# CP CONSERVATION
# IN THE STRONG INTERACTIONS

BJÖRN GARBRECHT, TUM

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CALTECH

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# Outline

- Introduction: QCD $\theta$-parameter, EFT, neutron EDM, topology
- Functional quantization and path integral contour
- Canonical quantization and $\theta$-vacua

# INTRODUCTION

CP-odd terms in effective field theories

Topology

# $CP$ violation in the strong interactions?

No empirical evidence—neutron electric dipole moment (EDM) strongly constrained:

$$d_n = (0.0 \pm 1.1_{\text{stat}} \pm 0.2_{\text{sys}}) \times 10^{-26}\, e\,\text{cm}$$ [2020 @ PSI]

QCD with massive quarks

$$\mathcal{L} \supset \frac{1}{2g^2} \operatorname{tr} F_{\mu\nu} F^{\mu\nu} + \sum_{j=1}^{N_f} \bar{\psi}_j \left( \mathrm{i} \not{D} - m_j \mathrm{e}^{\mathrm{i}\alpha_j \gamma^5} \right) \psi_j + \frac{1}{16\pi^2} \theta \operatorname{tr} F_{\mu\nu} \tilde{F}^{\mu\nu}$$

Believed to cause a neutron electric dipole moment (EDM) $d_n \sim 10^{-15}\, e\,\text{cm} \left( \theta + \sum_j \alpha_j \right)$
[Baluni (1979); Crewther, Di Vecchia, Veneziano, Witten (1979)]

*Or does it?*

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$\propto \vec{E}^2 + \vec{B}^2$

$\gamma^5$, parity-odd

$\propto \vec{E} \cdot \vec{B}$
parity-odd

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total derivative
→ EFT

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*Or does it?*

# Effective interactions with $\theta$

$\mathrm{SU}(N_f)_\mathrm{L} \times \mathrm{SU}(N_f)_\mathrm{R}$ of flavour global symmetry in the limit of massless quarks

Chiral $\mathrm{U}(1)_A$ symmetry of the quarks is anomalous however
$\longrightarrow \mathcal{L}$ invariant under [Fujikawa (1979,80)]

**chiral trafo**

$$\psi \to e^{i\beta\gamma_5}\psi$$
$$\bar{\psi} \to \bar{\psi}e^{i\beta\gamma_5}$$

**plus**

**“spurion” trafo**

$$m_j e^{i\alpha_j\gamma^5} \to m_j e^{i(\alpha_j-2\beta)\gamma^5}$$
$$\theta \to \theta + 2N_f\beta$$

Spurions break the symmetries explicitly. $\longrightarrow$ Approximate symmetries
This pattern should be replicated by any effective theory.

Rephasing invariant: $\bar{\theta} = \theta + \bar{\alpha}$, where $\bar{\alpha} = \sum_{j=1}^{N_f} \alpha_j$, $\longrightarrow$ $\theta$ is an angle

# Integrating out gauge fields: Effective ’t Hooft vertex

Tpological effects described by effective ’t Hooft vertex ($\Gamma_{N_f}$ some coefficient): ['t Hooft (1976,86)]

$$\mathcal{L} + \frac{1}{16\pi^2}\theta\,\mathrm{tr}\,F_{\mu\nu}\tilde{F}^{\mu\nu} \to \mathcal{L} - \Gamma_{N_f}\mathrm{e}^{\mathrm{i}\xi}\prod_{j=1}^{N_f}(\bar{\psi}_j P_\mathrm{L}\psi_j) - \Gamma_{N_f}\mathrm{e}^{-\mathrm{i}\xi}\prod_{j=1}^{N_f}(\bar{\psi}_j P_\mathrm{R}\psi_j)$$

- Effective interaction breaks $\mathrm{U}(1)_A$ explicitly $\longrightarrow$ $\eta'$-mass
- $\xi$ should be expressed in terms of parameters of the fundamental theory
- As a spurion, $\xi \to \xi + 2N_f\beta$

Two options:
$\xi = \theta$ (in general misaligned with masses) $\to$ $CP$ violation
$\xi = -\bar{\alpha}$ (present claim, aligned with mass terms) $\to$ no $CP$ violation

*So which one is it?*

In principle, we could have $\xi = c_\alpha\bar{\alpha} + c_\theta\theta$ ($\alpha$, $\theta$ are angular variables) with $-c_\alpha + c_\theta = 1$.

# Effective chiral Lagrangian ($\chi$PT)

$$U = U_0 \mathrm{e}^{\frac{\mathrm{i}}{f_\pi}\Phi} \qquad U_0 : \textit{chiral condensate}$$

$$\Phi = \begin{bmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta + \sqrt{\frac{2}{3}}\eta' & \sqrt{2}\,\pi^+ & \sqrt{2}\,K^+ \\ \sqrt{2}\,\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta + \sqrt{\frac{2}{3}}\eta' & \sqrt{2}\,K^0 \\ \sqrt{2}\,K^- & \sqrt{2}\,\bar{K}^0 & -\frac{2}{\sqrt{3}}\,\eta + \sqrt{\frac{2}{3}}\eta' \end{bmatrix}$$

Chiral Lagrangian (lowest order terms) inherits “spurious” symmetries:

$$\mathcal{L} = \frac{f_\pi^2}{4} \mathrm{Tr}\, \partial_\mu U \partial^\mu U^\dagger + \frac{f_\pi^2 B_0}{2} \mathrm{Tr}(M\, U + U^\dagger M^\dagger) + |\lambda| \mathrm{e}^{-\mathrm{i}\xi} f_\pi^4 \det U + |\lambda| \mathrm{e}^{\mathrm{i}\xi} f_\pi^4 \det U^\dagger$$
$$+ \mathrm{i}\bar{N}\partial\!\!\!/ N - \Big( m_N \bar{N}\, \tilde{U}\, P_{\mathrm{L}} N + \mathrm{i}c\bar{N}\, \tilde{U}^\dagger \partial\!\!\!/ P_{\mathrm{L}}\, \tilde{U}\, N + d\,\bar{N}\, \tilde{M}^\dagger P_{\mathrm{L}} N + e\,\bar{N}\, \tilde{U}\, \tilde{M}\, \tilde{U}\, P_{\mathrm{L}} N + \mathrm{h.c.}\Big)$$

$$M = \mathrm{diag}\{m_u \mathrm{e}^{\mathrm{i}\alpha_u}, m_d \mathrm{e}^{\mathrm{i}\alpha_d}, m_s \mathrm{e}^{\mathrm{i}\alpha_s}\}$$

$\tilde{M}$, $\tilde{U}$ reduced to subspace $(u, d)$

nucleon doublet $N = \begin{pmatrix} p \\ n \end{pmatrix}$

Effective interaction $\propto \det U$ cannot be quantitatively reliably handled in $\chi$PT but yet represents pattern of broken axial symmetry.

