



山东大学
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NNLO mixed QCD-EW corrections to W -pair production at electron-positron colliders

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Based on [JHEP 12 (2024) 038 and paper on preparing]

2025 Shanghai Symposium on Particle Physics and Cosmology

I. Research background

II. Calculation setup

III. Integral families of VWW/VW^*W^*

IV. Phenomenological results

Research background

W-mass anomaly

$m_W^{CDF} = 80.4335 \pm 0.0094 \text{ GeV}$

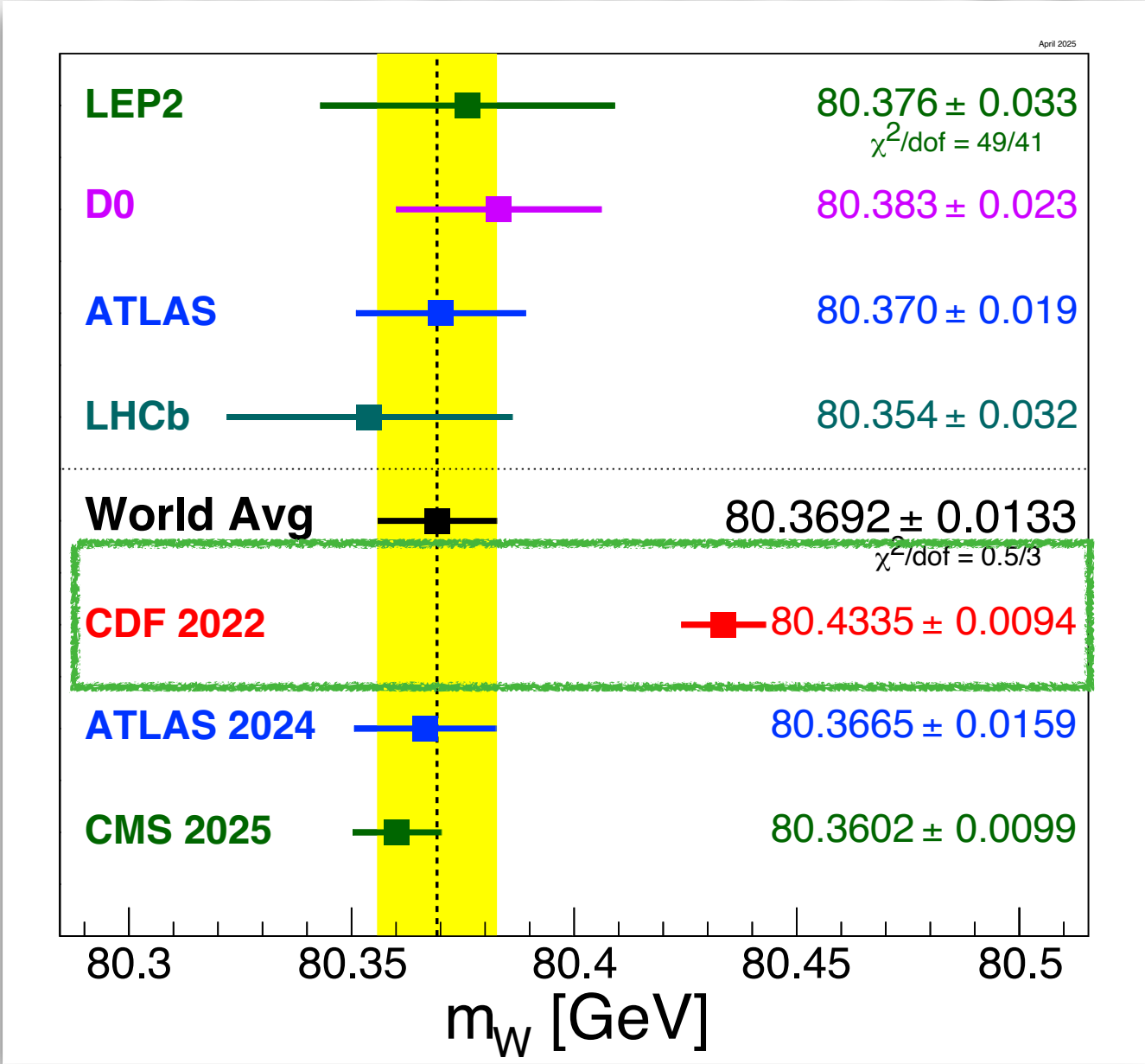
CDF II

7σ

SM

New physics beyond the SM ?

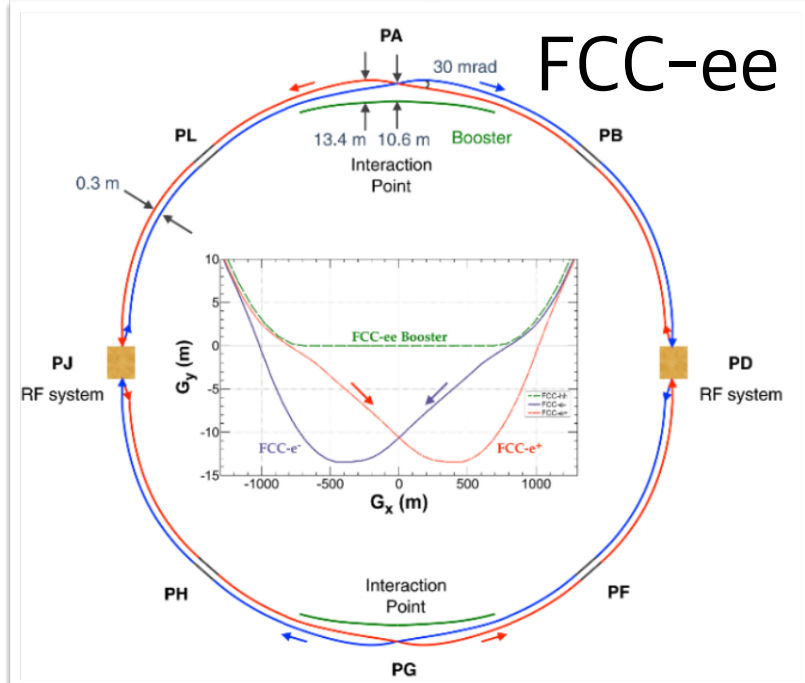
Requires a better understand of W physics!!!



