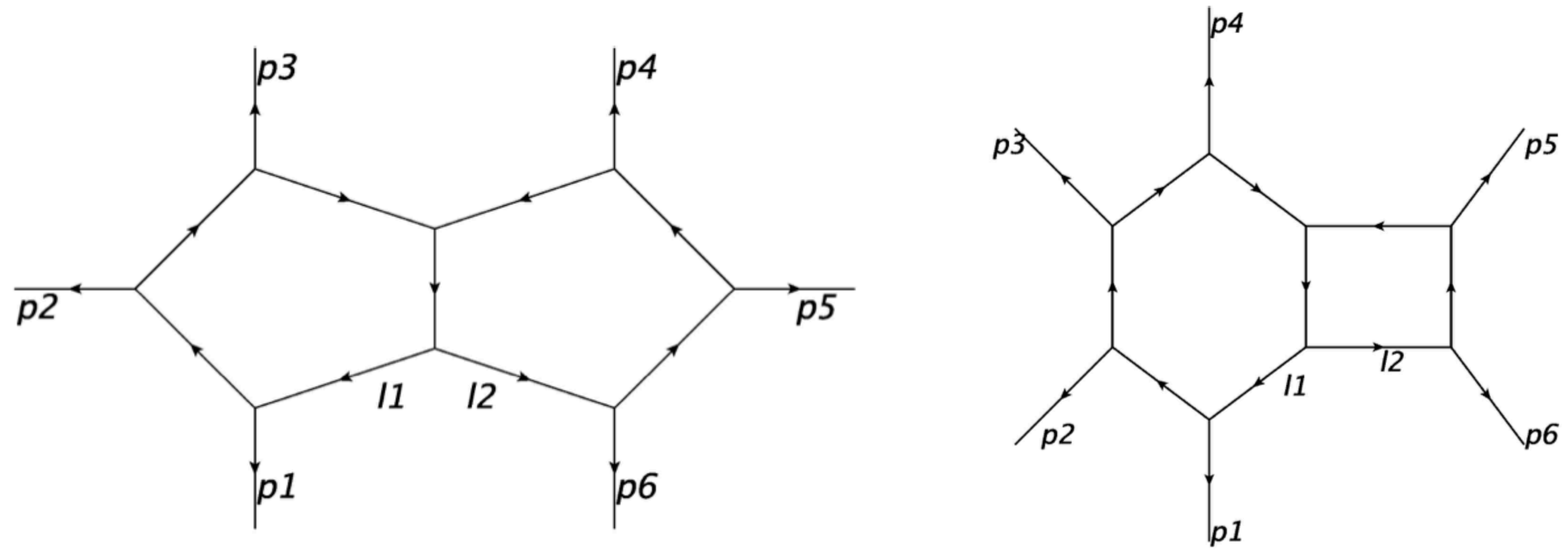
Frontiers of Multi-loop Analytic Feynman Integrals

Shanghai Symposium on Particle Physics and Cosmology 2025
2025.10.11

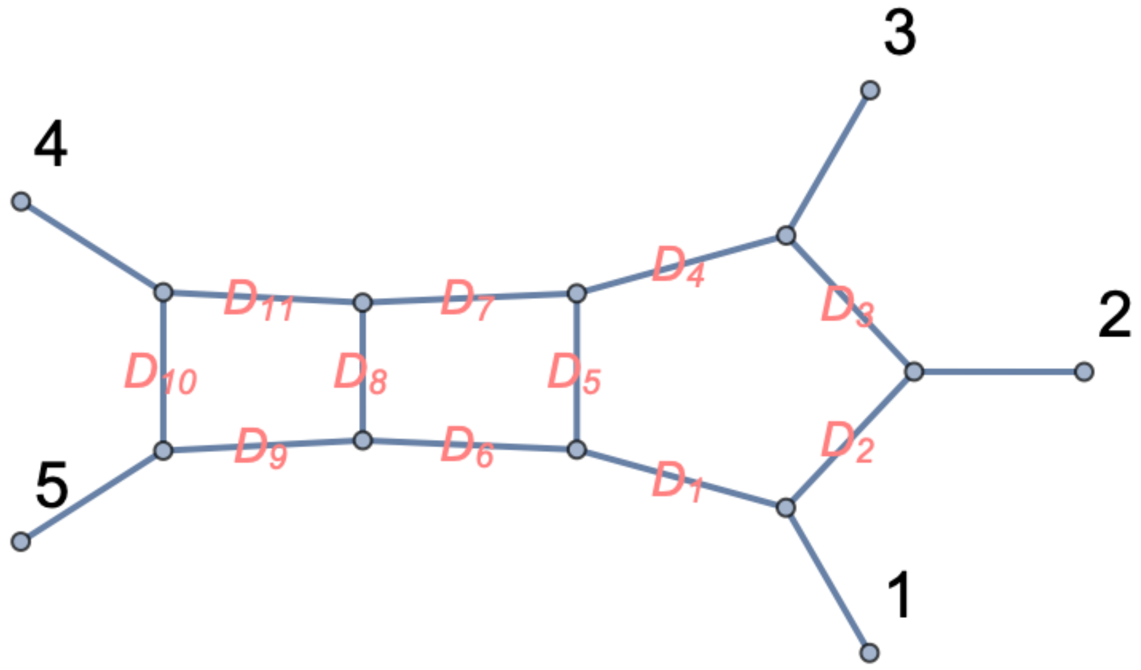
Yang Zhang
University of Science and Technology of Chi

Based on

Feynman integral Integral evaluation



analytic computation of all 2loop 6point massless planar integrals is done



The first analytic computation of 3loop 5-point Feynman integral family

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, Phys. Rev. Lett. 135, 031601

Liu, Matijasic, Miczajka, Xu, Xu, YZ, Phys.Rev.D 112 (2025) 1, 016021 (Editors' Suggestion)

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, JHEP 08(2024) 027

Based on package development

"**NeatIBP 1.0**, a package generating small-size integration-by-parts relations for Feynman integrals"
Wu, Boehm, Ma, Xu, YZ, *Comput. Phys. Commun. 295 (2024), 108999*

"Performing integration-by-parts reductions using NeatIBP 1.1 + Kira"
Wu, Boehm, Ma, Usovitsch, Xu, YZ, Comput.Phys.Commun. 316 (2025) 109798

NeatIBP **collaboration**

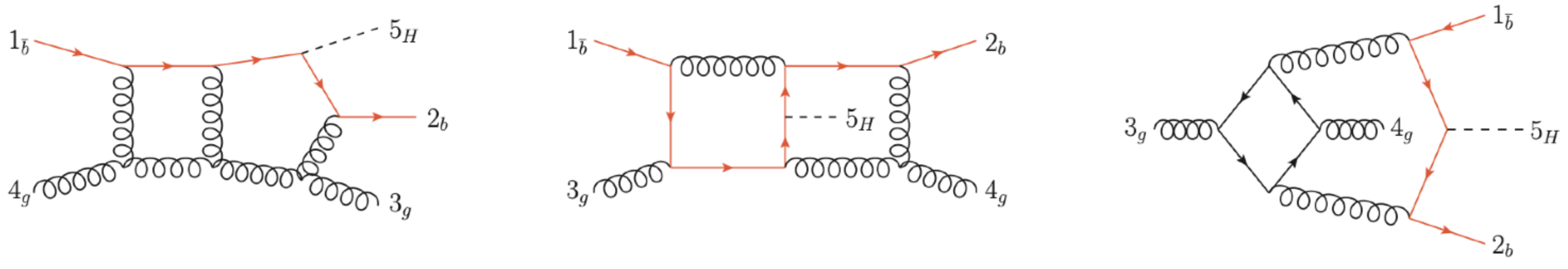
Based on perturbative QCD computations

"Two-loop amplitudes for $O(\alpha_s^2)$ corrections to W$\gamma\gamma$ production at the LHC"

Badger, Hartanto, Wu, YZ, Zoia, JHEP12(2024) 221

"Full-colour double-virtual amplitudes for associated production of a Higgs boson with a bottom-quark pair at the LHC"

Badger, Hartanto, Poncelet, Wu, YZ, Zoia, JHEP03(2025) 066

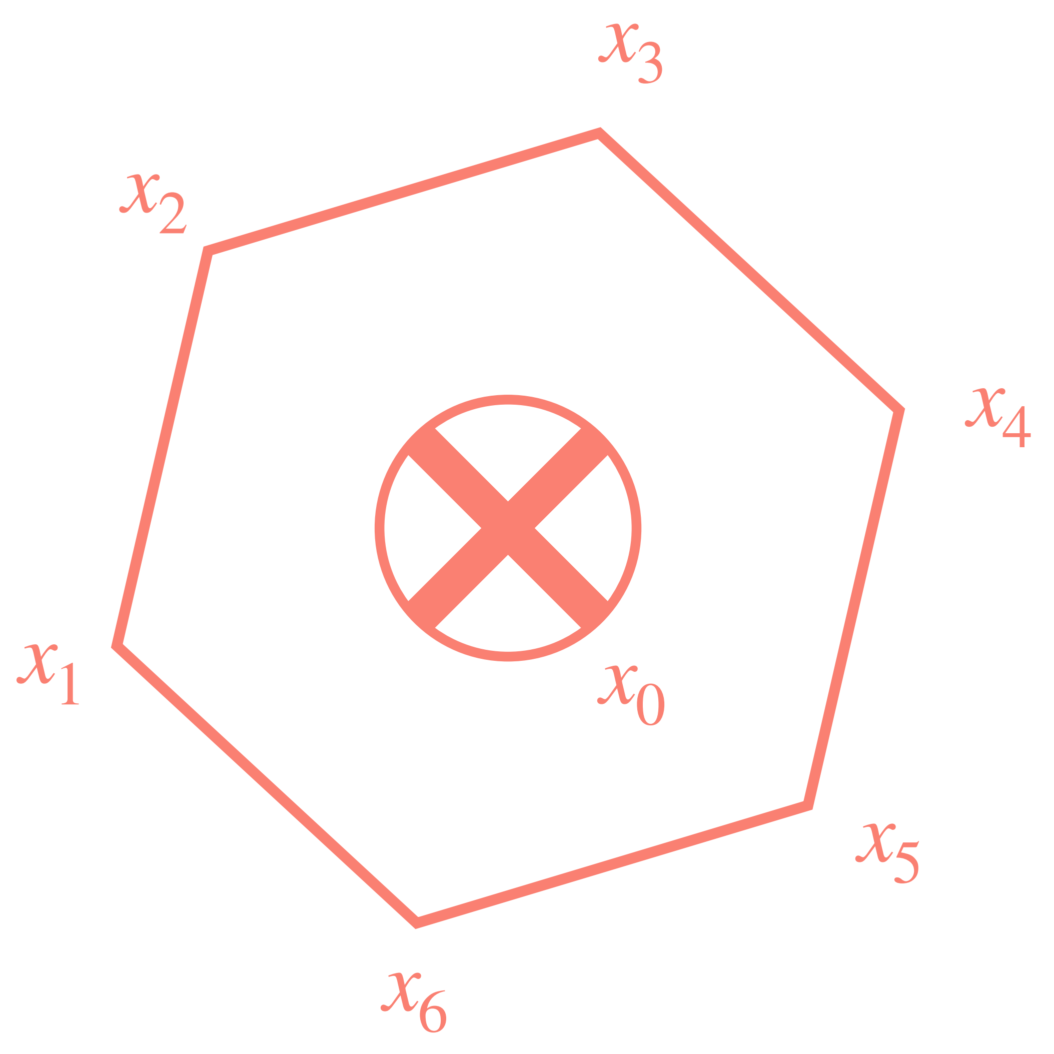


$A_{34}^{(2),N_c^2}, A_{34}^{(2),1}, A_{43}^{(2),1}$ $A_{34}^{(2),1}$ $A_{34}^{(2),n_f/N_c}, A_{43}^{(2),n_f/N_c}, A_{\delta}^{(2),n_f/N_c^2}$