# Neutron electric dipole moment

$\xi = \theta$: $\chi$ral condensate aligned with $\theta$

$\xi = -\bar{\alpha}$: $\chi$ral condensate aligned with quark mass phases

- Write $U_0 = \langle U \rangle = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\varphi_u}, \mathrm{e}^{\mathrm{i}\varphi_d}, \mathrm{e}^{\mathrm{i}\varphi_s})$, minimize $V(\langle U \rangle)$ $\rightarrow m_i(\varphi_i + \alpha_i) \approx \bar{m}(m_u, m_d, m_s)(\xi + \alpha_u + \alpha_d + \alpha_s)$ (for small angles)
- Substitute $\varphi_i$ back into $\mathcal{L}$ & suitably redefine $N \rightarrow \mathcal{N}(N, U)$

$$\mathcal{L}_{\text{neutron}} \supset -\frac{2c+1}{f_\pi}\partial_\mu \pi^a \bar{\mathcal{N}} T^a \gamma^\mu \gamma_5 \mathcal{N} \qquad CP \text{ even}$$

$$+\frac{2(d+e)\bar{m}}{f_\pi}(\xi + \alpha_u + \alpha_d + \alpha_s)\bar{\mathcal{N}} \pi^a T^a \mathcal{N} \qquad CP \text{ odd}$$

- $\chi$PT value: $d_n = 3.2 \times 10^{-16}(\xi + \bar{\alpha})e\,\mathrm{cm}$
- Calculations e.g. of neutron EDM implicitly *assume* $\xi = \theta$
  [e.g. Baluni (1979); Crewther, Di Vecchia, Veneziano, Witten (1979)]
- However $\xi = -\bar{\alpha}$ also perfectly valid by spurion arguments
- Another signature—weaker bounds: $\eta' \rightarrow \pi\pi$

# Topology in four-dimensional spacetime—winding number $\Delta n$

$$U = \begin{pmatrix} a_{\mathrm{R}} + \mathrm{i}a_{\mathrm{I}} & -b_{\mathrm{R}} + \mathrm{i}b_{\mathrm{I}} \\ b_{\mathrm{R}} + \mathrm{i}b_{\mathrm{I}} & a_{\mathrm{R}} - \mathrm{i}a_{\mathrm{I}} \end{pmatrix} \in \mathrm{SU}(2) \text{ for } a_{\mathrm{R}}^2 + a_{\mathrm{I}}^2 + b_{\mathrm{R}}^2 + b_{\mathrm{I}}^2 = 1$$

$\Rightarrow$Homotopy: local $\mathrm{SU}(3) \supset \mathrm{SU}(2) \cong S^3 \longrightarrow \pi_3(\mathrm{SU}(2)) = \pi_3(S^3) = \mathbb{Z}$

Theta-term/topological term is a total divergence:

gauge invariant

$$\frac{1}{4}\mathrm{tr}\, F_{\mu\nu}\tilde{F}_{\mu\nu} = \partial_\mu K_\mu$$

gauge dependent

$$K_\mu = \epsilon_{\mu\nu\alpha\beta}\mathrm{tr}\left[\frac{1}{2}A_\nu \partial_\alpha A_\beta + \frac{1}{3}A_\nu A_\alpha A_\beta\right]$$

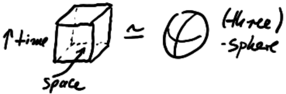
### Topological quantization for pure gauge $A_\mu \to -\frac{\mathrm{i}}{g}(\partial_\mu U)U^{-1}$ at $\partial\Omega \cong S^3$

$$\Delta n = \frac{1}{16\pi^2}\int_\Omega \mathrm{d}^4x F_{\mu\nu}\tilde{F}_{\mu\nu} = \frac{1}{4\pi^2}\oint_{\partial\Omega} \mathrm{d}^3\sigma K_\perp \in \mathbb{Z}$$

Haar measure for pure gauge

$K_\mu = \frac{1}{6}\varepsilon_{\mu\nu\lambda\rho}\mathrm{tr}[(U^{-1}\partial_\nu U)(U^{-1}\partial_\lambda U)(U^{-1}\partial_\rho U)]$

E.g. take boundary of $\Omega = \mathbb{R}^4$ as a sphere $S^3$:



Or $\Omega = T^4$(lattice), $\Omega = S^4$(Euclidean dS): $\Delta n \in \mathbb{Z}$ based on slightly more involved argument

# Topology on spatial hypersurfaces—point compactification, large gauge transformations

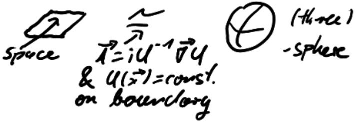
Consider temporal gauge $A^0 = 0$ (in view of canonical quantization)

Chern–Simons functional:

$$W[\vec{A}] = \frac{1}{4\pi^2}\varepsilon_{ijk}\int_V \mathrm{d}^3x\,\mathrm{tr}\left[\frac{1}{2}A_i\partial_j A_k - \frac{\mathrm{i}}{3}A_iA_jA_k\right] \equiv \frac{1}{4\pi^2}\int_V \mathrm{d}^3x\,K_0$$

Define $\vec{A}_U = U\vec{A}U^{-1} + \mathrm{i}\,U^{-1}\vec{\nabla}U$ (residual gauge freedom in temporal gauge)

With extra constraint $U(\vec{x}) \to$ const. on $\partial V$ (periodic on $T^3$)
→ Point compactification, homotopy $V \cong S^3$ ($V \cong T^3$)



$U^{(n)}$: “large” ($n \neq 0$) gauge transformation on spacelike ($\tau =$ const.) hypersurface $V \simeq S^3$ ($V \simeq T^3$) changing the Chern–Simons number by $n = W[\vec{A}_{U^{(n)}}] - W[\vec{A}] \in \mathbb{Z}$ units

# Topology on spatial hypersurfaces—point compactification, large gauge transformations

Consider temporal gauge $A^0 = 0$ (in view of canonical quantization)

Equivalence classes of $U^{(n)}$ (not connected with $\mathbb{1}$ for $n \neq 0$) only exist when we require these added *constraints* on $\vec{A}(\vec{x})$ (beyond $A^0(\vec{x}) = 0$)

Cannot impose these unless properly taken account in canonical formalism, e.g. through Lagrange multipliers

Unlike $\Delta n$, these equivalence classes are a result of a gauge choice.