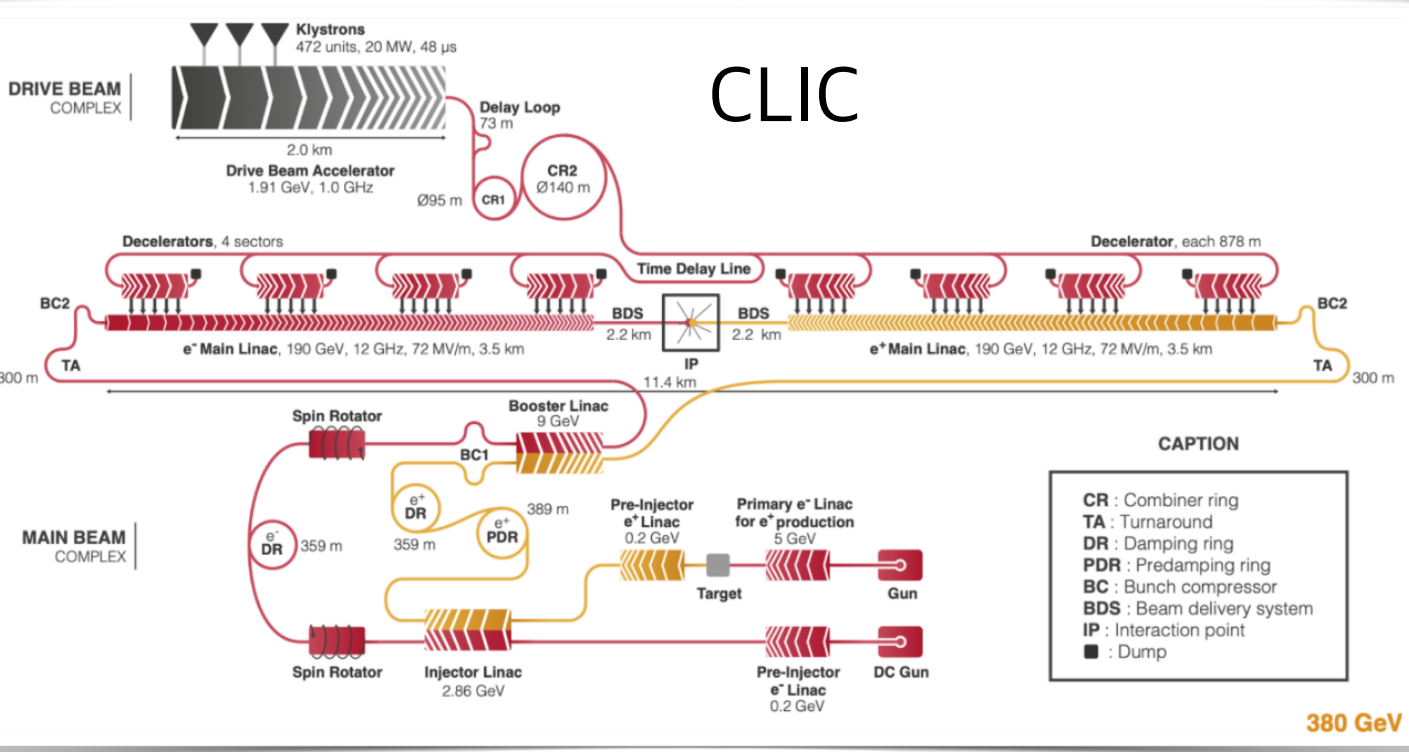
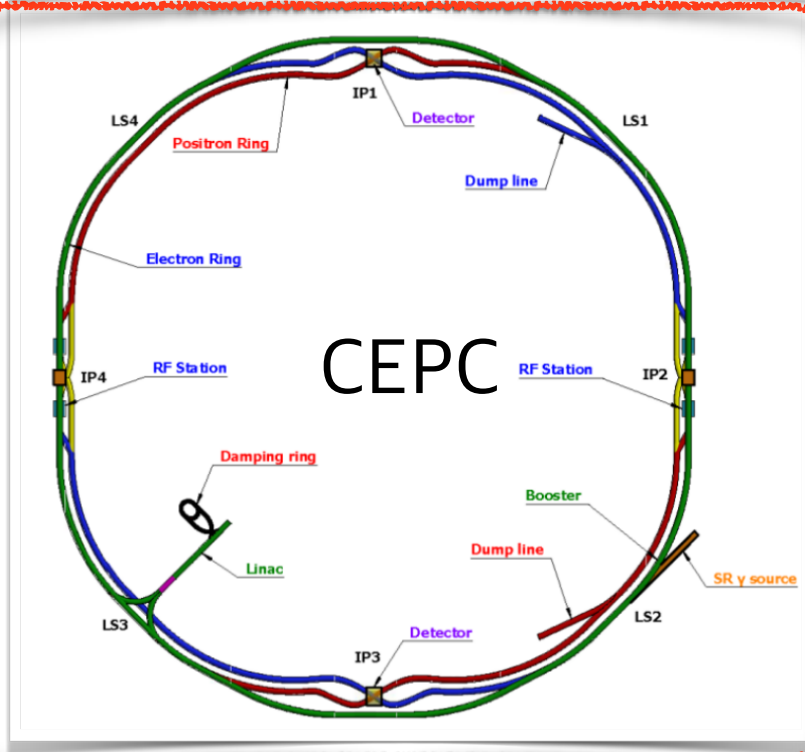