Based on new bootstrap ideas

"Hexagonal Wilson loop with Lagrangian insertion at two loops in N=4 super Yang-Mills theory"

Carrôlo, Chicherin, Henn, Yang, YZ, JHEP 07 (2025) 214



Outline

Introduction

Methodology

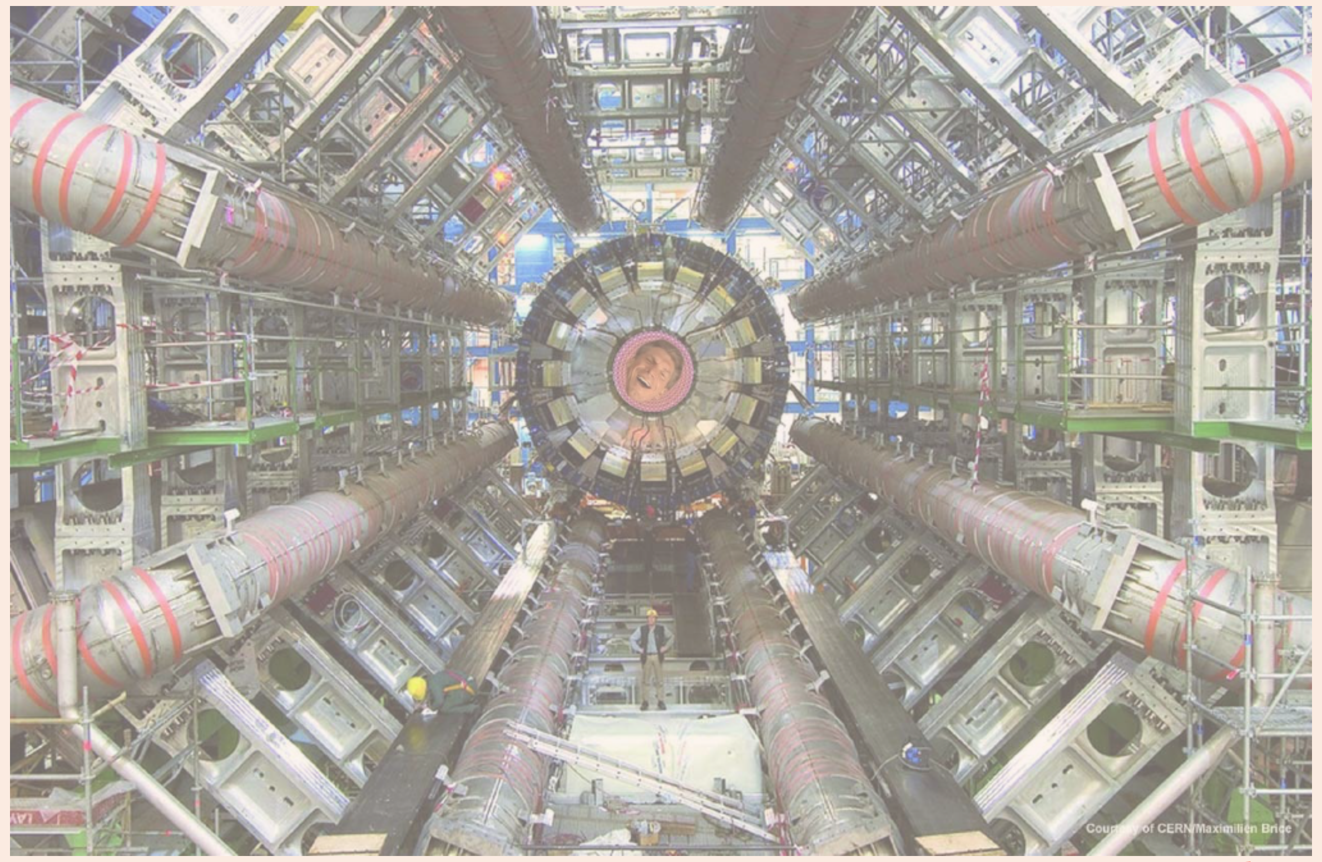
Analytic Feynman integral frontiers

Case 1: 2loop 6point Feynman integrals

Case 2: 3loop 5point Feynman integrals

Summary and Outlook

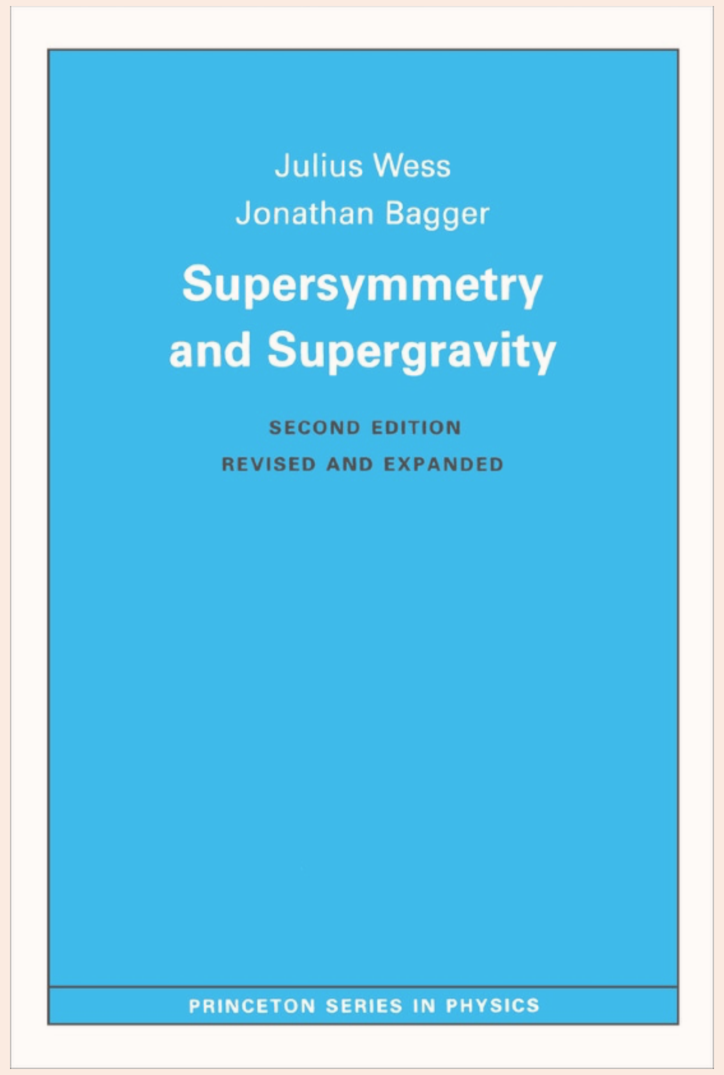
Introduction



Precision physics

$$\sigma = \sigma^{LO} + \sigma^{NLO} + \sigma^{NNLO}$$

Feynman integrals



Formal theory

N=8 supergravity UV finiteness

Gravitational wave template computations

for instance,
Driesse, Jakobsen , Klemm, Mogull, Nega, Plefka, Sauer, Usovitsch
Nature 641 (2025) 8063, 603-607

Why *analytic*?

- Analytic results can be very fast for numeric results
- Analytic methods can be sources for developing new numeric methods
- Theoretical aspects of quantum field theory

  for examples: 2loop N=4 SYM theory spacelike splitting amplitude

  Henn, Ma, Xu, Yan, *YZ*, Zhu, Phys.Rev.D 112 (2025) 7, 076003
- Quantum field theory computation of gravitational wave

  analytic continuation/ Fourier transform is sometimes needed
- Function space bootstrap for physical observable

  Carrôlo, Chicherin, Henn, Yang, YZ, JHEP 07 (2025) 214

Analytic Feynman integral computation and the methodology

Current status of analytic Feynman integral computation

	4-point	5-point	6-point	7-point	8-point
One-loop	known	known	known	known	known
Two-loop	most known	some results	?	?	?
Three-loop	some results	?	?	?	?
Four-loop	some results	?	?	?	?

with dimensional regulation

Current status of analytic Feynman integral computation

	4-point	5-point	6-point	7-point	8-point
One-loop	known	known	known	known	known
Two-loop	most known	some results	?	?	?
Three-loop	some results	?	?	?	?
Four-loop	some results	?	?	?	?

with dimensional regulation

Frontier

Current status of analytic Feynman integral computation

	4-point	5-point	6-point	7-point	8-point
One-loop	known	known	known	known	known
Two-loop	most known	some results	?	?	?
Three-loop	some results	?	?	?	?
Four-loop	some results	?	?	?	?

with dimensional regulation

Frontier

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, PRL. 135, 031601
Henn, Matijasic, Miczajka, Peraro, Xu, YZ, JHEP 08(2024) 027
Henn, Peraro, Xu, YZ, JHEP 03 (2022) 056

Liu, Matijasic, Miczajka, Xu, Xu, YZ,
Phys.Rev.D 112 (2025) 1, 016021, editors' suggestion

Canonical Differential Equation, new insights

ϵ-factorized differential equation algorithm — ϵ-collaboration 2506.09124

Better Integration-by-parts (IBP) reduction — **Blade,** Guan, Liu, Ma, Wu 2024
Comput.Phys.Commun. 310 (2025) 109538
NeatIBP, Wu, Boehm, Ma, Xu, *YZ* 2023
Comput.Phys.Commun. 295 (2024) 108999

Alphabet searching — **Effortless,** Matijasic, Miczajka to appear
https://github.com/antonela-matijasic/Effortless
BaikovLetter, Jiang, Liu, Xu, Yang, 2401.07632
SOFIA Correia, Giroux, Mizera 2503.16601

Solving differential equation — Novel representation of one-fold integration
Liu, Matijasic, Miczajka, Xu, Xu, YZ, 2411.18697

Using NeatIBP

"**NeatIBP 1.0**, a package generating small-size integration-by-parts relations for Feynman integrals"
Wu, Boehm, Ma, Xu, YZ, *Comput. Phys. Commun. 295 (2024), 108999*

"Performing integration-by-parts reductions using NeatIBP 1.1 + Kira"
Wu, Boehm, Ma, Usovitsch, Xu, YZ, Comput.Phys.Commun. 316 (2025) 109798

Using algebraic geometry to find short IBP system, 2 or 3 orders of magnitudes shorter than that from Laporta algorithm.