Chern–Simons

$W$ … $x\, K_0$

Define $\vec{A}_U =$ … uge)

With extra co

space

$A = iU^{-1}\vec{\nabla}U$
& $U(\vec{x}) = \text{const.}$
on boundary

(three)
-sphere

→ Point compactification, homotopy $V \cong S^3$ ($V \cong T^3$)

$U^{(n)}$: "large" ($n \neq 0$) gauge transformation on spacelike ($\tau = \text{const.}$) hypersurface $V \simeq S^3$ ($V \simeq T^3$) changing the Chern–Simons number by $n = W[\vec{A}_{U^{(n)}}] - W[\vec{A}] \in \mathbb{Z}$ units

# FUNCTIONAL QUANTIZATION

## Theta in infinite spacetime volume

# Euclidean path integral & topology

Topological term $F\tilde{F}$ total derivative—how can it contribute?

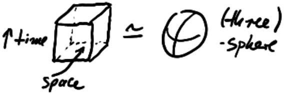
Recall: Euclidean path integral projects on ground state

$$\lim_{T\to\infty} \frac{\mathrm{e}^{-HT}}{\mathrm{e}^{-E_0 T}} \qquad \text{or} \qquad \lim_{T\to\infty} \frac{\mathrm{e}^{-\mathrm{i}HT(1-\mathrm{i}\varepsilon)}}{\mathrm{e}^{-\mathrm{i}E_0 T(1-\mathrm{i}\varepsilon)}}$$

$H$: Hamiltonian
$E_0$: ground state energy

$\longrightarrow$ Consider $\Omega = \mathbb{R}^4$ (or different *spatial* topologies)

Finite action $\longrightarrow$ Topological quantization $\longrightarrow$ Phases $\mathrm{e}^{\mathrm{i}\Delta n\theta}$



No reason for topological quantization in finite $\Omega \subset \mathbb{R}^4$ (finite temperature, see below)

Must take $T \to \infty$ before summing over sectors:

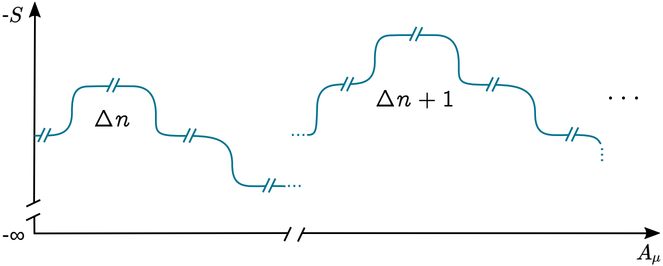
$$Z = \lim_{N\to\infty} \sum_{\Delta n=-N}^{N} \lim_{VT\to\infty} \int_{\Delta n} \mathcal{D}\phi\, \mathrm{e}^{-S_{\mathrm{E}}[\phi]}$$

Does interference of sectors have a material effect?

# More technically: Integration contour from Lefschetz thimbles

Parametrization of the path integral through steepest descent contours about classical saddle points ⟶ Contour integration on Lefschetz thimbles

$$\frac{\partial\phi(x;u)}{\partial u} = \overline{\frac{\delta S_{\mathrm{E}}[\phi(x;u)]}{\delta\phi(x;u)}} \Longrightarrow -\frac{\partial \mathrm{Re} S_{\mathrm{E}}[\phi(x;u)]}{\partial u} \leq 0 \ \text{ and } \ \frac{\partial \mathrm{Im} S_{\mathrm{E}}[\phi(x;u)]}{\partial u} = 0$$



Each thimble emerges from a critical point and corresponds to one $\Delta n \in \mathbb{Z}$

Keeping $VT$ finite while summing over different $\Delta n$ does *not* correspond to a nonsingular deformation of the Cauchy contour

Integration contour sweeps over full thimbles first, i.e. $VT \to \infty$ before sum over $\Delta n$

# So is it $\xi = -\bar{\alpha}$ or $\xi = \theta$?

- Take $\langle F(x)\tilde{F}(x)\rangle$ as measure for $CP$ violation
- Each element in the sequence over $N$ vanishes (not so when limits ordered the other way around):

$$\boxed{\langle F(x)\tilde{F}(x)\rangle = \lim_{\substack{N\to\infty\\ N\in\mathbb{N}}} \lim_{VT\to\infty} \frac{\sum_{\Delta n=-N}^{N} \frac{\Delta n}{VT} Z_{\Delta n}}{\sum_{\Delta n=-N}^{N} Z_{\Delta n}} = 0}$$

- Index theorem: $\Delta n$ is the difference of the numbers of right and left chiral zero modes
- Left/right chiral quasi-zero modes in spectral representation of fermion correlation regulated by $1/(m\,\mathrm{e}^{\mp\mathrm{i}\alpha})$

$$S(x,x') \underset{\substack{\uparrow\\ \boxed{\Delta n = -1}}}{=} \frac{\hat{\psi}_{0\mathrm{L}}(x)\hat{\psi}^{\dagger}_{0\mathrm{L}}(x')}{m\mathrm{e}^{-\mathrm{i}\alpha}} + \sum\!\!\!\!\!\!\!\int_{\lambda^{\mathrm{E}}\neq 0} \frac{\hat{\psi}_{\lambda}(x)\hat{\psi}^{\dagger}_{\lambda}(x')}{\lambda}$$

- No L/R imbalance in fermion quasi-zero modes →
  Quark correlations remain aligned with quark mass after interference of $\Delta n$-sectors

$$\to \boxed{\xi = -\bar{\alpha}}$$

- Order of limits matters because series is not positive definite due to phases $\mathrm{e}^{\mathrm{i}\Delta n\theta}$, not absolutely summable

# CANONICAL QUANTIZATION

## (in finite and infinite volumes)

# Theta vacuum, standard story

Take, $A^0 = 0$, *assume* in addition:

For $|\vec{x}| \to \infty$: $\vec{A}(\vec{x}) = \mathrm{i}\, U^{-1}(\vec{x}) \vec{\nabla} U(\vec{x})$ and $U(\vec{x}) \to \text{const.}$

But why? [cf. Jackiw (1980)]

Consider initial and final states, taking $x_4 \to \pm\infty$

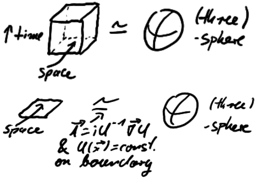
→ *Ansatz*: Construct from pure gauge configurations on these surfaces, with

$$n = \frac{1}{4\pi^2} \int_{x^4=\text{const.}} \mathrm{d}^3\sigma\, K_\perp \in \mathbb{Z}$$

