_2^2,q^2) = \int ds_1ds_2\frac{\rho_\mu^{pert}(s_1,s_2,q^2)}{(s_1-p_1^2)(s_2-p_2^2)}
$$

Here the spectrum function $\rho_\mu^{pert}(s_1,s_2,q^2)$ can be obtained by cutting rule:

$$
\begin{aligned}
\rho_\mu^{pert}(s_1,s_2,q^2) &= \frac{1}{(2\pi i)^2}(-12)(-2\pi i)^4\int\frac{d^4k_3}{(2\pi)^4}\frac{d^4k_4}{(2\pi)^4}d^4k_1d^4k_2\cdot N_\mu \\
&\times \delta(k_1^2-M_1^2)\delta(k_2^2-m_1^2)\delta(k_3^2-M_2^2)\delta(k_4^2-m_2^2)
\end{aligned}
$$

$$
\begin{aligned}
&\times \delta^4(p_2-k_3-k_4-k_2)\delta^4(p_1-k_3-k_4-k_1) \\
&= \frac{1}{(2\pi i)^2}(-12)(-2\pi i)^4 \int dm_{34}^2 \int d^4k_1 d^4k_2 d^4k_{34} \delta(k_1^2-M_1^2)\delta(k_2^2-m_1^2)\delta(k_{34}^2-m_{34}^2) \\
&\times \int \frac{d^4k_3}{(2\pi)^4}\frac{d^4k_4}{(2\pi)^4}\delta(k_3^2-M_2^2)\delta(k_4^2-m_2^2)\cdot N_\mu \cdot \delta^4(p_1-k_{34}-k_1)\delta^4(k_{34}-k_3-k_4)\delta^4(p_2-k_{34}-k_2) \\
&= \frac{1}{(2\pi i)^2}(-12)(-2\pi i)^4 \frac{1}{(2\pi)^2}\int dm_{34}^2 \\
&\times \int d^4k_1 d^4k_2 d^4k_{34}\delta(k_1^2-M_1^2)\delta(k_2^2-m_1^2)\delta(k_{34}^2-m_{34}^2)\delta^4(p_1-k_{34}-k_1)\delta^4(p_2-k_{34}-k_2) \\
&\times \int \frac{d^3k_3}{(2\pi)^3}\frac{1}{2E_3}\frac{d^3k_4}{(2\pi)^3}\frac{1}{2E_4}\delta^4(k_{34}-k_3-k_4)\cdot N_\mu \\
&= \frac{1}{(2\pi i)^2}(-12)(-2\pi i)^4\frac{1}{(2\pi)^2}\int dm_{34}^2 \int d\Phi_\Delta \int d\Phi_2 \cdot N_\mu \\
&= 12\int dm_{34}^2 \int d\Phi_\Delta \int d\Phi_2 \cdot N_\mu
\end{aligned}
$$

Now the full integral has been factorized into the convolution of a cutted triangle diagram integral and a two-body phase space integral.

Triangle diagram

In this section we calculate the cutted triangle diagram with the approach of Passarino-Veltman Integrals. We firstly consider the scalar integral where $N_\mu = 1$. Actually the dynamics of the triangle diagram are totally determined by the scalar integral, while other higher rank tensor integrals are just proportional to the scalar one.

Here we choose the CM frame of p_1+p_2, where $\vec{p}_1 = -\vec{p}_2$, the scalar integral is:

$$
\begin{aligned}
I_\Delta &= \int d\Phi_\Delta \cdot 1 \\
&= \int d^4k\delta(k^2-m^2)\delta((p_1-k)^2-m_1^2)\delta((p_2-k)^2-m_2^2) \\
&= \int d^3\vec{k}\frac{1}{2E_k}\delta(p_1^2-2p_1\cdot k+k^2-m_1^2)\delta(p_2^2-2p_2\cdot k+k^2-m_2^2) \\
&= \int d|\vec{k}||\vec{k}|^2 dcos\theta d\phi \frac{1}{2E_k}\delta(s_1+m^2-m_1^2-2E_kp_1^0+2|\vec{k}||\vec{p}_1|cos\theta)\delta(s_2+m^2-m_2^2-2E_kp_2^0-2|\vec{k}||\vec{p}_1|cos\theta) \\
&= \int \frac{d|\vec{k}||\vec{k}|^2}{2E_k}(2\pi)\frac{1}{2|\vec{k}||\vec{p}_1|}\delta(s_1+s_2+2m^2-m_1^2-m_2^2-2E_k(p_1^0+p_2^0)) \\
&= \frac{1}{4}\int dE_k\frac{1}{|\vec{p}_1|}(2\pi)\delta(2E_k(p_1^0+p_2^0)+m_1^2+m_2^2-2m^2-s_1-s_2) \\
&= \frac{\pi}{2}\frac{1}{|\vec{p}_1|}\frac{1}{2(p_1^0+p_2^0)}\times ThetaFunctions
\end{aligned}
$$