Studies that used NeatIBP	**Reference**
Differential Equations for Energy Correlators in Any Angle	arXiv:2506.02061
One-loop amplitudes for $t\bar{t}j$ and $t\bar{t}\gamma$ productions at the LHC through $O(\epsilon^2)$	arXiv:2505.10406
Two-loop Feynman integrals for leading color $Wt\bar{t}$ production	JHEP 07 (2025) 001
Two-loop QCD helicity amplitudes for $gg \to gt\bar{t}$ at leading color	JHEP 03 (2025) 070
Full-color double-virtual amplitudes for $q\bar{q} \to b\bar{b}H$	JHEP 03 (2025) 066
Three-loop five-point pentagon-box-box Feynman diagram	arXiv:2411.18697
Two-loop QCD corrections for $pp \to t\bar{t}j$	arXiv:2411.10856
Two-loop amplitudes for $W\gamma\gamma$ production at LHC	JHEP 12 (2025) 221
NLO corrections to $J/\Psi c\bar{c}$ photoproduction	Phys.Rev.D 110 (2024) 9, 094047
Two-loop five-point two-mass planar integrals	JHEP 10 (2024) 167
Two-loop integrals for $t\bar{t}j$ production at hadron colliders in the leading color approximation	JHEP 07 (2024) 073

phenomenology application of NeatIBP

2loop 6point Feynman integrals

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, *Phys. Rev. Lett. 135, 031601*

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, *JHEP 08(2024) 027*

2loop Feynman integral: Scale frontier

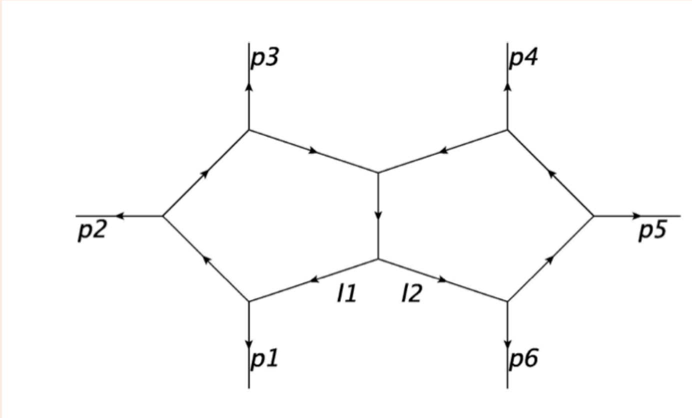
2loop 5point massless	*Gehrmann, Henn, Lo Presti 2015* *Chicherin, Gehrmann, Henn, Wasser, YZ, Zoia 2019*	5 scales
2loop 5point one-mass	*Papadopoulos, Tommasini, Wever 2019* *Abreu, Ita, Moriello, Page, Tschernow, Zeng 2020* *Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia 2023* *Jiang,Liu,Xu,Yang 2024* *Badger, Becchetti,Giraudo, Zoia 2024*	6 scales
2loop 5point two-mass	*Cordero, Figueiredo, Kraus, Page and Reina 2023* for leading-Color pp→ttH amplitudes with a light-quark loop	7 scales
2loop 6point massless	*Henn, Matijasic, Miczajka, Peraro, Xu, YZ, 2025* *for NNLO 4 jets production, 2 jets+ 2photons*	8 scales!

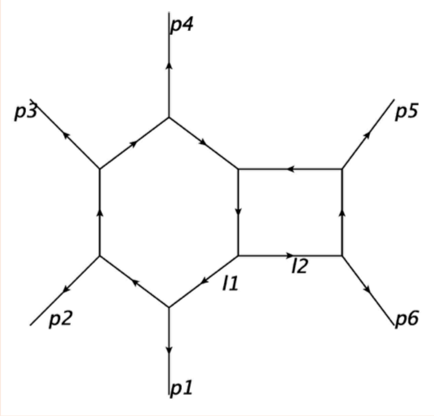
$s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{16}, s_{123}, s_{345}$

All planar 2loop 6point integrals

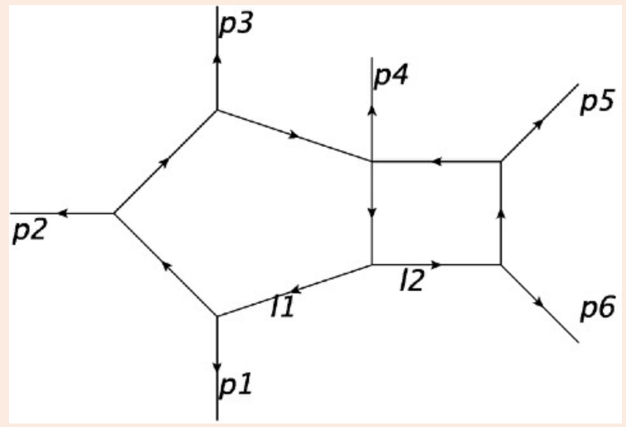
J. Henn, A. Matijasic, J. Miczajka, T. Peraro, Y. Xu, YZ, *Phys. Rev. Lett*. 135, 031601

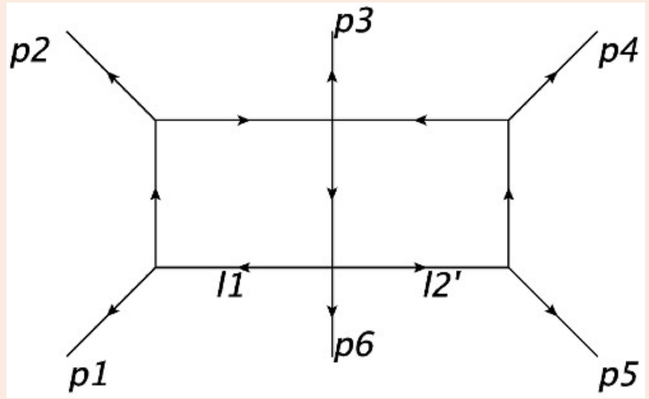
267
UT integrals

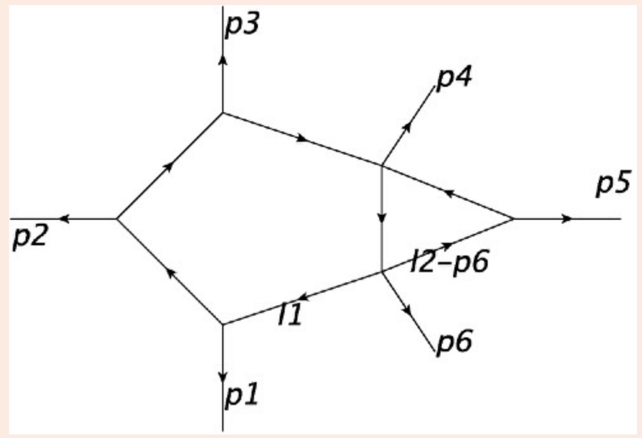


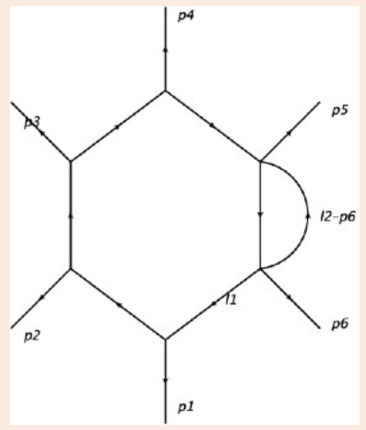


202
UT integrals









Momentum Twistor

external momenta $d=4$

$$p_i = x_{i+1} - x_i$$

dual coordinates

$$\epsilon_{\dot\beta\dot\alpha} x^{\dot\alpha\gamma} \lambda_{A,\gamma} = \mu_{A,\dot\beta}$$
$$\epsilon_{\dot\beta\dot\alpha} x^{\dot\alpha\gamma} \lambda_{B,\gamma} = \mu_{B,\dot\beta}$$

$$(Z_A, Z_B) \to x \qquad (Z_i, Z_{i+1}) \to x_{i+1}$$

$$Z_i = \begin{pmatrix} \lambda_{i,\alpha} \\ \mu_{i,\dot\beta} \end{pmatrix}, \quad i = 1, \ldots 6$$