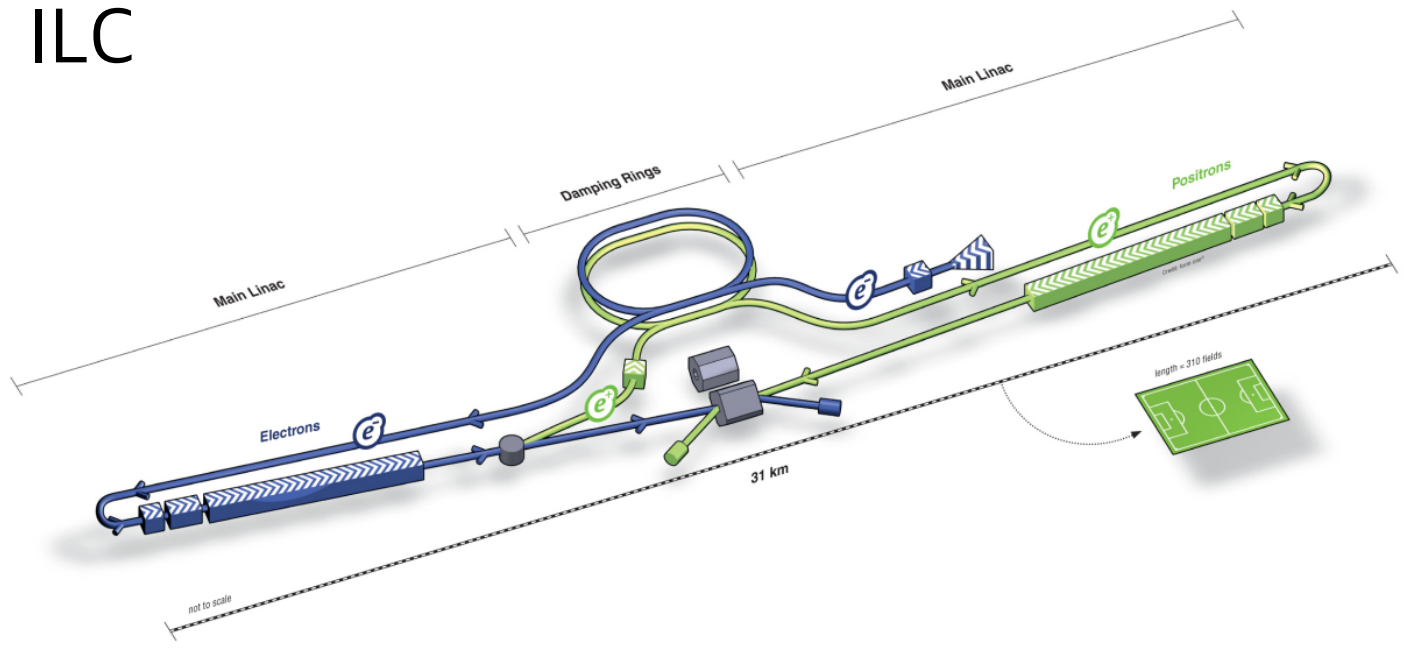
Research background