Chern–Simons number

**not** gauge invariant

point compactification



Construct ground states from prevacua $|n\rangle$ (field eigenstates)

Gauge invariant (up to phase) state $|\theta\rangle = \sum_n \mathrm{e}^{-in\theta} |n\rangle$ [Callan, Dashen, Gross (1976); Jackiw, Rebbi (1976); Jackiw (1980)]

# Standard story: two loose ends

The prevacua $|n\rangle$ are field eigenstates, very different from the ground state

Resolutions:

- Take $T \to \infty$ in the path integral to project on the ground state:
  $|\mathrm{vac}\rangle = \mathrm{e}^{-HT} \sum_n \mathrm{e}^{-in\theta} |n\rangle$, $T \to \infty$ (cf. $VT \to \infty$ in previous part)
- Or use the symmetries and no further properties of the wave functional
  [Jackiw, Rebbi (1976); Jackiw (1980)]

States are not normalizable in the proper sense because $\langle \theta^{(i)} | \theta^{(j)} \rangle = \delta(\theta^{(i)} - \theta^{(j)})$
[cf. e.g. Callan, Dashen, Gross (1976); issue taken by Okubo, Marshak (1992)]

Without ado, this contradicts 1st Dirac–von Neumann axiom of quantum mechanics.
[Dirac 1932, von Neumann 1932]

Possible resolutions:

- Construct wave packets—not acceptable however because gauge invariance should be exact
- Use gauge fixing in order to normalize states (which is what we will do here)

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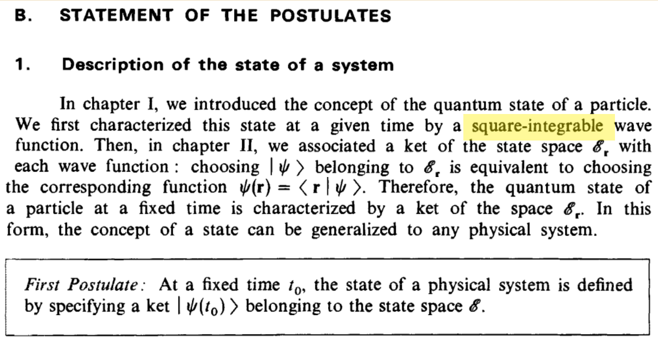

**B. STATEMENT OF THE POSTULATES**

**1. Description of the state of a system**

In chapter I, we introduced the concept of the quantum state of a particle. We first characterized this state at a given time by a square-integrable wave function. Then, in chapter II, we associated a ket of the state space $\mathscr{E}_\mathbf{r}$ with each wave function: choosing $|\psi\rangle$ belonging to $\mathscr{E}_\mathbf{r}$ is equivalent to choosing the corresponding function $\psi(\mathbf{r}) = \langle \mathbf{r} | \psi \rangle$. Therefore, the quantum state of a particle at a fixed time is characterized by a ket of the space $\mathscr{E}_\mathbf{r}$. In this form, the concept of a state can be generalized to any physical system.

*First Postulate:* At a fixed time $t_0$, the state of a physical system is defined by specifying a ket $|\psi(t_0)\rangle$ belonging to the state space $\mathscr{E}$.

[Cohen-Tanoudji, Diu, Laloë]

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# Canonical quantization of the gauge field

Minkowski spacetime, temporal gauge $A^0 = 0$, no sources $\longrightarrow$

$$g\vec{E}^a = -\partial/\partial t\,\vec{A}^a$$

$$g\vec{B}^a = \vec{\nabla} \times \vec{A}^a - 1/2\, f^{abc}\vec{A}^a \times \vec{A}^b$$

Canonical momentum conjugate to $\vec{A}^a$:

$$g\vec{\Pi}^a = -\vec{E}^a + \frac{g^2}{8\pi^2}\theta\vec{B}^a$$

- **No constraints on $\partial V$ accounted for $\longrightarrow$ $\Psi[\vec{A}]$ must be defined for $U(\vec{x}) \neq \text{const.}$ on $\partial V$**
- **Residual gauge dofs.: Throw out unphysical states “First quantize, then constrain”**

The corresponding operator must observe the commutation relations:

$$[A^{a,i}(\vec{x}), \Pi^{b,j}(\vec{x}\,')] = \mathrm{i}\delta^{ij}\delta^{ab}\delta^3(\vec{x} - \vec{x}\,')\,, \quad [\Pi^{a,i}(\vec{x}), \Pi^{b,j}(\vec{x}\,')] = 0$$

These commutators hold for ($\theta_\Pi$ arbitrary) $\vec{\Pi}^a = \dfrac{\delta}{\mathrm{i}\delta\vec{A}^a} + \theta_\Pi \dfrac{g}{8\pi^2}\vec{B}^a$

Hamiltonian density:

$$\mathcal{H} = \frac{1}{2}\left((\vec{E}^a)^2 + (\vec{B}^a)^2\right) = \frac{1}{2}\left(\left(g\frac{\delta}{\mathrm{i}\delta\vec{A}^a} - \frac{g^2}{8\pi^2}(\theta - \theta_\Pi)\vec{B}^a\right)^2 + (\vec{B}^a)^2\right)$$

[e.g. Jackiw (1980)]

# Wave functional in gauge theory (temporal gauge $A_0 = 0$)

Since $[U^{(n)}, H] = 0$, can find states $\Psi_{\theta^{(i)}}[\vec{A}_{(U^{(1)})^n}] = \mathrm{e}^{\mathrm{i}n\theta^{(i)}}\Psi_{\theta^{(i)}}[\vec{A}]$

Wave functionals not properly normalizable

$$\int \mathcal{D}\vec{A}\,\Psi^{(a)*}_{\theta^{(i)}}[\vec{A}]\Psi^{(b)}_{\theta^{(j)}}[\vec{A}] ==2\pi\delta(\theta^{(i)} - \theta^{(j)})\delta_{ab} \qquad \text{Bloch theorem}$$

*What about the first postulate?*

Cf. $T^4$/lattice:

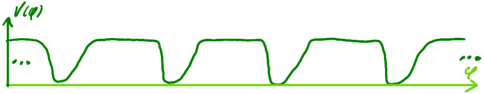
$$Z = \sum\!\!\!\!\!\!\int_a \int \mathcal{D}\vec{A}\,\Psi^{(a)*}_{\theta^{(i)}}[\vec{A}]\mathrm{e}^{-\beta H}\Psi^{(b)}_{\theta^{(i)}}[\vec{A}]$$

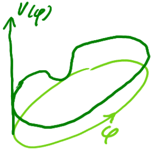
Not properly normalizable either

# Crystal or circle?

The functionals $\Psi_\theta(\vec{A})$ with above periodicity properties can be viewed as Bloch states.