momentum twistor

$\langle Z_A Z_B Z_C Z_D \rangle$ is dual conformally invariant

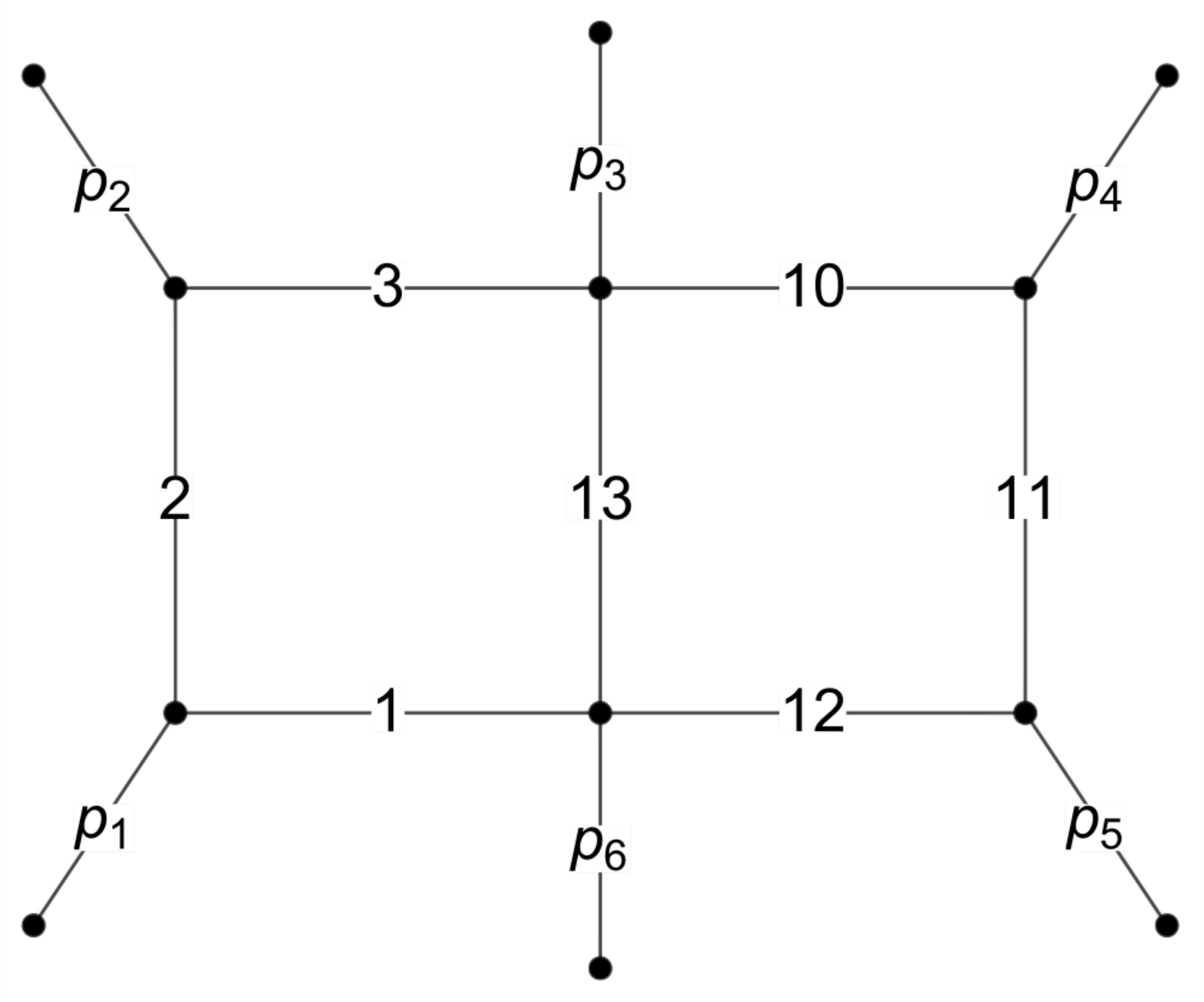
$$\tilde\lambda_i = \frac{\langle i,\, i+1\rangle \mu_{i-1} + \langle i+1,\, i-1\rangle \mu_i + \langle i-1,\, i\rangle \mu_{i+1}}{\langle i,\, i+1\rangle \langle i-1,\, i\rangle},$$

$$(\lambda_{i,\alpha}, \tilde\lambda_{i,\dot\alpha})$$

spinor helicity

Uniformly transcendental (UT) basis determination

key step



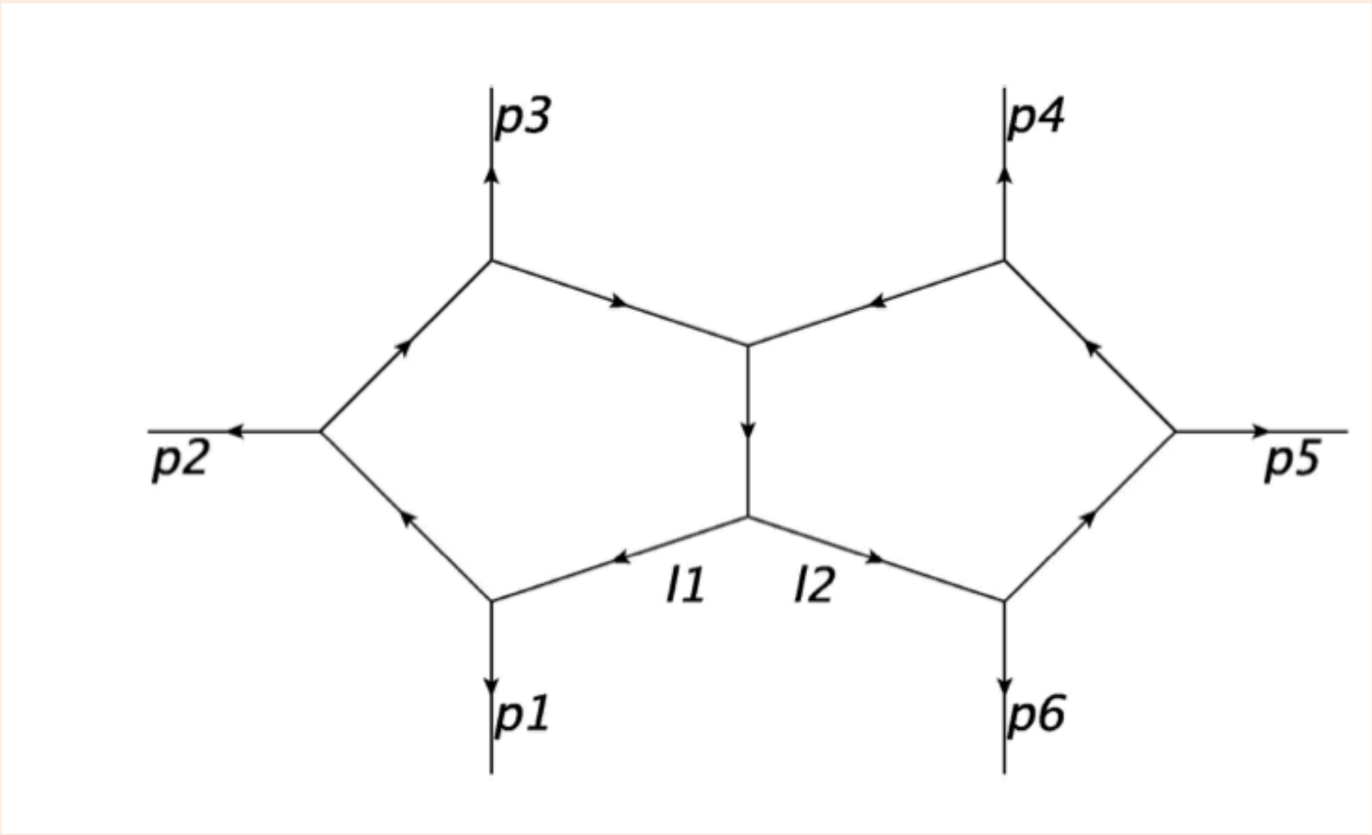
$$I_{\mathrm{db},i} = \int \frac{d^{4-2\epsilon} l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon} l_2}{i\pi^{2-\epsilon}} \frac{N_i}{D_1 D_2 D_3 D_{10} D_{11} D_{12} D_{13}}\,, \qquad i = 1, \ldots, 7$$

$$\begin{aligned}
N_1 &= -s_{12} s_{45} s_{156}\,,\\
N_2 &= -s_{12} s_{45} (l_1 + p_5 + p_6)^2\,,\\
N_3 &= \frac{s_{45}}{\epsilon_{5126}} G\begin{pmatrix} l_1 & p_1 & p_2 & p_5 + p_6 \\ p_1 & p_2 & p_5 & p_6 \end{pmatrix},\\
N_4 &= \frac{s_{12}}{\epsilon_{1543}} G\begin{pmatrix} l_2 - p_6 & p_5 & p_4 & p_1 + p_6 \\ p_1 & p_5 & p_4 & p_3 \end{pmatrix},\\
N_5 &= -\frac{1}{4} \frac{\epsilon_{1245}}{G(1,2,5,6)} G\begin{pmatrix} l_1 & p_1 & p_2 & p_5 & p_6 \\ l_2 & p_1 & p_2 & p_5 & p_6 \end{pmatrix},\\
N_6 &= \frac{1}{8} G\begin{pmatrix} l_1 & p_1 & p_2 \\ l_2 - p_6 & p_4 & p_5 \end{pmatrix} + \frac{D_2 D_{11} (s_{123} + s_{126})}{8}\,,\\
N_7 &= -\frac{1}{2\epsilon} \frac{\Delta_6}{G(1,2,4,5) D_{13}} G\begin{pmatrix} l_1 & p_1 & p_2 & p_4 & p_5 \\ l_2 & p_1 & p_2 & p_4 & p_5 \end{pmatrix}.
\end{aligned}$$

Chiral numerator
(Arkani-Hamed, Bourjaily, Cachazo, Trnka 2011)
/ Gram determinant
correspondence

$$\Delta_6 = \langle 12\rangle[23]\langle 34\rangle[45]\langle 56\rangle[61] - \langle 23\rangle[34]\langle 45\rangle[56]\langle 61\rangle[12]\,.$$

2loop 6point top sector, UT integrals



J. Henn, A. Matijasic, J. Miczajka, T. Peraro, Y. Xu, YZ, PRL 135, 031601

UT integrals list

5 MIs (this sector)
267 MIs (whole family)

$$I_1^{\text{DP-a}} = \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{N_1^{\text{DP-a}} - N_4^{\text{DP-a}}}{D_1 \dots D_9}$$ → evanescent

$$I_2^{\text{DP-a}} = \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{N_2^{\text{DP-a}} - N_3^{\text{DP-a}}}{D_1 \dots D_9}$$

$$I_3^{\text{DP-a}} = F_3 \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{\mu_{12}}{D_1 \dots D_9}$$ → evanescent

$$I_4^{\text{DP-a}} = F_4 \epsilon^2 \int \frac{d^{6-2\epsilon}l_1}{i\pi^{3-\epsilon}} \frac{d^{6-2\epsilon}l_2}{i\pi^{3-\epsilon}} \frac{1}{D_1 \dots D_9}$$ → evanescent, 6D weight-6 integral

$$I_5^{\text{DP-a}} = \int \frac{d^{4-2\epsilon}l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon}l_2}{i\pi^{2-\epsilon}} \frac{N_1^{\text{DP-a}} + N_4^{\text{DP-a}} + F_5 \mu_{12}}{D_1 \dots D_9}$$

245 letters in total
except the 6D ones

"evanescent": vanishing up to ϵ^0

N_1, N_2, N_3 and N_4 are chiral numerators — Arkani-Hamed, Bourjaily, Cachazo, Trnka 2010

Complete canonical differential equation for 2l6p planar integrals

Use momentum twistor Variables

267 $\times$ 267 for double pentagon
202 $\times$ 202 for hexagon box

$$\frac{\partial}{\partial x_i} I(x,\epsilon) = \epsilon A_i(x) I(x,\epsilon)$$

$$A_i = \frac{\partial}{\partial x_i}\tilde{A}, \quad \tilde{A} = \sum_k \tilde{a}_k \log(W_k)$$

use alphabet to fit the canonical differential equation

Even letter, Odd letter and the more complicated ...