Future Lepton Colliders

Clean background



ILC



Research background



See Z. Qian's talk

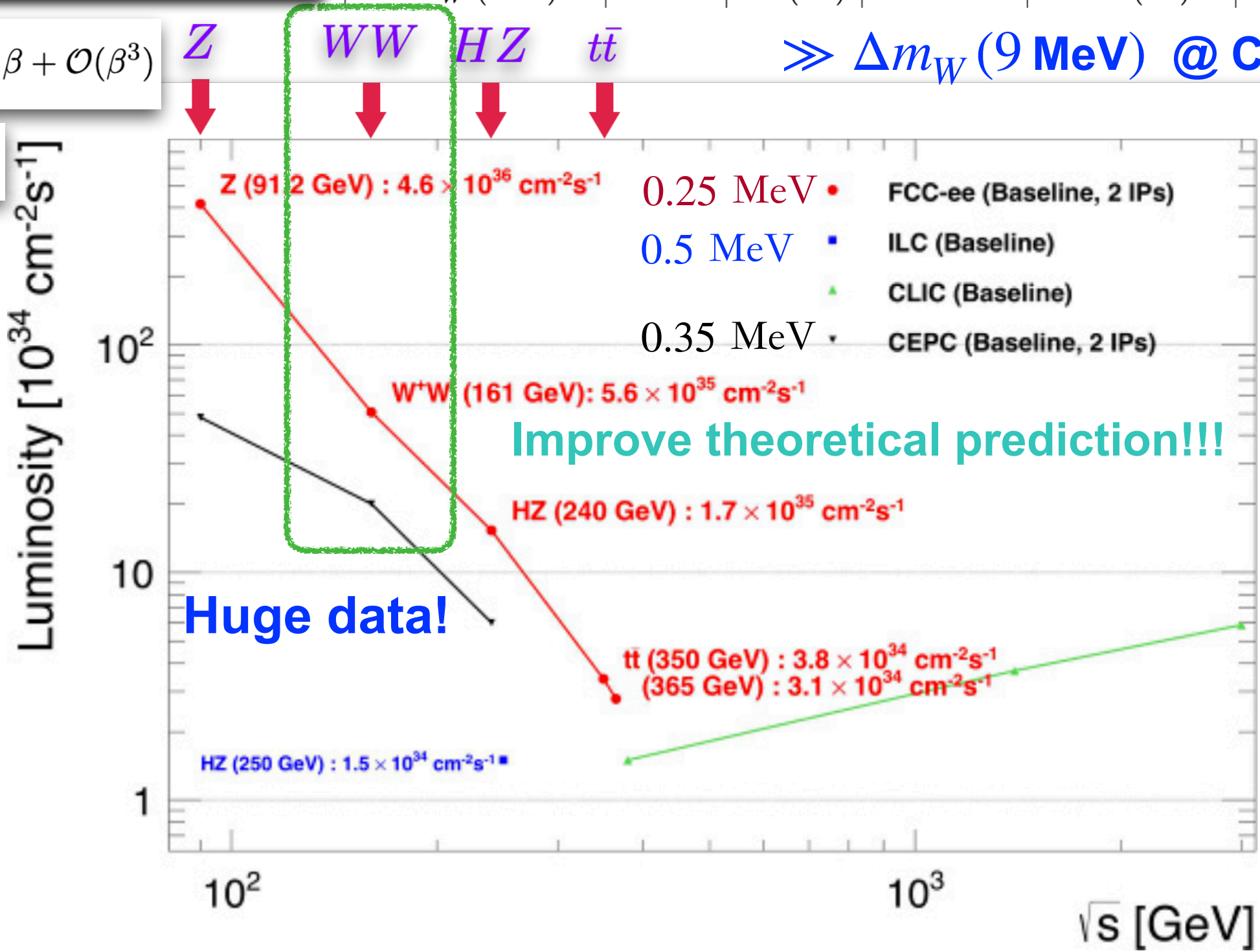
Future Lepton Colliders

Quantity	current	ILC250	ILC-GigaZ	FCC-ee	CEPC
$\Delta\alpha(m_Z)^{-1} (\times 10^3)$	17.8*	17.8*		3.8 (1.2)	17.8*
Δm_W (MeV)	12*	0.5 (2.4)		0.25 (0.3)	0.35 (0.3)

$$\sigma_{\text{Born}} = \frac{\pi\alpha^2}{s} \frac{1}{\sin^2\theta_W} \beta + \mathcal{O}(\beta^3)$$

$$\beta = \sqrt{(1 - m_W^2/E^2)}$$

$\gg \Delta m_W$ (9 MeV) @ CDF



[ILC, '13]

[CEPC, '18]

[FCC, '19]

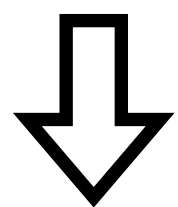
Research background

State-of-the-art

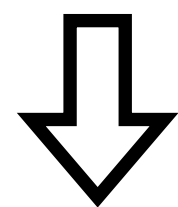
$$e^+(p_1, \lambda_1) + e^-(p_2, \lambda_2) \rightarrow W^+(p_3, \lambda_3) + W^-(p_4, \lambda_4),$$

❖ Perturbative expansion

$$\sigma = \boxed{\sigma_{\text{LO}} + \Delta\sigma_{\text{NLO}}^{\mathcal{O}(\alpha)}} + \boxed{\Delta\sigma_{\text{NNLO}}^{\mathcal{O}(\alpha\alpha_s)} + \Delta\sigma_{\text{NNLO}}^{\mathcal{O}(\alpha^2)} + \dots}$$