Bloch states live on a crystal:
$\vec{A}_{U_4^{(1)}}$ is a *different* site than $\vec{A}$





In contrast: In gauge theory
$\vec{A}_{U_4^{(1)}}$ is a *redundant* description of the configuration $\vec{A}$—corresponding to $\varphi \to \varphi + 2\pi n$ on a circle

On a crystal: Bloch states do not correpsond to normalized wave functions, these are rather wave packets made up of Bloch states. Packets, however, not translation (gauge) invariant

On a circle: Truncation of the inner product according to a single period leads to properly normalizable states, corresponding here to *gauge fixing* $\vec{A} \in \mathcal{A}$ so that each physical configuration appears one time and one time only:

$$\int_{\mathcal{A}} \underbrace{\mathcal{D}\vec{A}\, f_{\mathcal{A}}[\vec{A}]}_{\substack{\text{gauge invariant} \\ \text{under change of } \mathcal{A}}} \quad \Psi^{(a)*}_{\theta^{(i)}}[\vec{A}]\, \Psi^{(b)}_{\theta^{(j)}}[\vec{A}]$$

Note: Under gauge fixed inner product, $\Psi^{(a)}_{\theta^{(i)}}$, $\Psi^{(b)}_{\theta^{(j)}}$ no longer orthogonal for $\theta^{(i)} \neq \theta^{(j)}$

# Form of the wave functional & Constraining the Hilbert space

Require: Gauge invariance & $\frac{\delta}{\mathrm{i}\delta \vec{A}(\vec{x})}$ should remain Hermitian under restricted inner product

$\Longrightarrow \Psi^{(a)}[\vec{A}] \overset{(*)}{=} \Psi^{(a)}[\vec{A}]_{\text{g.i.}} \exp(\mathrm{i}\varphi[\vec{A}])$ valid for *all* $U(\vec{x})$ (also nonconstant on boundary)

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$\longrightarrow$ "Diagonalize" $H$:

$$
\begin{aligned}
\Psi'[\vec{A}] &= \mathrm{e}^{-\mathrm{i}(\theta-\theta_{\Pi})W[\vec{A}]}\Psi[\vec{A}]\,,\\
H' &= \mathrm{e}^{-\mathrm{i}(\theta-\theta_{\Pi})W[\vec{A}]}H\mathrm{e}^{\mathrm{i}(\theta-\theta_{\Pi})W[\vec{A}]} = \frac{1}{2}\int \mathrm{d}^3x\,\mathrm{tr}\left[-g^2\frac{\delta^2}{\delta\vec{A}^2}+\vec{B}^2\right]\\
&= -\frac{g^2}{2}\sum\!\!\!\!\!\!\int_{\sigma}\frac{\delta^2}{\delta A^2(\sigma)}+\frac{1}{2}\int \mathrm{d}^3x\,\mathrm{tr}\,\vec{B}^2\,,\ \sigma\in\{\sigma_{\mathrm{gauge}},\sigma_{||}\}
\end{aligned}
$$

$$\frac{\delta}{\delta\vec{A}(\vec{x})}W[\vec{A}] = \frac{g}{8\pi^2}\vec{B}(\vec{x})$$

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**Only trivial one-dimensional representations of SU(2)**

$$\Psi[\vec{A}_U] = \mathrm{e}^{\mathrm{i}\varphi[\vec{A}_U]}\Psi[\vec{A}]\ \ (\text{eigenstate of } U)\,,\qquad U_3 = U_2\,U_1$$

$$\mathrm{e}^{\mathrm{i}\varphi[\vec{A}_{U_3}]} = \mathrm{e}^{\mathrm{i}\varphi[\vec{A}_{U_2}]}\mathrm{e}^{\mathrm{i}\varphi[\vec{A}_{U_1}]} \Rightarrow \mathrm{e}^{\mathrm{i}\varphi[\vec{A}_{U_2U_1}]} - \mathrm{e}^{\mathrm{i}\varphi[\vec{A}_{U_1U_2}]} = 0$$

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Throw states not satisfying $(*, **)$ out of the Hilbert space
→ $CP$ conserved

# Gauß' law in the constrained Hilbert space

For $\omega(\vec{x})$ an infinitesimal generator of gauge transformations
$\longrightarrow$ Noether charge:

$$Q(\omega) = \frac{1}{g} \int \mathrm{d}^3x \, \mathrm{tr} \left[ \Pi^i (D^i \omega) \right] = \int_V \mathrm{d}^3x \, \mathrm{tr} \left[ \left( -E^i + \frac{g^2}{8\pi^2} (\theta - \theta_\Pi) B^i \right) D^i \omega \right]$$

$$= \int \mathrm{d}^3x \, \mathrm{tr} \left[ \omega D^i \left( E^i - \frac{g^2}{8\pi^2} (\theta - \theta_\Pi) B^i \right) \right] + \int_{\partial V} \mathrm{d}a^i \, \mathrm{tr} \left[ \omega \left( -E^i + \frac{g^2}{8\pi^2} (\theta - \theta_\Pi) B^i \right) \right]$$

For $\omega(\vec{x}) = 0$ when $\vec{x} \in \partial V$ and since $\Psi'$ is gauge invariant
$\rightarrow$ Gauß' law: $\vec{D} \cdot \vec{E} \, \Psi'[\vec{A}] = 0$

Usually, the argument is made the other way around: Impose Gauß' law to throw states out of the Hilbert space

Since $[Q(\omega), W[\vec{A}]] = 0$ for $\omega(\vec{x}) = 0$ when $\vec{x} \in \partial V$ this also holds when $\Psi'[\vec{A}] \rightarrow \mathrm{e}^{\mathrm{i}\tilde{\theta} W[\vec{A}]} \Psi'[\vec{A}]$, so imposing Gauß' law does not fix $\tilde{\theta}$, does not tell us about large gauge transformations

# Nondiagonal basis

Redefining derivatives w.r.t. $\vec{A}$ as

$$\vec{D}_{\vec{A}}\Psi[\vec{A}] = \mathrm{i}\left(\frac{\delta}{\mathrm{i}\delta\vec{A}} - (\theta-\theta_{\Pi})\frac{g}{8\pi^2}\vec{B}\right)\Psi[\vec{A}]$$

corresponds to a canonical transformation of the momentum operator.