Even letter $F(s)$ a polynomial in Mandelstam variables or homogeneously linear in square roots

Conjecture: a Feynman integrals' even letters are all from Landau singularity?

Odd letter $\frac{P(s)-\sqrt{Q(s)}}{P(s)+\sqrt{Q(s)}}$ $\log(W) \mapsto -\log(W)$ under the sign change of the square root

"square roots": ϵ_{ijkl}, Δ_6, , $\sqrt{\lambda(s_{12}, s_{34}, s_{56})}$

pseudo scalar

leading singularity hexagon

Källin function from massive triangle diagrams

More complicated letter $\frac{P(s)-\sqrt{Q_1(s)}\sqrt{Q_2(s)}}{P(s)+\sqrt{Q_1(s)}\sqrt{Q_2(s)}}$

Even letter, Odd letter and the more complicated …

245 letters

156 Even letters

$$s_{12}, \quad s_{123}$$
$$s_{12} - s_{123}$$
$$\ldots$$
$$-s_{12}s_{45} + s_{123}s_{345}$$
$$\ldots$$
$$\sqrt{\lambda(s_{12}, s_{34}, s_{56})}, \quad \epsilon_{ijkl}$$

79 Odd letters

$$\frac{s_{12} + s_{34} - s_{56} - \sqrt{\lambda(s_{12}, s_{34}, s_{56})}}{s_{12} + s_{34} - s_{56} + \sqrt{\lambda(s_{12}, s_{34}, s_{56})}}$$
$$\ldots$$
$$\frac{s_{12}s_{23} - s_{23}s_{34} + s_{23}s_{56} + s_{34}s_{123} - s_{234}(s_{12} + s_{123}) - \epsilon_{1234}}{s_{12}s_{23} - s_{23}s_{34} + s_{23}s_{56} + s_{34}s_{123} - s_{234}(s_{12} + s_{123}) + \epsilon_{1234}},$$
$$\ldots \quad \ldots$$
$$\frac{-s_{12}s_{45}s_{234} + s_{34}s_{61}s_{123} + s_{345}(-s_{23}s_{56} + s_{123}s_{234}) - \Delta_6}{-s_{12}s_{45}s_{234} + s_{34}s_{61}s_{123} + s_{345}(-s_{23}s_{56} + s_{123}s_{234}) + \Delta_6}$$

Then the canonical DE is derived analytically after ~50 times of numeric IBP running

10 More complicated letters

$$\frac{P - \epsilon_{1234}\sqrt{\lambda(s_{12}, s_{34}, s_{56})}}{P + \epsilon_{1234}\sqrt{\lambda(s_{12}, s_{34}, s_{56})}} \qquad \ldots$$

Boundary Values

Numeric boundary values

It is fine to use the package AMFlow to get ~100 digits as the boundary value for double-box, pentagon-triangle, hexagon-bubble diagrams *Liu, Wang, Ma, 2018* *Liu, Ma 2022*

Analytic boundary values

It is still possible to get *fully analytic* boundary values

$$X_0 : \quad \{s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{16}, s_{123}, s_{234}, s_{345}\} \to \{-1, -1, -1, -1, -1, -1, -1, -1, -1\}$$

Solve the canonical DE on a curve starting with X_0 and require the finite solution
Some known integrals' boundary values
} analytic boundary value

boundary value for a point in the physical region also obtained

Boundary Values

Analytic boundary values

Boundary values at the initial point, are combination of poly-logarithm of roots of unity

$$
\begin{aligned}
\epsilon^4 I_{\mathrm{db},1}(X_0) &= 1 + \frac{\pi^2}{6}\epsilon^2 + \frac{38}{3}\zeta_3\epsilon^3 + \left(\frac{49\pi^4}{216} + \frac{32}{3}\ \mathrm{Im}\left[\mathrm{Li}_2(\rho)\right]^2\right)\epsilon^4,\\
\epsilon^4 I_{\mathrm{db},2}(X_0) &= 1 + \frac{\pi^2}{6}\epsilon^2 + \frac{34}{3}\zeta_3\epsilon^3 + \left(\frac{71\pi^4}{360} + 20\ \mathrm{Im}\left[\mathrm{Li}_2(\rho)\right]^2\right)\epsilon^4,\\
I_{\mathrm{db},3}(X_0) &= I_{\mathrm{db},4}(X_0) = I_{\mathrm{db},5}(X_0) = 0,\\
\epsilon^4 I_{\mathrm{db},6}(X_0) &= -\left(\frac{\pi^4}{540} + \frac{4}{3}\ \mathrm{Im}\left[\mathrm{Li}_2(\rho)\right]^2\right)\epsilon^4,\\
\epsilon^4 I_{\mathrm{db},7}(X_0) &= 0.
\end{aligned}
$$

from the ordinary differential equation

spurious pole asymptotic analysis

$$\rho = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

Solution of canonical DE

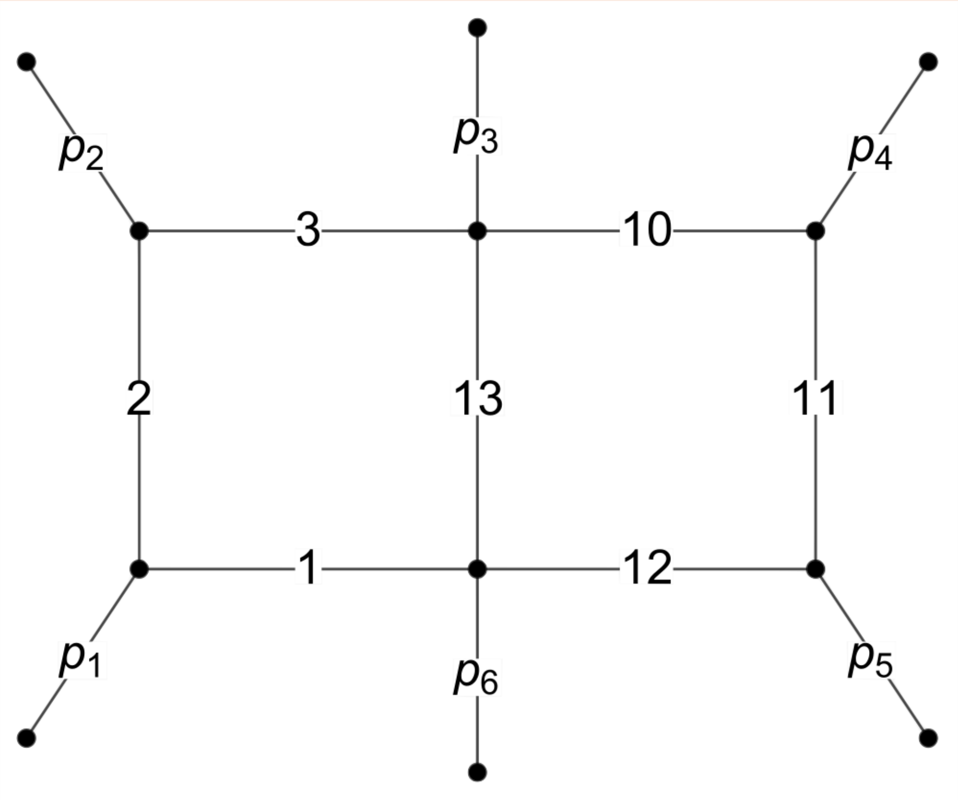
$$dI = \epsilon (d\tilde{A}) I$$

$$I = \frac{1}{\epsilon^4}\left(I^{(0)} + \epsilon I^{(1)} + \epsilon^2 I^{(2)} + \epsilon^3 I^{(3)} + \epsilon^4 I^{(4)} + \dots\right)$$

$$I^{(n)} = I^{(n)}_{X_0} + \int_\gamma (d\tilde{A}) I^{(n-1)}$$

weight-1, weight-2

All in logarithm and classical poly-logarithm



$$\begin{aligned} I^{(2)}_{\text{db},1} = & -\log(-v_1)\log(-v_2) - \log(-v_1)\log(-v_3) + \log(-v_1)\log(-v_4) - \log(-v_1)\log(-v_5) - \\ & \log(-v_1)\log(-v_6) + 4\log(-v_1)\log(-v_8) + \frac{1}{2}\log^2(-v_1) + \log(-v_2)\log(-v_3) - \\ & \log(-v_2)\log(-v_4) - \text{Li}_2\left(1 - \frac{v_2 v_5}{v_7 v_8}\right) + \log(-v_2)\log(-v_6) + \log(-v_2)\log(-v_7) - \\ & 2\text{Li}_2\left(1 - \frac{v_2}{v_8}\right) - \log(-v_2)\log(-v_8) - \log^2(-v_2) - \log(-v_3)\log(-v_4) + \log(-v_3)\log(-v_5) - \\ & \text{Li}_2\left(1 - \frac{v_3 v_6}{v_8 v_9}\right) - 2\text{Li}_2\left(1 - \frac{v_3}{v_8}\right) - \log(-v_3)\log(-v_8) + \log(-v_3)\log(-v_9) - \\ & \log^2(-v_3) - \log(-v_4)\log(-v_5) - \log(-v_4)\log(-v_6) + 4\log(-v_4)\log(-v_8) + \\ & \frac{1}{2}\log^2(-v_4) + \log(-v_5)\log(-v_6) + \log(-v_5)\log(-v_7) - 2\text{Li}_2\left(1 - \frac{v_5}{v_8}\right) - \log(-v_5)\log(-v_8) - \\ & \log^2(-v_5) - 2\text{Li}_2\left(1 - \frac{v_6}{v_8}\right) - \log(-v_6)\log(-v_8) + \log(-v_6)\log(-v_9) - \log^2(-v_6) - \\ & \log(-v_7)\log(-v_8) - \frac{1}{2}\log^2(-v_7) - \log(-v_8)\log(-v_9) + 3\log^2(-v_8) - \frac{1}{2}\log^2(-v_9) + \\ & \frac{\pi^2}{6} \end{aligned}$$