The integration over $cos\theta$ produce a condition:

$$
-1 < \frac{s_2+m^2-m_2^2-2E_kp_2^0}{2|\vec{k}||\vec{p}_1|} < 1
$$

Which can be rewritten as:

$$
4s_2E_k^2 - 4p_2^0E_k(m^2+s_2)+(m^2+s_2)^2+4m^2|\vec{p}_1|^2 < 0
$$

The last integration over E_k is nonzero since the pole of the delta function is positive:

$$E_k = \frac{s_1 + s_2 + 2m^2 - m_1^2 - m_2^2}{2(p_1^0 + p_2^0)} > 0$$

Pluging it into the above condition, and using the relations:

$$\begin{aligned}
|\vec{p}_1|^2 &= \frac{\lambda}{4(p_1+p_2)^2} = \frac{\lambda}{4s} \\
(p_2^0)^2 &= \vec{p}_1^{\,2} + s_2 = \frac{\lambda}{4s} + s_2 \\
s &= 2(s_1+s_2) - q^2 \\
\lambda &= (s_1+s_2-q^2)^2 - 4s_1s_2
\end{aligned}$$

We have

$$s_1 > \frac{m_1^4 s_2 - m_1^2 q^2 s_2 + m_1^2 s_2^2 - m_1^2 q^2 m^2 + (q^2)^2 m^2 - m_1^2 s_2 m^2 - q^2 s 2m^2 + q^2 m^4}{(m_1^2 - q^2)(s_2 - m^2)}$$

In this problem m_2 is the mass of light quark and has been taken to be zero in the theta functions.

$$I_\Delta = \frac{\pi}{2\sqrt{\lambda}}\theta(s_1 - \frac{m_1^4 s_2 - m_1^2 q^2 s_2 + m_1^2 s_2^2 - m_1^2 q^2 m^2 + (q^2)^2 m^2 - m_1^2 s_2 m^2 - q^2 s 2m^2 + q^2 m^4}{(m_1^2 - q^2)(s_2 - m^2)})\theta(s_2 - m^2)$$

Then consider higher rank tensor integrals. For rank one integral:

$$\begin{aligned}
I_\Delta^\mu &= \int d\Phi_\Delta \cdot k^\mu \\
&= \int d^4k k^\mu \delta(k^2 - m^2)\delta((p_1-k)^2 - m_1^2)\delta((p_2-k)^2 - m_2^2) \\
&= a_1(s_1, s_2, q^2)p_1^\mu + a_2(s_1, s_2, q^2)p_2^\mu
\end{aligned}$$

To obtain the coefficients a_1 and a_2, we need to contract $p_{1\mu}$ and $p_{2\mu}$ on the both sides of the equation. Using the on-shell condition:

$$\begin{aligned}
&(p_1 - k)^2 - m_1^2 = 0 \\
&\qquad \Longrightarrow p_1 \cdot k = \frac{1}{2}(s_1 + m^2 - m_1^2)
\end{aligned}$$

and

$$p_1 \cdot p_2 = \frac{1}{2}(s_1 + s_2 - q^2)$$

$$p_{1\mu} I_\Delta^\mu = \frac{1}{2}(s_1 + m^2 - m_1^2)\int d^4k \delta(k^2 - m^2)\delta((p_1-k)^2 - m_1^2)\delta((p_2-k)^2 - m_2^2) = \frac{1}{2}(s_1 + m^2 - m_1^2) I_\Delta$$