Induces translation as

$$T[\Delta\vec{A}]\Psi[\vec{A}] = \mathrm{e}^{-\mathrm{i}(\theta-\theta_{\Pi})\left(W[\vec{A}+\Delta\vec{A}]-W[\vec{A}]\right)}\Psi[\vec{A}+\Delta\vec{A}]$$

For a shift $\Delta\vec{A}_{\mathrm{gauge}}$ corresponding to a *general* gauge transformation:

gauge invariant

$$T[\Delta\vec{A}_{\mathrm{gauge}}]\Psi[\vec{A}] = \Psi[\vec{A}] \quad \text{if} \quad \Psi[\vec{A}] = \mathrm{e}^{\mathrm{i}(\theta-\theta_{\Pi})W[\vec{A}]}\Psi_{\mathrm{g.i.}}[\vec{A}]$$

Agrees with reasoning & result in the diagonal basis

$\theta-\theta_{\Pi}$ in $\Psi_{\theta-\theta_{\Pi}}$ is pinned to $\theta-\theta_{\Pi}$ in $H$ so that $CP$ is conserved

# Conclusion

There is no $CP$ violation in QCD.

Challenges to the standard calculation and resolutions:

- **Taking $T \to \infty$ after summing over sectors corresponds to an inequivalent deformation of the integration contour**
  Maintain Cauchy contour and order of limits
- **No point compactification/topology in temporal gauge (w/o extra constraint)**
  Drop the constraint, define $\Psi$ for all spatial gauges
- **$\theta$-vacua are not properly normalizable $\to$ not physical states according to postulates of QM**
  Integrate over each physical configuration one time and one time only
  →No need to give up Dirac–von Neumann axioms

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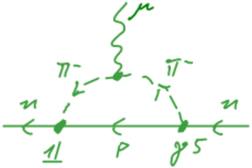
**THANK YOU!**

# Neutron electric dipole moment

$$\mathrm{i}\mathcal{M} = -\,2\mathrm{i}D(q^2)\varepsilon^*_\mu(\vec{q})\bar{u}_{s'}(\vec{p}')\frac{\mathrm{i}}{4}[\gamma^\mu,\gamma^\nu]\,q_\nu\mathrm{i}\gamma_5\,u_s(\vec{p})$$

$$\rightarrow\mathcal{L}_{\text{eff}} = D(0)\bar{n}\,\underbrace{F_{\mu\nu}\frac{\mathrm{i}}{4}[\gamma^\mu,\gamma^\nu]\mathrm{i}\gamma_5}_{\subset\vec{S}\cdot\vec{E}}\,n$$



- $\chi$PT value: $d_n = 3.2 \times 10^{-16}(\xi + \bar{\alpha})\,e\,\mathrm{cm}$
- Experimental bound: $|d_n| < 1.8 \times 10^{-26}\,e\,\mathrm{cm}$ (90% c.l.) [nEDM/PSI (2020)]
- Calculations e.g. of neutron EDM implicitly *assume* $\xi = \theta$
  [e.g. Baluni (1979); Crewther, Di Vecchia, Veneziano, Witten (1979)]
- However $\xi = -\bar{\alpha}$ also perfectly valid by arguments used to this end
- Another signature—weaker bounds: $\eta' \rightarrow \pi\pi$

# Topology—instantons

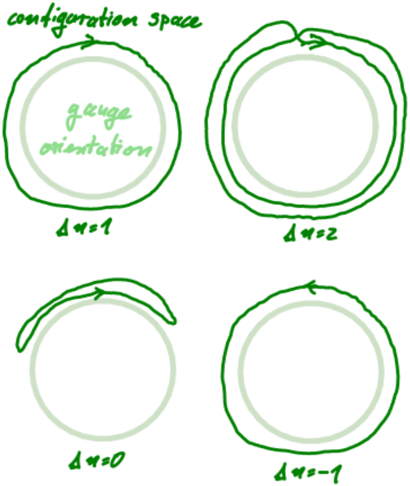
Does $\Delta n \neq 0$ imply nontrivial physical field configurations?

Yes, cf. anti-instanton: $A_\mu{}^u{}_v = -\dfrac{\sigma_{\mu\nu}{}^u{}_v x_\nu}{x^2+\rho^2}$

(extended solution to *Euclidean* EOMs)

[Belavin, Polyakov, Schwarz, Tyupkin (1975)]

Surface term decays as $1/|x|^3 \rightarrow$ surface integral does not need to vanish



Theta term contributes to the action though being a total derivative

# Why $T \to \infty$

(Implying $\Omega = VT \to \infty$ as opposed to a finite spacetime volume)

To evaluate amplitudes at finite $T$, project path integral on the state in terms of a wave function(al) $\Psi[\phi(\vec{x})] = \langle \phi(\vec{x}) | \Psi \rangle$ (more on this later):

$$\langle \Psi_f, t_f | \Psi_i, t_i \rangle = \int \mathcal{D}\phi_f \mathcal{D}\phi_i \langle \Psi_f, t_f | \phi_f \rangle \int_{\substack{\phi(t_f,\vec{x})=\phi_f(\vec{x}) \\ \phi(t_i,\vec{x})=\phi_i(\vec{x})}} \mathcal{D}\phi \, \mathrm{e}^{\mathrm{i}S[\phi]} \langle \phi_i | \Psi_i, t_i \rangle$$

$|\phi_{i,f}\rangle$ field eigenstates, not energy eigenstates

Problem: Neither know $\Psi[\phi(\vec{x})]$ nor kernels of Schrödinger equation

Way out: Euclidean path integral/project on ground state

$$\lim_{T\to\infty} \frac{\mathrm{e}^{-HT}}{\mathrm{e}^{-E_0 T}} \quad \text{or} \quad \lim_{T\to\infty} \frac{\mathrm{e}^{-\mathrm{i}HT(1-\mathrm{i}\varepsilon)}}{\mathrm{e}^{-\mathrm{i}E_0 T(1-\mathrm{i}\varepsilon)}}$$

$H$: Hamiltonian

$E_0$: ground state energy

→ Obtain vacuum correlations without bothering about $\Psi[\phi(\vec{x})]$

# Fermion correlations and instantons

Dilute instanton gas (DIGA) picture (to determine phase of ’t Hooft vertex—not quantitatively accurate for actual QCD)

Leading contribution to two-point function (no instantons)

$$\langle\psi(x)\bar{\psi}(x')\rangle = \mathrm{i}S_{0\mathrm{inst}}(x,x')$$

$$\mathrm{i}S_{0\mathrm{inst}}(x,x') = (-\gamma^{\mu}\partial_{\mu} + \mathrm{i}m_{i}\mathrm{e}^{-\mathrm{i}\alpha_{i}\gamma^{5}})\int\frac{\mathrm{d}^{4}p}{(2\pi)^{4}}\frac{\mathrm{e}^{-\mathrm{i}p(x-x')}}{p^{2}-m_{i}^{2}+\mathrm{i}\epsilon}$$