Solution of canonical DE

J. Henn, A. Matijasic, J. Miczajka, T. Peraro, Y. Xu, YZ, *Phys. Rev. Lett* 135, 031601

$$dI = \epsilon(d\tilde{A})I$$

$$I = \frac{1}{\epsilon^4}\left(I^{(0)} + \epsilon I^{(1)} + \epsilon^2 I^{(2)} + \epsilon^3 I^{(3)} + \epsilon^4 I^{(4)} + \dots\right)$$

$$I^{(n)} = I^{(n)}_{X_0} + \int_\gamma (d\tilde{A}) I^{(n-1)}$$

weight-3, weight-4

$$\begin{aligned}\vec{I}^{(4)} &= \vec{I}^{(4)}(\vec{x}_0) + \int_0^1 \mathrm{d}t \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t} \vec{I}^{(3)}(\vec{x}_0) + \int_0^1 \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \frac{\mathrm{d}\tilde{A}}{\mathrm{d}t_1}\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t_2} \vec{f}^{(2)}(t_2) \\ &= \vec{I}^{(4)}(\vec{x}_0) + \int_0^1 \mathrm{d}t \left(\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t}\vec{I}^{(3)}(\vec{x}_0) + \left(\tilde{A}(1) - \tilde{A}(t)\right)\frac{\mathrm{d}\tilde{A}}{\mathrm{d}t}\vec{f}^{(2)}(t)\right).\end{aligned}$$

one-fold integration

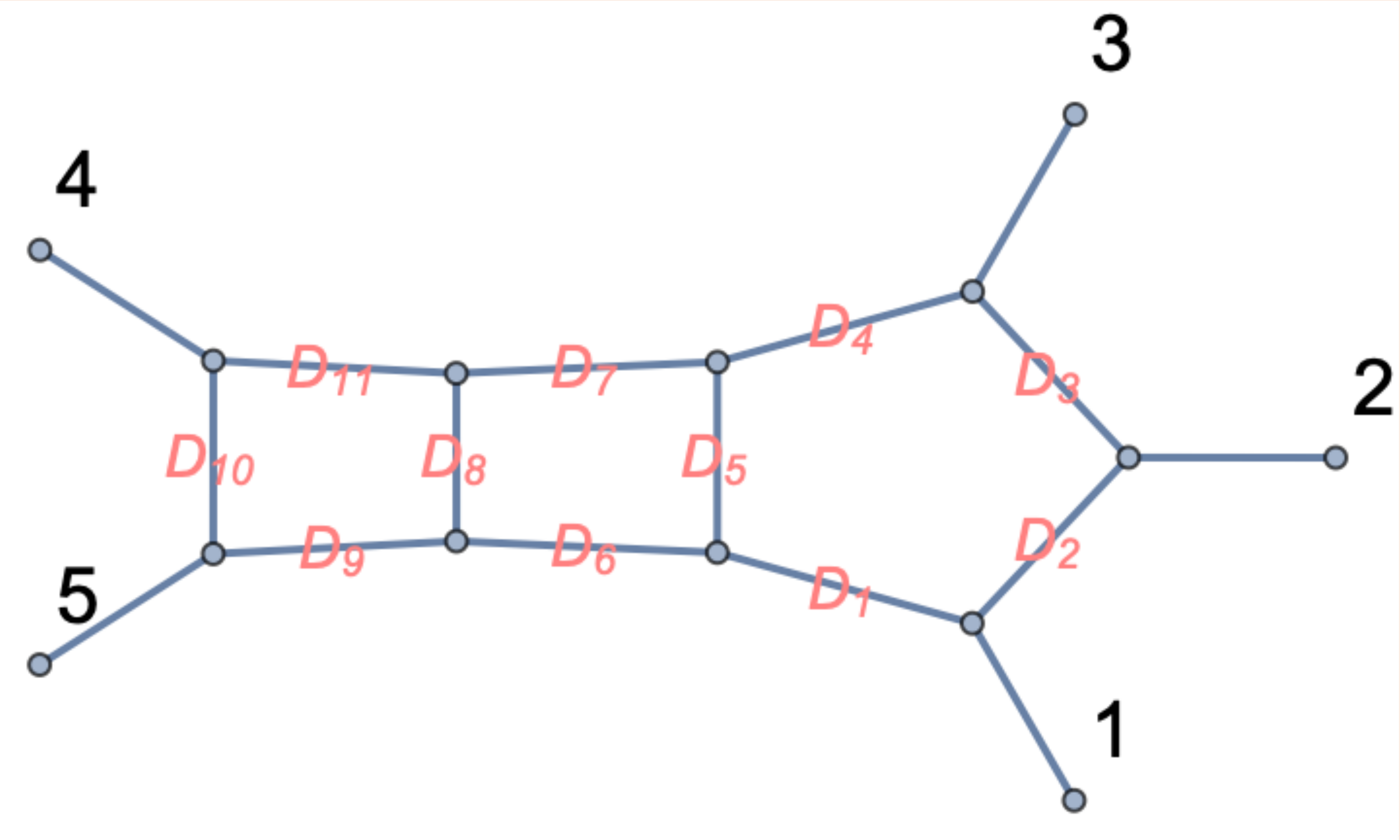
It takes minutes on a laptop to get 14 digits for all 2loop 6point integrals from our solution in both Euclidean and Physical regions

3loop 5point Feynman integrals

Liu, Matijasic, Miczajka, Xu, Xu, YZ,

Phys.Rev.D 112 (2025) 1, 016021, editors' suggestion

3loop 5point planar family



5 scales $s_{12}, s_{23}, s_{34}, s_{45}, s_{15}$

316 Master Integrals

Liu, Matijasic, Miczajka, Xu, Xu, YZ, Phys.Rev.D 112 (2025) 1, 016021

✓ UT basis found!

Baikov analysis
Gram determinant

Canonical differential equation complicated ?

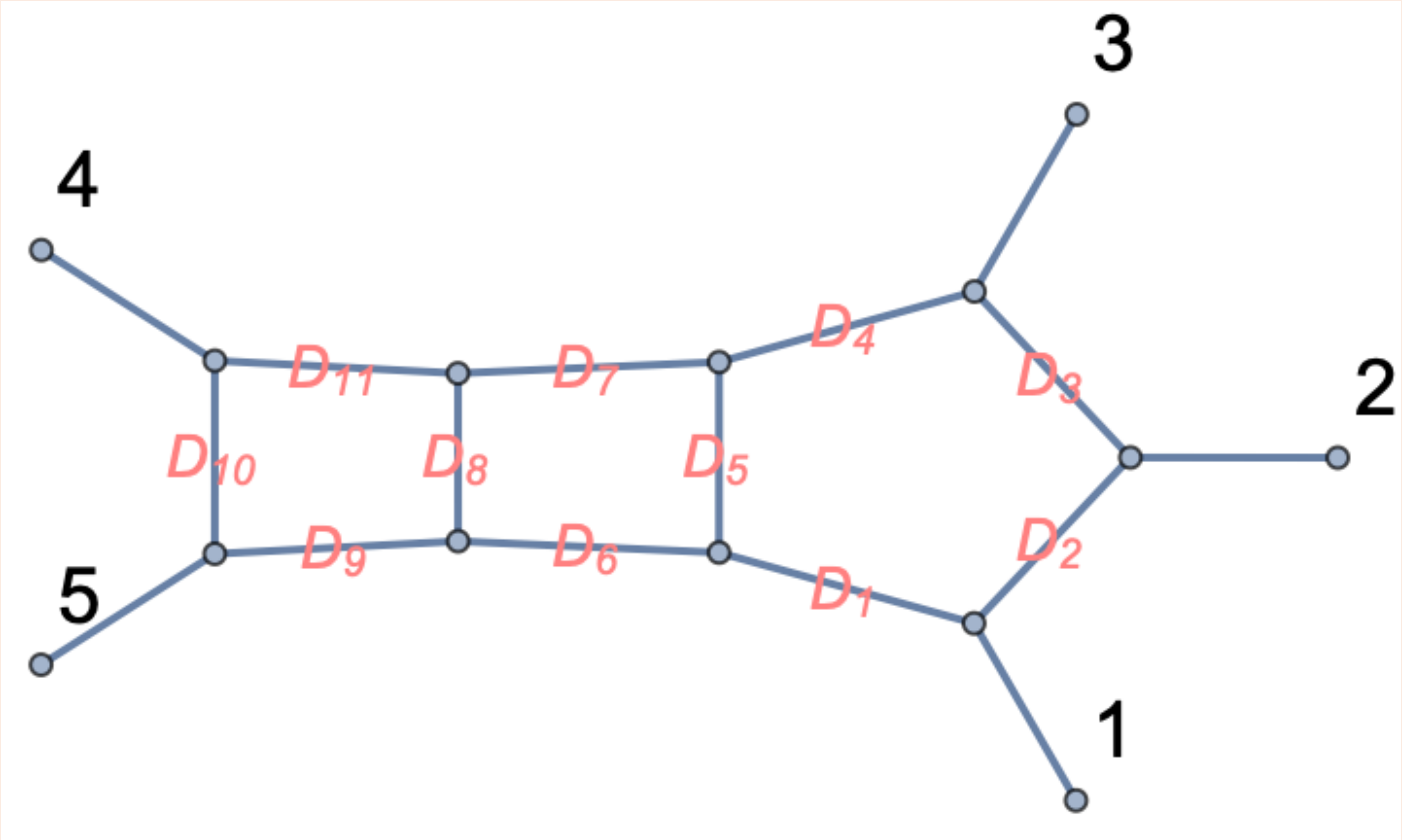
We use NeatIBP to derive the differential equation

~100 million IBPs → 85000 IBPs

hard to integrate to weight-6?

A novel one-fold representation

First family of 3loop 5point calculated



✓ UT basis found!

✓ Canonical differential equation found with NeatIBP

31 letters …

All boundary values up to weight-6 are obtained by spurious pole analysis

5 scales $s_{12}, s_{23}, s_{34}, s_{45}, s_{15}$

316 Master Integrals

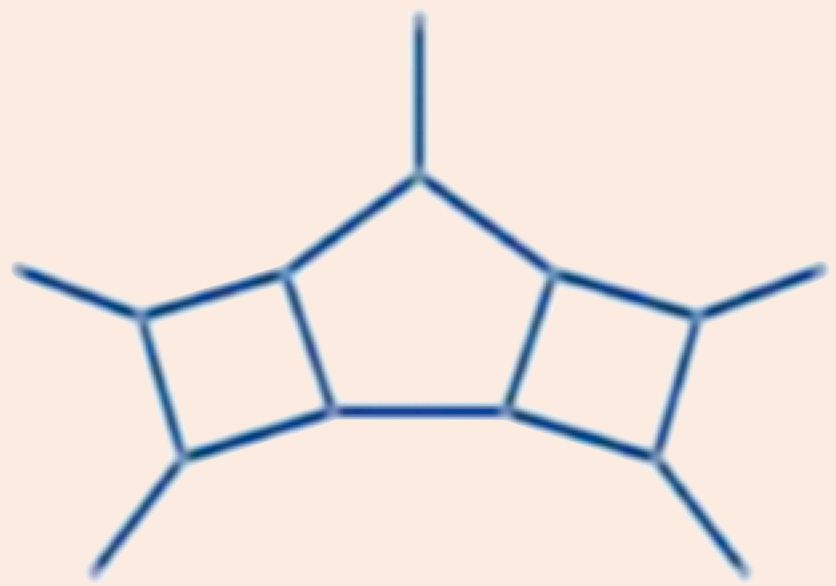
✓ weight-1,2,3 classical polylogarithm, weight-4,5,6 one-fold integration
It takes 2 minutes on a laptop to get 10 digits from our analytic solution