Well Studied



Still Unknown

❖ Theoretical overview

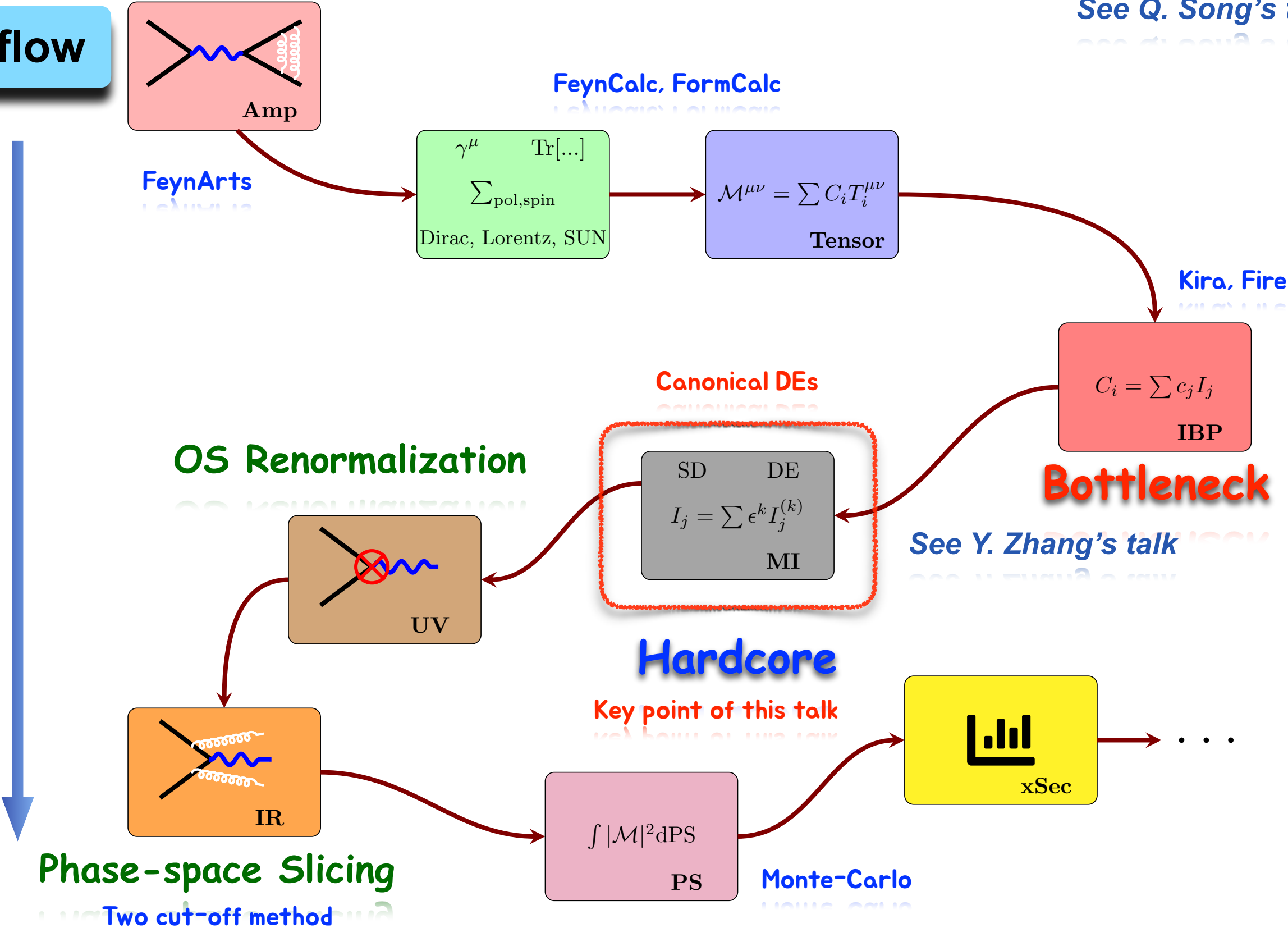
- **LO results:** [W. Alles, C. Boyer, A.J. Buras '77, A. Blondel et al. '86, K. Hagiwara et al. '87]
- **Radiative corrections :** [M. Lemoine '80, R. Philippe '82, M. Bohm '88, W. Beenakker '91]
- **Hard bremsstrahlung :** [J. Fleischer, K. Kołodziej '93, W. Beenakker, A. Denner '94]
- **LL ISR corrections :** [A. Denner '93, W. Beenakker '94, W. Beenakker '96]

Calculation setup

See Q. Song's talk

Workflow

Time ordering



Calculation setup

NLO EW corrections

❖ EW corrections

$$\Delta\sigma_{\text{EW}} = \sigma_{\text{virtual}} + \sigma_{\text{real}} + \sigma_{\text{h.o.ISR}}$$

❖ Virtual corrections **On-shell renormalization** [A.Denner '97]

$$\frac{d\sigma_{\text{virtual}}}{d\Omega} = \frac{\beta}{64\pi^2 s} \frac{1}{4} \sum_{\lambda_1, \dots, 4} 2 \operatorname{Re} \mathcal{M}_0^* \delta \mathcal{M}^{\mathcal{O}(\alpha)}$$

❖ Real photon radiation **Phase space slicing method** [B. W. Harris, '02]

$$\sigma_{\text{real}} = \sigma_{\text{S}}(\delta_s) + \sigma_{\text{HC}}(\delta_s, \delta_c) + \sigma_{\text{H}\bar{\text{C}}}(\delta_s, \delta_c)$$

❖ Initial-state radiation **Structure functions** [W. Beenakker, '96]