## Green’s function in $n$-instanton, $\bar{n}$-anti-instanton background (DIGA)

$$\mathrm{i}S_{n,\bar{n}}(x,x') \approx \mathrm{i}S_{0\mathrm{inst}}(x,x') + \sum_{\bar{\nu}=1}^{\bar{n}}\frac{\hat{\psi}_{0\mathrm{L}}(x-x_{0,\bar{\nu}})\hat{\psi}_{0\mathrm{L}}^{\dagger}(x'-x_{0,\bar{\nu}})}{m\mathrm{e}^{-\mathrm{i}\alpha}} + \sum_{\nu=1}^{n}\frac{\hat{\psi}_{0\mathrm{R}}(x-x_{0,\nu})\hat{\psi}_{0\mathrm{R}}^{\dagger}(x'-x_{0,\nu})}{m\mathrm{e}^{\mathrm{i}\alpha}}$$

$\hat{\psi}_{0\mathrm{L,R}}$: ‘t Hooft zero modes

Alignment of $\alpha$ in Lagrangian mass and instanton-induced $\chi$SB ⟶ No $CP$ violation here

## Sum/interference over DIGA configurations

$$\langle\psi(x)\bar{\psi}(x')\rangle = \lim_{\substack{N\to\infty\\ N\in\mathbb{N}}}\ \lim_{VT\to\infty}\frac{\sum_{\Delta n=-N}^{N}\langle\psi(x)\bar{\psi}(x')\rangle_{\Delta n}}{\sum_{\Delta n=-N}^{N} Z_{\Delta n}}$$

$$= \mathrm{i}S_{0\mathrm{inst}}(x,x') + \mathrm{i}\kappa\bar{h}(x,x')m^{-1}\mathrm{e}^{-\mathrm{i}\alpha\gamma^{5}}$$

$$\longrightarrow \xi = -\alpha \text{ (alignment)}$$

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$$= \mathrm{i}S_{0\mathrm{inst}}(x,x') + \mathrm{i}\kappa\bar{h}(x,x')m^{-1}\mathrm{e}^{\mathrm{i}\theta\gamma^{5}}$$

$$\longrightarrow \xi = \theta \text{ (destructive interference)}$$

# Boundary configurations & topological quantization

The parameter $\theta$ can be viewed as an angular variable (forced by the anomalous chiral current). $\longrightarrow$

Requires $\Delta n \in \mathbb{Z}$ (“topological quantization”) $\rightarrow \exp(\mathrm{i}S)|_\theta = \exp(\mathrm{i}S)|_{\theta+2\pi}$

Readily built into the path integral for $VT \rightarrow \infty$ without constraining boundary conditions by hand:

(Relatively) nonvanishing contributions in *infinite* spacetime only from classical local minima of the Euclidean action & fluctuations about these—these configurations must go to pure gauges at $\infty$

Indeed, for pure gauge configurations at $\infty \rightarrow \Delta n \in \mathbb{Z}$ (as discussed above)

*There is no such restriction/principle to fixed physical bcs. for finite $VT$.*

Consequence: $\Delta n \in \mathbb{Z}$ requires $T \rightarrow \infty$ first $\rightarrow$ In the path integral, take $T \rightarrow \infty$, then sum over all topological sectors $\Delta n$ weighted $\exp(\mathrm{i}\Delta n\theta)$

## Fermion correlations

The effective vertex generates the following correlation functions at tree level:

$$\langle\prod_{j=1}^{N_f}\psi_j(x_j)\bar{\psi}_j(x_j')\rangle_{\mathrm{inst}}=\left(\mathrm{e}^{-\mathrm{i}\xi}\prod_{j=1}^{N_f}P_{\mathrm{L}j}+\mathrm{e}^{\mathrm{i}\xi}\prod_{j=1}^{N_f}P_{\mathrm{R}j}\right)\bar{H}(x_1,\ldots,x_1',\ldots)$$

Goal: Compute correlation function and compare with EFT answer above to fix $\xi$

Cf. leading contribution to two-point function

$$\begin{aligned}\langle\psi_i(x)\psi_j(x')\rangle =&\mathrm{i}S_{0\mathrm{inst}\,ij}(x,x')\\ \mathrm{i}S_{0\mathrm{inst}\,ij}(x,x') =&(-\gamma^\mu\partial_\mu+\mathrm{i}m_i\mathrm{e}^{-\mathrm{i}\alpha_i\gamma^5})\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}p(x-x')}\frac{\delta_{ij}}{p^2-m_i^2+\mathrm{i}\epsilon}\end{aligned}$$

So $\xi=\theta/\xi=-\bar{\alpha}$ implies $CP$-violation/no $CP$-violation

Only one explicit calculation based on dilute instanton gas (DIGA) finding $\xi=\theta$
['t Hooft (1986)]

# Fermion correlations

- Obtain correlation functions from Green's functions in fixed background of instantons and anti-instantons
- Interfere all instanton configurations
  - First, within one topological sector
  - Then over the different sectors

DIGA to dermine $CP$ phase of 't Hooft vertex—not quantitatively accurate for actual QCD

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$\hat{\psi}_{0\mathrm{L,R}}$: 't Hooft zero modes

Comments:

- For small masses, zero modes dominate close to the cores of instantons, far away from instantons the solution goes to the zero-instanton configuration [Diakonov, Petrov (1986)]
- Alignment of phase $\alpha$ between Lagrangian mass and instanton-induced $\chi$SB ⟶ No indication of $CP$ violation here

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## Interferences within the topological sectors

Within a topological sector, interfere/sum/integrate over

- all instanton/anti-instanton numbers $n + \bar{n}$ with $\Delta n = n - \bar{n}$ fixed
- locations of all instantons/anti-instantons
- remaining collective coordinates

$\longrightarrow$ Dilute instanton gas approximation (skip technicalities)

Can also obtain coincident fermion correlations using the index theorem and anomalous current only