Liu, Matijasic, Miczajka, Xu, Xu, YZ, Phys.Rev.D 112 (2025) 1, 016021 editors' suggestion

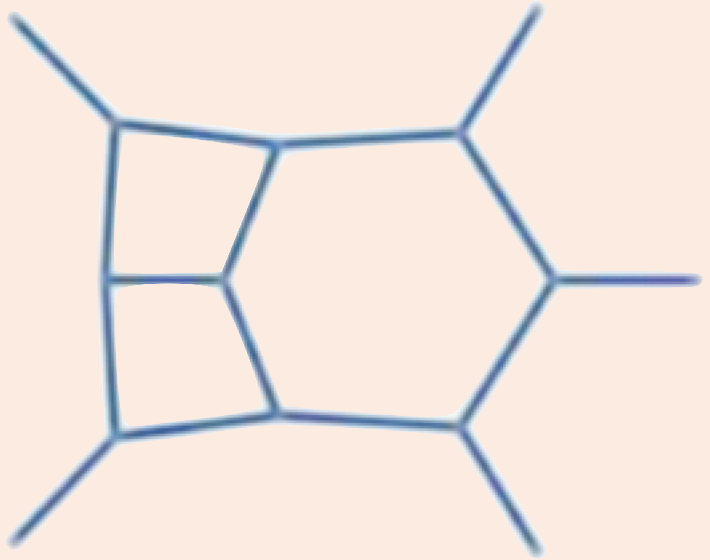
What's more

All 3loop 5point integrals, canonical DE obtained

Symbol calculated



367 MIs



431 MIs

734 MIs

56 Letters (2loop 5point, 31 Letters)

to appear

small number

good news for bootstrap

Weight	1	2	3	4	5	6
3loop 5point Symbols	5	20	76	285	1000	2220

Bootstrap

Carrôlo, Chicherin, Henn, Yang, YZ
JHEP 07 (2025) 214

Bootstrap

Traditional bootstrap

Admissible letters → Integrable symbols → Ansatz

for N=4 sYM
planar amplitudes bootstrap

dual conformal: 9 letters for six points

Caron-Huot, Dixon, McLeod, von Hippel, 2016 (5loop)
Caron-Huot, Dixon, Dulat, von Hippel, McLeod, Papathanasiou, 2020 (6loop & 7loop)

Our bootstrap

Feynman integrals → symbols in integrals → Ansatz

aims for QCD amplitudes

no IBP needed

Carrôlo, Chicherin, Henn, Yang, YZ

Bootstrap

Traditional bootstrap

Admissible letters

↓

Integrable symbols

↓

Ansatz

for N=4 sYM
planar amplitudes bootstrap

larger number than

Our bootstrap

Feynman integrals

↓

symbols in integrals

↓

Ansatz

aims for QCD amplitudes

dual conformal: 9 letters for six points

Caron-Huot, Dixon, McLeod, von Hippel, 2016 (5loop)
Caron-Huot, Dixon, Dulat, von Hippel, McLeod, Papathanasiou, 2020 (6loop & 7loop)

no IBP needed

Carrôlo, Chicherin, Henn, Yang, YZ

Wilson loop and scattering amplitudes

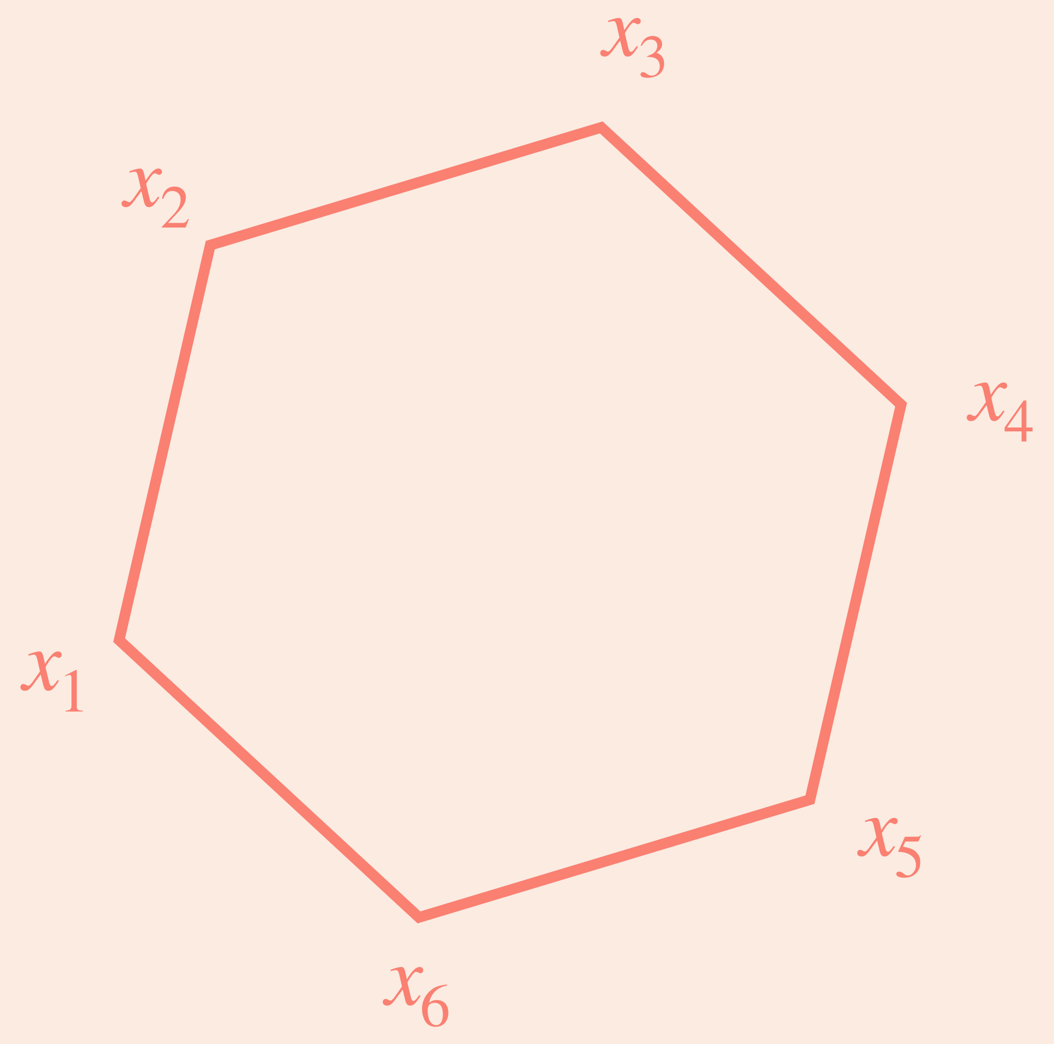
Alday and Maldacena, 2007

N=4 planar sYM,
Lightlike boundary Wilson loop expectation value is dual to MHV scattering amplitude

$$\log \frac{A_n^{\mathrm{MHV}}(p_1,\ldots,p_n)}{A_n^{\mathrm{MHV,\ tree}}(p_1,\ldots,p_n)} = \log\langle W(x_1,\ldots,x_n)\rangle \qquad p_i = x_{i+1} - x_i$$

infrared divergence

UV divergence from the gluon exchange near the corner



In the x-space, it has anomalous (dual) conformal invariance DCI

$$K^\mu \ \log\langle W(x_1,\ldots,x_n)\rangle_{\mathrm{finite}} = \frac{1}{2}\Gamma_{\mathrm{cusp}} \sum_{i=1}^{n} x^\mu_{i,i+1} \log\left(\frac{x^2_{i,i+2}}{x^2_{i-1,i+1}}\right)$$

Bern-Dixon-Smirnov (BDS) Ansatz is a solution for this equation,
However the physical solution is BDS Ansatz plus a DCI remainder function

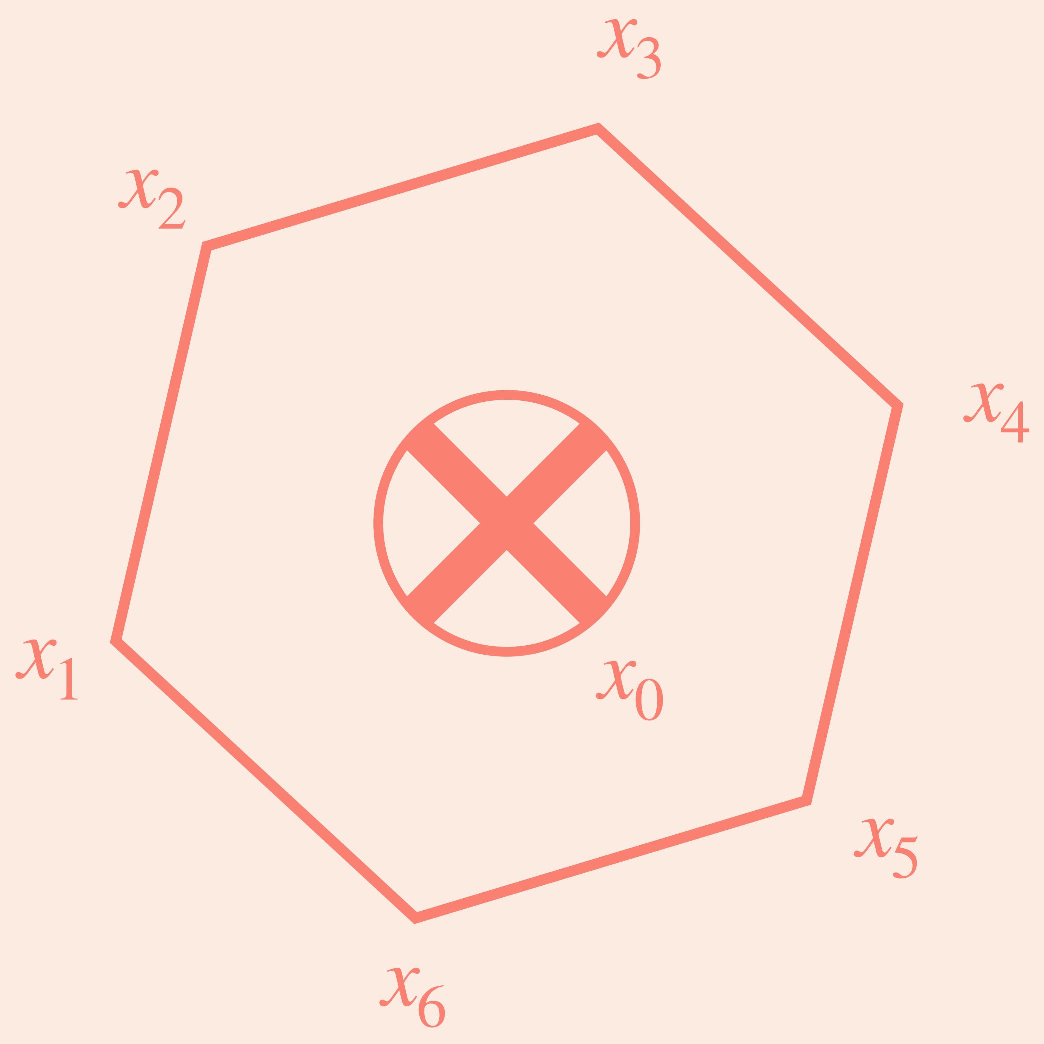
Wilson loop + Lagrangian insertion and scattering amplitudes

$$F_n(x_1,\ldots,x_n;x_0) \equiv \pi^2 \frac{\langle W(x_1,\ldots,x_n)\mathcal{L}(x_0)\rangle}{\langle W(x_1,\ldots,x_n)\rangle}$$

Alday, Buchbinder and Tseytlin 2011
Alday, Heslop and Sikorowski 2013

N=4 sYM on-shell Lagrangian

This quantity is finite



Surprisingly, it is dual to the maximally transcendental part of all-plus helicity amplitude in pure Yang-Mills theory!