$$\sigma_{\text{h.o.ISR}} = \sigma_{\text{ISR-LL}} - \sigma_{\text{LL,sub}}$$

Calculation setup

NNLO QCD $\otimes$ EW corrections

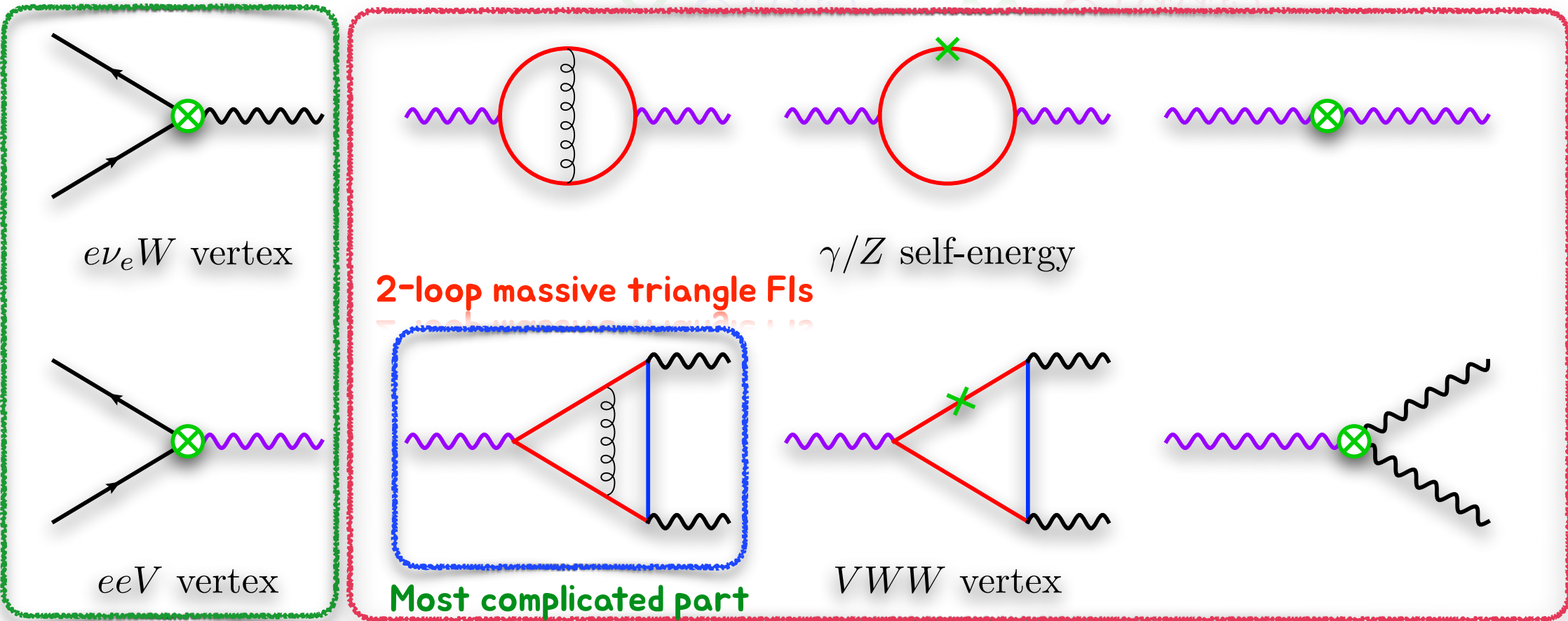
✿ $\mathcal{O}(\alpha\alpha_s)$ corrections to amplitudes

$$\delta\mathcal{M}^{\mathcal{O}(\alpha\alpha_s)} = \delta\mathcal{M}_{e\nu_e W}^{\mathcal{O}(\alpha\alpha_s)} + \delta\mathcal{M}_{eeV}^{\mathcal{O}(\alpha\alpha_s)} + \delta\mathcal{M}_{\gamma/Z\text{-S.E.}}^{\mathcal{O}(\alpha\alpha_s)} + \delta\mathcal{M}_{VWW}^{\mathcal{O}(\alpha\alpha_s)}$$

✿ Feynman diagrams

✕ $\mathcal{O}(\alpha_s)$

⊗ $\mathcal{O}(\alpha\alpha_s)$



Integral family

❖ Two-loop triangle integrals

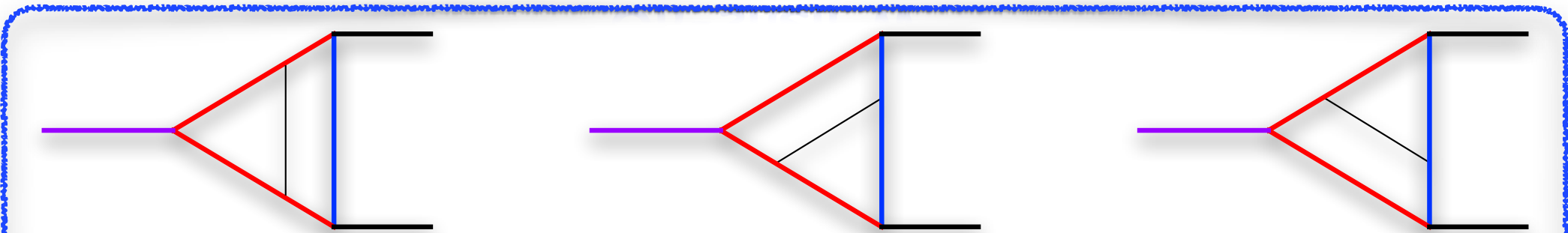
$$F(\alpha_1, \dots, \alpha_7) = \int \mathcal{D}^d l_1 \mathcal{D}^d l_2 \frac{1}{D_1^{\alpha_1} \dots D_7^{\alpha_7}}$$

❖ Propagators **Full mass dependence!**

$$D_1 = l_1^2 - m_t^2, \quad D_2 = l_2^2 - m_t^2, \quad D_3 = (l_1 - p_3)^2 - m_b^2, \quad D_4 = (l_2 - p_3)^2 - m_b^2$$

$$D_5 = (l_1 - p_3 - p_4)^2 - m_t^2, \quad D_6 = (l_2 - p_3 - p_4)^2 - m_t^2, \quad D_7 = (l_1 - l_2)^2.$$

❖ Topologies **On-shell W**



$$\mathcal{S} \equiv (1, 1, 1, 0, 1, 1, 1)_F \cup (1, 0, 1, 1, 1, 1, 1)_F \cup (1, 1, 1, 1, 1, 0, 1)_F$$