Similarly we have $\bar{\nu}$

$$p_{2\mu} I_\Delta^\mu = \frac{1}{2}(s_2 + m^2 - m_2^2) I_\Delta$$

a_1 and a_2 are obtained by solving the equations:

$$
\begin{aligned}
s_1 a_1 + \frac{1}{2}(s_1 + s_2 - q^2) b_1 &= \frac{1}{2}(s_1 + m^2 - m_1^2) I_\Delta \\
s_2 b_1 + \frac{1}{2}(s_1 + s_2 - q^2) a_1 &= \frac{1}{2}(s_2 + m^2 - m_2^2) I_\Delta
\end{aligned}
$$

$$
a_1 = -\frac{\frac{1}{2} I_\Delta s_2 \left(m^2 - m_1^2 + s_1\right) - \frac{1}{4} I_\Delta \left(m^2 - m_2^2 + s_2\right)\left(-q^2 + s_1 + s_2\right)}{\frac{1}{4}\left(-q^2 + s_1 + s_2\right)^2 - s_1 s_2}
$$

$$
b_1 = -\frac{\frac{1}{2} I_\Delta s_1 \left(m^2 - m_2^2 + s_2\right) - \frac{1}{4} I_\Delta \left(m^2 - m_1^2 + s_1\right)\left(-q^2 + s_1 + s_2\right)}{\frac{1}{4}\left(-q^2 + s_1 + s_2\right)^2 - s_1 s_2}
$$

The rank two integral is parameterized as

$$
\begin{aligned}
I_\Delta^{\mu\nu} &= \int d\Phi_\Delta \cdot k^\mu k^\nu \\
&= \int d^4k k^\mu k^\nu \delta(k^2 - m^2)\delta((p_1 - k)^2 - m_1^2)\delta((p_2 - k)^2 - m_2^2) \\
&= a_2 p_1^\mu p_1^\nu + b_2 p_2^\mu p_2^\nu + c_2(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu) + d_2 g^{\mu\nu}
\end{aligned}
$$

$$
\begin{aligned}
a_2 = &-\frac{I_\Delta}{16\left(s_1 s_2 - \frac{1}{4}\left(-q^2 + s_1 + s_2\right)^2\right)^2} \\
&\times \Big(-\left(m^2 - m_2^2 + s_2\right)^2\left(-q^2 + s_1 + s_2\right)^2 - 2m^2 s_2\left(-q^2 + s_1 + s_2\right)^2 + 6s_2\left(m^2 - m_1^2 + s_1\right)\left(m^2 - m_2^2 + s_2\right)\left(-q^2 + s_1 + s_2\right) \\
&- 6s_2^2\left(m^2 - m_1^2 + s_1\right)^2 + 8m^2 s_1 s_2^2 - 2s_1 s_2\left(m^2 - m_2^2 + s_2\right)^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
b_2 = &-\frac{I_\Delta}{16\left(s_1 s_2 - \frac{1}{4}\left(-q^2 + s_1 + s_2\right)^2\right)^2} \\
&\times \Big(-\left(m^2 - m_1^2 + s_1\right)^2\left(-q^2 + s_1 + s_2\right)^2 - 2m^2 s_1\left(-q^2 + s_1 + s_2\right)^2 + 6s_1\left(m^2 - m_1^2 + s_1\right)\left(m^2 - m_2^2 + s_2\right)\left(-q^2 + s_1 + s_2\right) \\
&-6s_1^2\left(m^2 - m_2^2 + s_2\right)^2 + 8m^2 s_2 s_1^2 - 2s_2 s_1\left(m^2 - m_1^2 + s_1\right)^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
c_2 = &-\frac{I_\Delta}{16\left(s_1 s_2 - \frac{1}{4}\left(-q^2 + s_1 + s_2\right)^2\right)^2} \\
&\times \Big(m^2\left(-q^2 + s_1 + s_2\right)^3 - 2\left(m^2 - m_1^2 + s_1\right)\left(m^2 - m_2^2 + s_2\right)\left(-q^2 + s_1 + s_2\right)^2 + 3s_1\left(m^2 - m_2^2 + s_2\right)^2\left(-q^2 + s_1 + s_2\right) \\
&-4m^2 s_1 s_2\left(-q^2 + s_1 + s_2\right) + 3s_2\left(m^2 - m_1^2 + s_1\right)^2\left(-q^2 + s_1 + s_2\right) - 4s_1 s_2\left(m^2 - m_1^2 + s_1\right)\left(m^2 - m_2^2 + s_2\right)\Big)
\end{aligned}
$$