# Correlation function for fixed $\Delta n$

$$\langle\psi(x)\bar{\psi}(x')\rangle_{\Delta n} = \sum_{\substack{\bar{n},n\geq 0\\ n-\bar{n}=\Delta n}} \frac{1}{\bar{n}!n!}\left[\bar{h}(x,x')\left(\frac{\bar{n}}{m\mathrm{e}^{-\mathrm{i}\alpha}}P_{\mathrm{L}} + \frac{n}{m\mathrm{e}^{\mathrm{i}\alpha}}P_{\mathrm{R}}\right)(VT)^{\bar{n}+n-1} + \mathrm{i}S_{0\mathrm{inst}}(x,x')(VT)^{\bar{n}+n}\right] \times (\mathrm{i}\kappa)^{\bar{n}+n}(-1)^{n+\bar{n}}\mathrm{e}^{\mathrm{i}\Delta n(\alpha+\theta)}$$

$$= \left[\left(\mathrm{e}^{\mathrm{i}\alpha}I_{\Delta n+1}(2\mathrm{i}\kappa VT)P_{\mathrm{L}} + \mathrm{e}^{-\mathrm{i}\alpha}I_{\Delta n-1}(2\mathrm{i}\kappa VT)P_{\mathrm{R}}\right)\frac{\mathrm{i}\kappa}{m}\,\bar{h}(x,x') + I_{\Delta n}(2\mathrm{i}\kappa VT)\mathrm{i}S_{0\mathrm{inst}}(x,x')\right] \times (-1)^{\Delta n}\mathrm{e}^{\mathrm{i}\Delta n(\alpha+\theta)}$$

Instantons per spacetime volume: $\mathrm{i}\kappa \propto \mathrm{e}^{-S_{\mathrm{E}}}$

$\chi$SB rank-two spinor-tensor from integrating quark zero-modes over the locations of the instantons: $\bar{h}(x,x')$

Modified Bessel function: $I_\nu(x)$

### Sum is dominated by particular value of $n \approx \bar{n}$: [Diakonov, Petrov (1986)]

$$\langle n\rangle = \frac{\sum_{n=0}^{\infty} n\frac{(\kappa VT)^n}{n!}}{\sum_{n=0}^{\infty}\frac{(\kappa VT)^n}{n!}} = \kappa VT\,, \quad \frac{\sqrt{\langle(n-\langle n\rangle)^2\rangle}}{\langle n\rangle} = \frac{1}{\sqrt{\kappa VT}}\,, \quad \text{cf.} \lim_{x\to\infty}\frac{I_{\Delta n}(\mathrm{i}x\,\mathrm{e}^{-\mathrm{i}0^+})}{I_{\Delta n'}(\mathrm{i}x\,\mathrm{e}^{-\mathrm{i}0^+})} = 1$$

⟶ No relative $CP$ phase between mass and instanton induced breaking of $\chi$ral symmetry—alignment in infinite-volume limit

Correspondingly, partition function for fixed $\Delta n$: [cf. Leutwyler, Smilga (1992)]

$$Z_{\Delta n} = I_{\Delta n}(2\mathrm{i}\kappa VT)\,(-1)^{\Delta n}\mathrm{e}^{\mathrm{i}\Delta n(\alpha+\theta)}$$

**Note:** The topological phase $\mathrm{e}^{\mathrm{i}\Delta n(\alpha+\theta)}$ multiplies $\langle\psi(x)\bar{\psi}(x')\rangle_{\Delta n}$ and $Z_{\Delta n}$ entirely—not just the contributions induced by instantons.

Other correlation functions ($n$ point, stress-energy, for some observer,...) are calculated from the Feynman diagram with the Green's function in the $n$ instanton, $\bar{n}$ anti-instanton background.
Then it remains to average over $n$, $\bar{n}$, locations and remaining collective coordinates.

There is no $CP$ violation/misalignment of phases to this end. It remains to consider the interference between the topological sectors.

# Interferences among topological sectors (are immaterial)

Topological quantization ↔ Interference between sectors for $VT \to \infty$

## Fermion correlator

$$\begin{aligned}\langle \psi(x)\bar\psi(x')\rangle =& \lim_{\substack{N\to\infty\\ N\in\mathbb{N}}} \lim_{VT\to\infty} \frac{\sum_{\Delta n=-N}^{N}\langle \psi(x)\bar\psi(x')\rangle_{\Delta n}}{\sum_{\Delta n=-N}^{N} Z_{\Delta n}}\\ =& \mathrm{i}S_{0\mathrm{inst}}(x,x') + \mathrm{i}\kappa\bar h(x,x')m^{-1}\mathrm{e}^{-\mathrm{i}\alpha\gamma^5} \quad \text{(same as for fixed } \Delta n\text{)}\end{aligned}$$

Recall: $\mathrm{i}S_{0\mathrm{inst}}(x,x') = (-\gamma^\mu\partial_\mu + \mathrm{i}m\mathrm{e}^{-\mathrm{i}\alpha\gamma^5})\int \frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}p(x-x')}\frac{1}{p^2-m^2+\mathrm{i}\epsilon}$

⟶ No relative $CP$-phase between mass and instanton term

⟶ $\xi = -\alpha$

⟶ $CP$ is conserved

# Limits ordered the other way around

First sum over all $\Delta n$ as well:

$$\sum_{\bar{n},n\geq 0}\frac{1}{\bar{n}!n!}\Big[\bar{h}(x,x')(\bar{n}\,m^{-1}\mathrm{e}^{\mathrm{i}\alpha}P_{\mathrm{L}}+n\,m^{-1}\mathrm{e}^{-\mathrm{i}\alpha}P_{\mathrm{R}})\,(VT)^{\bar{n}+n-1}+\mathrm{i}S_{0\mathrm{inst}}(x,x')\,(VT)^{\bar{n}+n}\Big]$$
$$\times\,(-mi\kappa)^{\bar{n}+n}\mathrm{e}^{\mathrm{i}\Delta n(\alpha+\theta)}$$
$$=\Big[-\left(\mathrm{e}^{-\mathrm{i}\theta}P_{\mathrm{L}}+\mathrm{e}^{\mathrm{i}\theta}P_{\mathrm{R}}\right)\frac{\mathrm{i}\kappa}{m}\bar{h}(x,x')+\mathrm{i}S_{0\mathrm{inst}}(x,x')\Big]\,\mathrm{e}^{-2\mathrm{i}\kappa\,VT\cos(\alpha+\theta)}$$

$$Z\to\sum_{n,\bar{n}}\frac{1}{n!\bar{n}!}(-\mathrm{i}\kappa\,VT)^{\bar{n}+n}\mathrm{e}^{-\mathrm{i}(\bar{n}-n)(\alpha+\theta)}=\mathrm{e}^{-2\mathrm{i}\kappa\,VT\cos(\alpha+\theta)}$$

Then, $VT\to\infty$ trivial as $VT$-dependence cancels

⟶ Relative $CP$ phase leading to $CP$-violating observables

However: Changing the order does not correspond to a nonsingular integration contour.