Integral families of VWW

Canonical DEs

❖ Differential equations

$$\frac{\partial}{\partial x_i} I(\vec{x}, \epsilon) = B_i(\vec{x}, \epsilon) I(\vec{x}, \epsilon)$$

❖ Canonical DEs

$$I = T(\vec{x}, \epsilon) g$$

[J. Henn, '13]

$$\frac{\partial}{\partial x_i} g(\vec{x}, \epsilon) = \epsilon A_i(\vec{x}) g(\vec{x}, \epsilon)$$

ϵ factorization

❖ Dlog-form

$$dg(\vec{x}, \epsilon) = \epsilon dA(\vec{x}) g(\vec{x}, \epsilon) = \epsilon \left[\sum_i M_i d \log \eta_i(\vec{x}) \right] g(\vec{x}, \epsilon)$$

See Y. Zhang's talk

- **Magnus series** [M. Argeri, '14]
- **Lee's algorithm** [R.N. Lee, '14]
- **Leading singularities** [J. Henn, '13]
- **Dlog form** [J. Henn, '20]
- **Intersection theory** [Chen, Yang, '20]

- **Epsilon, Fuchsia** [Prausa, '17, Gituliar, '17]
- **Libra** [R.N. Lee, '20]
- **DlogBasis** [J. Henn, '20]
- **CANONICA** [C. Meyer, '16]
- **INITIAL** [C. Dlapa, '20, '22]

constant matrix

symbol letters

Integral families of VWW

Canonical basis

✿ Pre-canonical basis

✿ Canonical DEs

• Linear-form

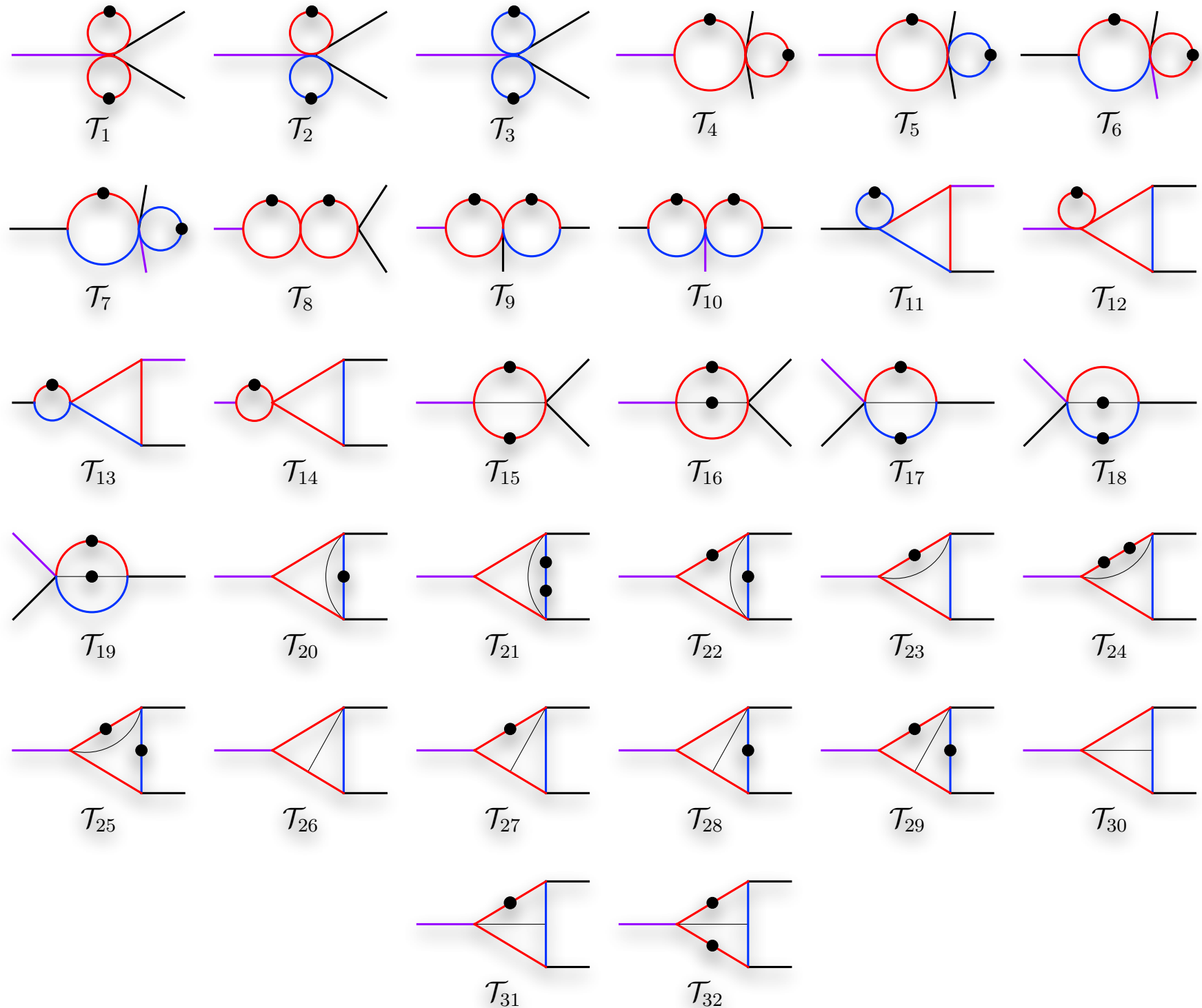
$$dF(\vec{x}, \epsilon) = d[B_0(\vec{x}) + \epsilon B_1(\vec{x})] F(\vec{x}, \epsilon),$$