$$
\begin{aligned}
d_2 = &\frac{I_\Delta}{8\left(\frac{1}{4}\left(-q^2 + s_1 + s_2\right)^2 - s_1 s_2\right)} \\
&\times \Big(m^2\left(-q^2 + s_1 + s_2\right)^2 - \left(m^2 - m_1^2 + s_1\right)\left(m^2 - m_2^2 + s_2\right)\left(-q^2 + s_1 + s_2\right) - 4m^2 s_1 s_2 \\
&+ s_1\left(m^2 - m_2^2 + s_2\right)^2 + s_2\left(m^2 - m_1^2 + s_1\right)^2\Big)
\end{aligned}
$$

The rank three integral is parameterized as

$$
\begin{aligned}
I_\Delta^{\mu\nu\rho} &= \int d\Phi_\Delta \cdot k^\mu k^\nu k^\rho \\
&= \int d^4k k^\mu k^\nu k^\rho \delta(k^2 - m^2)\delta((p_1 - k)^2 - m_1^2)\delta((p_2 - k)^2 - m_2^2) \\
&= a_3 p_1^\mu p_1^\nu p_1^\rho + b_3(p_1^\mu g^{\nu\rho} + p_1^\nu g^{\mu\rho} + p_1^\rho g^{\mu\nu})p_1^2 \\
&+ c_3 p_2^\mu p_2^\nu p_2^\rho + d_3(p_2^\mu g^{\nu\rho} + p_2^\nu g^{\mu\rho} + p_2^\rho g^{\mu\nu})p_2^2 \\
&+ e_3(p_1^\mu p_2^\nu p_2^\rho + p_1^\nu p_2^\mu p_2^\rho + p_1^\rho p_2^\mu p_2^\nu) \\
&+ f_3(p_2^\mu p_1^\nu p_1^\rho + p_2^\nu p_1^\mu p_1^\rho + p_2^\rho p_1^\mu p_1^\nu)
\end{aligned}
$$

$$
\begin{aligned}
e_3(p_1^\mu p_2^\nu p_2^\rho + p_1^\nu p_2^\mu p_2^\rho + p_1^\rho p_2^\mu p_2^\nu) e_3(p_1^\mu p_2^\nu p_2^\rho + p_1^\nu p_2^\mu p_2^\rho + p_1^\rho p_2^\mu p_2^\nu) &= e_3(p_1^\mu p_2^\nu p_2^\rho + p_1^\nu p_2^\mu p_2^\rho + p_1^\rho p_2^\mu p_2^\nu) \\
e_3(p_1^\mu p_2^\nu p_2^\rho + p_1^\nu p_2^\mu p_2^\rho + p_1^\rho p_2^\mu p_2^\nu) e_3(p_1^\mu p_2^\nu p_2^\rho + p_1^\nu p_2^\mu p_2^\rho + p_1^\rho p_2^\mu p_2^\nu) & \\
e_3(p_1^\mu p_2^\nu p_2^\rho + p_1^\nu p_2^\mu p_2^\rho + p_1^\rho p_2^\mu p_2^\nu) e_3(p_1^\mu p_2^\nu p_2^\rho + p_1^\nu p_2^\mu p_2^\rho + p_1^\rho p_2^\mu p_2^\nu) &
\end{aligned}
$$