L-loop F_n is dual to (L+1)-loop all-plus helicity amplitude in pure Yang-Mills theory

Chicherin and Henn, 2202.05596

Bootstrap

Set up an ansatz and fit the coefficients

ansatz from all functions (all integrable symbols)

usually too many functions …

ansatz from functions in Feynman integrals!

leading singularity (rational function)

$$F_n = \sum_{i=1}^{22} \sum_j c_{ij} R_i f_j$$

transcendental function
from Feynman integrals

Wilson loop + Lagrangian leading singularity

Chicherin and Henn, 2022
Brown, Henn, Mazzucchelli, Trnka 2025

$$B_{ijklm} := \frac{\langle AB(mij) \cap (jkl)\rangle^2}{\langle ABjm\rangle\langle ABij\rangle\langle ABjk\rangle\langle ABlj\rangle\langle ABmi\rangle\langle ABkl\rangle},$$

$$(abc) \cap (def) := (ab)\langle cdef\rangle - (ac)\langle bdef\rangle + (bc)\langle adef\rangle$$

$$B_{ijkl} := \frac{\langle ijkl\rangle^2}{\langle ABij\rangle\langle ABjk\rangle\langle ABkl\rangle\langle ABli\rangle}.$$

Kermit functions

Take the gauge, $\langle ABij\rangle \to \langle ij\rangle$ rational function in momentum twistor variables

There are only 20 (22) linearly independent leading singularities

Function space: planar 2loop 6point massless integrals

weight-3
integrable symbols
$528 > 266$

A counting of functions, with the dihedral symmetry

Transcendental weight	1	2	3	4
# All symbols	9	62	319	945
# Two-loop six-point symbols	9	62	266	639
# Two-loop five-point one-mass symbols	9	59	263	594
# One-loop squared symbols	9	59	221	428
# Genuine two-loop six-point symbols	0	0	3	45

Surprisingly,
the number of genuinely two-loop six-point functions are very small …
It is a good news for bootstrap

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, 2501.01847

First Application:
Hexagonal Wilson loop with Lagrangian insertion at two loops in N=4 sYM

Carrôlo, Checherin, Henn, Yang, YZ, JHEP 07 (2025) 214

$$F_n(x_1,\ldots,x_n;x_0) := \pi^2 \frac{\langle 0|W_n[x_1,\ldots,x_n]L(x_0)|0\rangle}{\langle 0|W_n[x_1,\ldots,x_n]|0\rangle}$$

Using group representation to construct an Ansatz with dihedral symmetry →

weight	0	1	2	3	4
unknowns in dihedral ansatz	5	22	139	644	1892
genuine **unknowns**	4	20	125	585	1718
constraints:					
soft	3	20	116	515	1439
collinear	3	20	121	551	1539
spurious $s_{24}=0$	1	12	76	360	1044
spurious $s_{25}=0$	1	6	36	165	483
scaling dimension	0	4	20	125	585
triple collinear	1	5	31	134	353
total **constraints**	4	20	125	585	1718
unfixed **unknowns**	0	0	0	0	0

Bootstrap

$F_2(x_1\ldots x_6;x_0)$ is fixed in the symbol level!

First Application:
Two-loop six-point QCD bootstrap

Carrôlo, Chicherin, Henn, Yang, YZ

to appear

$$- - + + + +$$

two-loop six-point
pure-Yang-mills bootstrap done!

no supersymmetry

Summary and Outlook

Analytic computation of all 2loop 6point planar massless integrals is done
The first computation on 3loop 5point family is done; all families' result is coming

NeatIBP, a powerful package for cutting-edge IBP reduction

a lot of future applications

The dawn of bootstrap

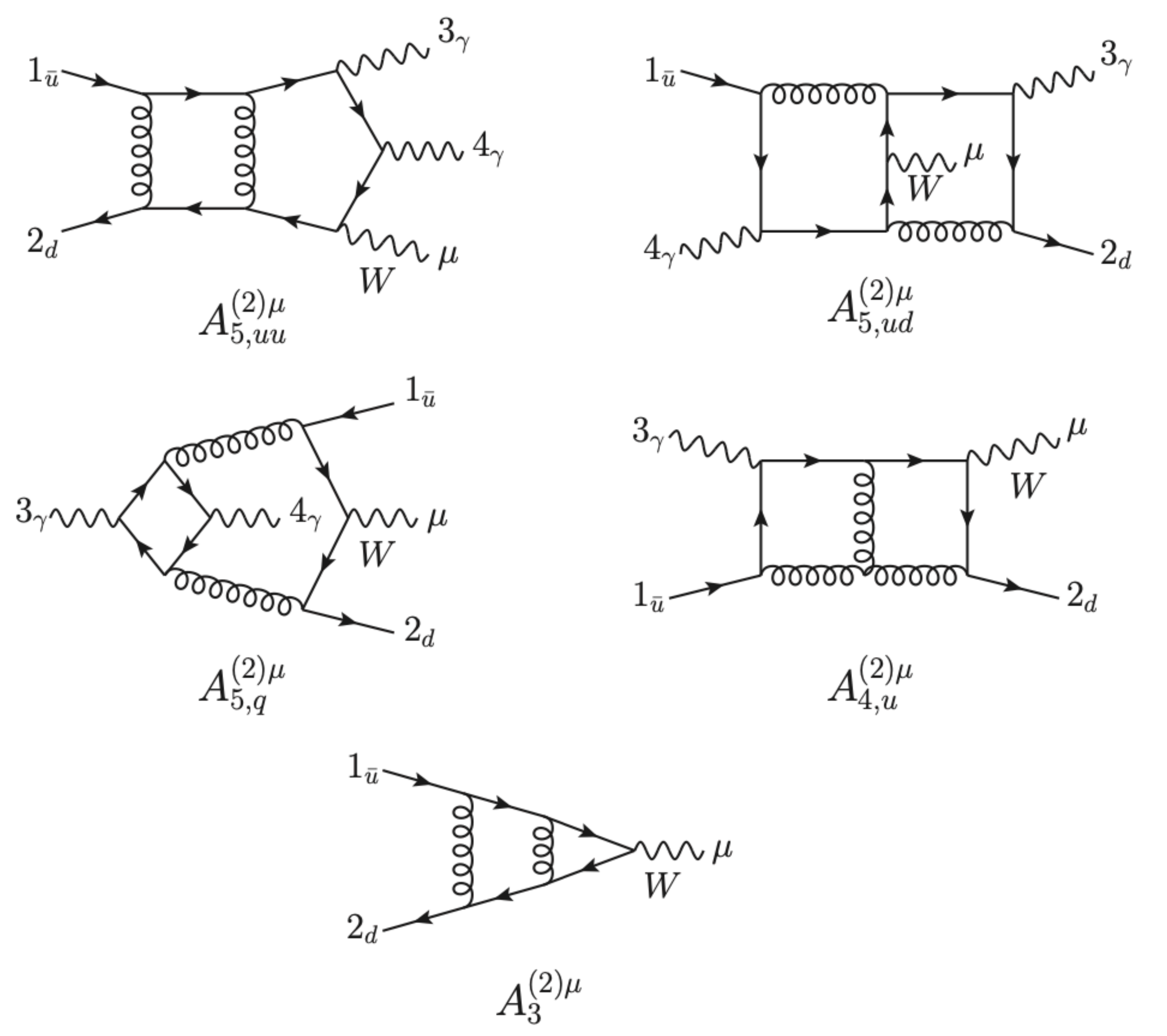
two-loop hexagonal Wilson loop + Lagrangian insertion
with the help of the function space of Feynman integrals

QCD amplitude bootstrap on the way

Multi-loop multi-leg Feynman integrals are no longer that difficult!

NNLO QCD correction to W + 2 photon production

Badger, Hartanto, Wu, *YZ*, Zoia, JHEP 12 (2025) 221



Most complicated diagram would be the two-loop five-point with one massive external leg (W boson)

6 scales: 5 Mandelstam + 1 mass

Laporta algorithm needs >10 million IBP reductions
Traditional IBP reduction is difficult ...

NNLO QCD correction to W + 2 photon production

Badger, Hartanto, Wu, *YZ*, Zoia, JHEP 12 (2025) 221

Using NeatIBP 1.0 to generate short IBP systems

family	deg.	# IBPs	# integrals	IBP disk size	running time
DPmz	5	26673	27432	71.4 MB	7h5m
DPzz	5	61777	63880	375.8 MB	21h54m*
HBmzz	5	15428	15916	17.0 MB	5h38m
HBzmz	5	10953	11289	13.4 MB	5h45m
HBzzz	5	21126	21766	38.0 MB	9h32m
PBmzz	5	10224	10329	7.8 MB	2h23m
PBzmz	5	11610	11791	6.5 MB	2h50m
PBzzz	5	8592	8752	5.8 MB	2h50m
HTmzzz	4	3120	3176	1.5 MB	1h12m
HTzmzz	4	6594	6650	2.5 MB	1h31m
HTzzzz	4	4680	4631	4.0 MB	2h31m

Laporta algorithm needs >10 million IBP reductions

$\sim$10 million $\rightarrow$ $\sim$10000

NeatIBP 1,0's IBP system has the size $\sim$10000

Planar:
NeatIBP+ Finiteflow 8 times faster,
3 times lower RAM usage than Finiteflow

non-Planar:
NeatIBP+ Finiteflow works
Finiteflow itself does not provide the result