Two-body phase space integrals

Here the two-body phase space will be shown

$$
\begin{aligned}
& \int \frac{d^3k_1}{2E_1} \frac{d^3k_2}{2E_2} \delta^4(p - k_1 - k_2) \\
= & \int \frac{d^3k_1}{2E_1 2E_2} \delta(M - E_1 - E_2) \\
= & \int \frac{d|\vec{k}_1||\vec{k}_1|^2 d\cos\theta d\phi}{2E_1 2E_2} \delta(M - E_1 - E_2) \\
= & \frac{4\pi}{4} \int d|\vec{k}_1| \frac{|\vec{k}_1|^2}{\sqrt{|\vec{k}_1|^2 + m_1^2}\sqrt{|\vec{k}_1|^2 + m_2^2}} \delta(M - \sqrt{|\vec{k}_1|^2 + m_1^2} - \sqrt{|\vec{k}_1|^2 + m_2^2}) \\
= & \frac{4\pi}{4} \int d|\vec{k}_1| \frac{|\vec{k}_1|^2}{E_1 E_2} \frac{1}{\frac{|\vec{k}_1|}{E_1} + \frac{|\vec{k}_1|}{E_2}} \delta(|\vec{k}_1| - \cdots) \\
= & \frac{4\pi}{4} \int d|\vec{k}_1| \frac{|\vec{k}_1|}{E_1 + E_2} \delta(|\vec{k}_1| - \cdots) \\
= & \pi \frac{|\vec{k}_1|}{M} \\
= & \frac{\pi\sqrt{\lambda}}{2M^2}
\end{aligned}
$$

$$p = k_3 + k_4$$

$$I_0(p^2) = \int d\phi_2 \cdot 1 = \int \frac{d^3k_3}{(2\pi)^3} \frac{1}{2E_3} \frac{d^3k_4}{(2\pi)^3} \frac{1}{2E_4} \delta^4(p - k_3 - k_4) \cdot 1$$

$$= \frac{\pi\sqrt{(p^2-(M_2+m_2)^2)\,(p^2-(M_2-m_2)^2)}}{(2\pi)^6 2p^2}$$

$$I_3^\mu(p^2) = \int d\phi_2 \cdot k_3^\mu = A_1^{(3)} p^\mu, \qquad I_4^\mu(p^2) = \int d\phi_2 \cdot k_4^\mu = A_1^{(4)} p^\mu$$

$$A_1^{(3)} = \frac{I_0\left(-m_2^2+M_2^2+p^2\right)}{2p^2}, \qquad A_1^{(4)} = \frac{I_0\left(m_2^2-M_2^2+p^2\right)}{2p^2}$$

$$\begin{aligned} I_{34}^{\mu\nu} &= \int d\phi_2 \cdot k_3^\mu k_4^\nu = A_2^{(34)} p^\mu p^\nu + B_2^{(34)} g^{\mu\nu} p^2 \\ I_{44}^{\mu\nu} &= \int d\phi_2 \cdot k_4^\mu k_4^\nu = A_2^{(44)} p^\mu p^\nu + B_2^{(44)} g^{\mu\nu} p^2 \end{aligned}$$

$$\begin{aligned} A_2^{(34)} &= -\frac{\frac{1}{2} I_0 p^2\left(-m_2^2-M_2^2+p^2\right) - I_0\left(m_2^2-M_2^2+p^2\right)\left(-m_2^2+M_2^2+p^2\right)}{3(p^2)^2} \\ B_2^{(34)} &= -\frac{\frac{1}{4} I_0\left(m_2^2-M_2^2+p^2\right)\left(-m_2^2+M_2^2+p^2\right) - \frac{1}{2} I_0 p^2\left(-m_2^2-M_2^2+p^2\right)}{3(p^2)^2} \end{aligned}$$

$$A_2^{(44)} = -\frac{I_0 m_2^2 p^2 - I_0\left(m_2^2-M_2^2+p^2\right)^2}{3(p^2)^2}, \qquad B_2^{(44)} - \frac{\frac{1}{4} I_0\left(m_2^2-M_2^2+p^2\right)^2 - I_0 m_2^2 p^2}{3(p^2)^2}$$

$$\begin{aligned} I_{334}^{\mu\nu\rho} &= \int d\phi_2 \cdot k_3^\mu k_3^\nu k_4^\rho \\ &= A_3^{(334)} p^\mu p^\nu p^\rho + B_3^{(334)} g^{\mu\nu} p^\rho p^2 + C_3^{(334)} (g^{\mu\rho} p^\nu + g^{\nu\rho} p^\mu) p^2 \end{aligned}$$

$$\begin{aligned} I_{344}^{\mu\nu\rho} &= \int d\phi_2 \cdot k_3^\mu k_4^\nu k_4^\rho \\ &= A_3^{(344)} p^\mu p^\nu p^\rho + B_3^{(344)} g^{\mu\nu} p^\rho p^2 + C_3^{(344)} (g^{\mu\rho} p^\nu + g^{\nu\rho} p^\mu) p^2 \end{aligned}$$

$$\begin{aligned} A_3^{(334)} &= -\frac{1}{3(p^2)^3}[\frac{1}{2} I_0 M_2^2 p^2\left(m_2^2-M_2^2+p^2\right) + \frac{1}{2} I_0 p^2\left(-m_2^2-M_2^2+p^2\right)\left(-m_2^2+M_2^2+p^2\right) \\ &\quad - \frac{3}{4} I_0\left(m_2^2-M_2^2+p^2\right)\left(-m_2^2+M_2^2+p^2\right)^2] \\ B_3^{(334)} &= -\frac{\frac{1}{8} I_0\left(m_2^2-M_2^2+p^2\right)\left(-m_2^2+M_2^2+p^2\right)^2 - \frac{1}{2} I_0 M_2^2 p^2\left(m_2^2-M_2^2+p^2\right)}{3(p^2)^3} \\ C_3^{(334)} &= -\frac{\frac{1}{8} I_0\left(m_2^2-M_2^2+p^2\right)\left(-m_2^2+M_2^2+p^2\right)^2 - \frac{1}{4} I_0 p^2\left(-m_2^2-M_2^2+p^2\right)\left(-m_2^2+M_2^2+p^2\right)}{3(p^2)^3} \end{aligned}$$

$$\begin{aligned} A_3^{(344)} &= -\frac{\frac{3}{4} I_0 p^2\left(-m_2^2-M_2^2+p^2\right)\left(m_2^2-M_2^2+p^2\right) - \frac{3}{4} I_0\left(m_2^2-M_2^2+p^2\right)^2\left(-m_2^2+M_2^2+p^2\right)}{3(p^2)^3} \\ B_3^{(344)} &= -\frac{\frac{1}{8} I_0\left(m_2^2-M_2^2+p^2\right)^2\left(-m_2^2+M_2^2+p^2\right) - \frac{1}{4} I_0 p^2\left(-m_2^2-M_2^2+p^2\right)\left(m_2^2-M_2^2+p^2\right)}{3(p^2)^3} \\ C_3^{(344)} &= -\frac{\frac{1}{8} I_0\left(m_2^2-M_2^2+p^2\right)^2\left(-m_2^2+M_2^2+p^2\right) - \frac{1}{4} I_0 p^2\left(-m_2^2-M_2^2+p^2\right)\left(m_2^2-M_2^2+p^2\right)}{3(p^2)^3} \end{aligned}$$

$\langle \bar{q}q \rangle$ condensate

The light quark condensate can be expanded in the power of space time distance:

$$\langle 0|\bar{q}^a_\alpha(x) q^b_\beta(y)|0\rangle = \delta^{ab}\langle\bar{q}q\rangle[\frac{1}{12}\delta_{\beta\alpha} + i\frac{m_q}{48}(\not{x}-\not{y})_{\beta\alpha} - \frac{m_q^2}{96}(x-y)^2\delta_{\beta\alpha}\cdots]$$

Since the light quarks' mass can be neglected, here we only need the first term.

There's two light quark lines and each one can condensate in the background of QCD vaccum. We first consider the condensate of q_2.

$$\begin{aligned}
\Pi^{\langle\bar{q}_2q_2\rangle}_\mu(p_1^2,p_2^2,q^2) &= i^2\int d^4x d^4y e^{ip_2\cdot x - ip_1\cdot y}\langle 0|T\{j_h(x)j_\mu(0)\bar{j}_H(y)\}|0\rangle^{\langle\bar{q}_2q_2\rangle} \\
&= 12i^2\int d^4x d^4y e^{ip_2\cdot x - ip_1\cdot y}\frac{1}{12}\langle\bar{q}q\rangle \\
&\times [\gamma_\alpha\gamma_5 S_Q(x-y)\gamma^\beta C S_Q^T(-y) C(1-\gamma_5)\gamma_\mu C S_{q_1}^T(x) C\gamma^\alpha\gamma_\beta\gamma_5] \\
&= i\langle\bar{q}q\rangle\int\frac{d^4k_3}{(2\pi)^4}\frac{N^{\langle\bar{q}_2q_2\rangle}_\mu}{(k_3^2-M_2^2)(k_1^2-M_1^2)(k_2^2-m_1^2)}
\end{aligned}$$

Where

$$k_1 = p_1 - k_3, \quad k_2 = p_2 - k_3$$

$$N^{\langle\bar{q}_2q_2\rangle}_\mu = \gamma_\alpha\gamma_5(\not{k}_3 + m_Q)\gamma^\beta(\not{k}_1 - m_Q)(1-\gamma_5)\gamma_\mu(\not{k}_2 - m_1)\gamma^\alpha\gamma_\beta\gamma_5$$

$$\begin{aligned}
\rho^{\langle\bar{q}_2q_2\rangle}_\mu(p_1^2,p_2^2,q^2) &= i\langle\bar{q}q\rangle(-2\pi i)^3\int\frac{d^4k_3}{(2\pi)^4}\delta(k_3^2-M_2^2)\delta((p_1-k_3)^2-M_1^2)\delta((p_2-k_3)^2-m_1^2)N^{\langle\bar{q}_2q_2\rangle}_\mu \\
&= i\langle\bar{q}q\rangle(-2\pi i)^3\frac{1}{(2\pi)^4}\int d\Phi_\Delta N^{\langle\bar{q}_2q_2\rangle}_\mu
\end{aligned}$$

We then consider the condensate of q_1.

$$\Pi^{\langle\bar{q}_1q_1\rangle}_\mu(p_1^2,p_2^2,q^2) \sim \int\frac{d^4k_3}{(2\pi)^4}\frac{N^{\langle\bar{q}_1q_1\rangle}_\mu}{(q^2-M_1^2)(k_3^2-M_2^2)((p_2-k_3)^2-m_2^2)}$$

It can be found that this expression is independent of p_1^2. So when we translate it into the form of double dispersion integral, the double discontinuity $\rho^{\langle\bar{q}_1q_1\rangle}_\mu(p_1^2,p_2^2,q^2)$ vanishes since there's no discontinuity over the variable p_1^2.

Form Factors

$f_1(q^2)$

There are four Lorentz structures can be used to extract $f_1(q^2)$:γ_μ, $\gamma_\mu\not{p}_1$, $\gamma_\mu\not{p}_2$, $\gamma_\mu\not{p}_2\not{p}_1$. Here we firstly use γ_μ.

$$\frac{f_h f_H m_h m_H}{(p_2^2-m_h^2)(p_1^2-m_H^2)}f_1^{pert}(q^2) + \int_{s_0^{(1)}} ds_1\int_{s_0^{(2)}} ds_2\frac{\rho^h(s_1,s_2,q^2)}{(s_1-p_1^2)(s_2-p_2^2)} = \int_{(M_1+M_2)^2} ds_1\int_{M_2^2} ds_2\frac{\rho^{pert}(s_1,s_2,q^2)}{(s_1-p_1^2)(s_2-p_2^2)}$$

$$\frac{f_h f_H m_h m_H}{(p_2^2 - m_h^2)(p_1^2 - m_H^2)} f_1(q^2) = \int_{(M_1+M_2)^2}^{s_0^{(1)}} ds_1 \int_{M_2^2}^{s_0^{(2)}} ds_2 \frac{\rho^{pert}(s_1, s_2, q^2)}{(s_1 - p_1^2)(s_2 - p_2^2)}$$

Where

Borel transformation:

$$f_1^{pert}(q^2) = \frac{12}{f_h f_H m_h m_H} exp(\frac{m_H^2}{M_1^2}) exp(\frac{m_h^2}{M_2^2}) \int m_{34}^2 \int_{(M_1+M_2)^2}^{s_0^{(1)}} ds_1 \int_{M_2^2}^{s_0^{(2)}} ds_2 exp(-\frac{s_1}{M_1^2}) exp(-\frac{s_2}{M_2^2}) [\int d\Phi_\Delta \int d\Phi_2 \cdot N_{pert}]$$