Magnus series

[M. Argeri, '14]

• Canonical-form

$$dg(\vec{x}, \epsilon) = \epsilon dA(\vec{x})g(\vec{x}, \epsilon)$$



Integral families of VWW

General solutions

See Y. Zhang's talk

❖ Path-ordered exponential solution

$$g(\vec{x}, \epsilon) = \mathcal{P} \exp \left\{ \epsilon \int_{\gamma} dA \right\} g(\vec{x}_0, \epsilon)$$

❖ Order by order

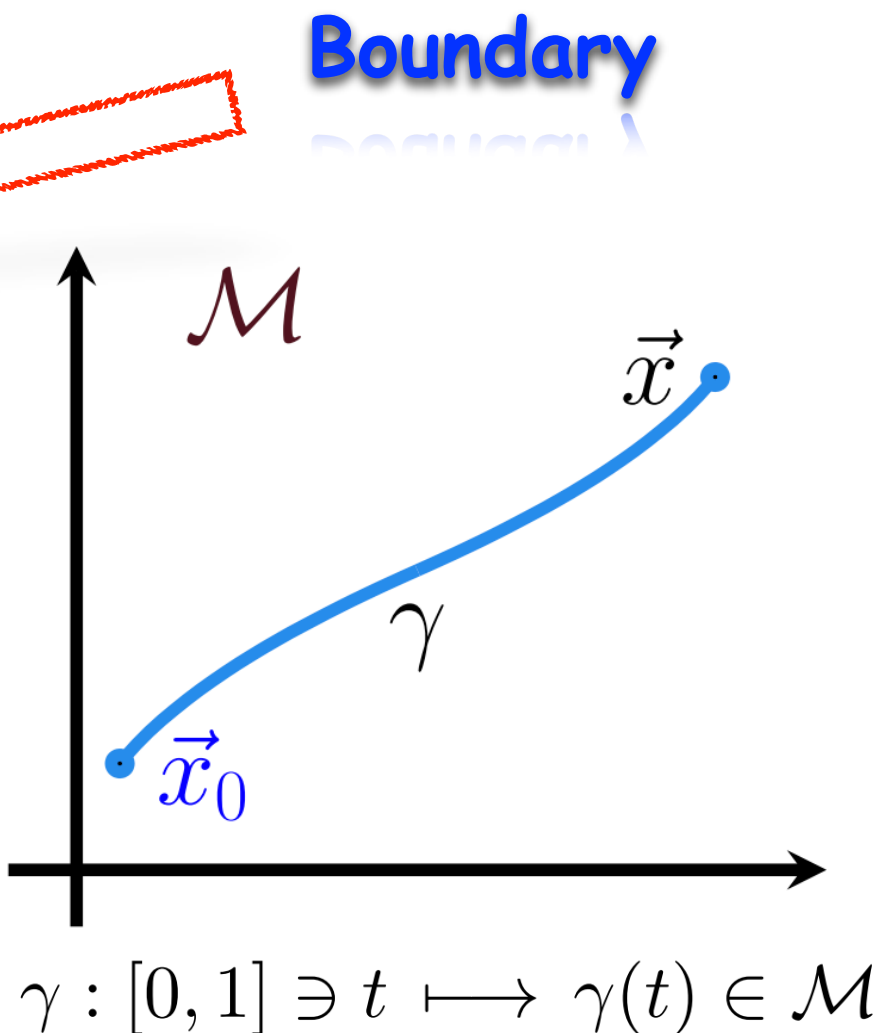
$$g^{(n)}(\vec{x}) = \begin{cases} g^{(0)}(\vec{x}_0) & n = 0 \\ g^{(n)}(\vec{x}_0) + \int_{\gamma} dA g^{(n-1)}(\vec{x}) & n > 0 \end{cases}$$

❖ Chen's iterated integrals

$$K(t) = dA/dt$$

$$\int_{\gamma} \underbrace{dA \cdots dA}_{n \text{ times}} = \int_0^1 dt_1 K(t_1) \int_0^{t_1} dt_2 K(t_2) \cdots \int_0^{t_{n-1}} dt_n K(t_n)$$

[K. Chen, '77]



Integral families of VWW

Rationalization

❖ Square roots

$$\lambda_1 = \sqrt{s(s - 4m_t^2)}, \quad \lambda_2 = \sqrt{s(s - 4m_W^2)},$$

$$\lambda_3 = \sqrt{(m_W^2 - m_t^2 - m_b^2)^2 - 4m_t^2 m_b^2}.$$

❖ Dimensionless kinematics

$$\tau_1 = -\frac{s}{m_t^2}, \quad \tau_2 = -\frac{m_W^2}{m_t^2}, \quad \tau_3 = \frac{m_b^2}{m_t^2}.$$

❖ Change of variables

$$(\tau_1, \tau_2, \tau_3) \longmapsto (x, y, z) : \begin{cases} \tau_1 = \frac{(1-x)^2}{x} \\ \tau_2 = \frac{(1-x)^2 z}{x(1+z)^2} \\ \tau_3 = \left(1 + \frac{y}{1+z}\right) \left[1 - \frac{(1-x)^2 z}{xy(1+z)}\right] \end{cases}$$

❖ Rationalization

$$\bar{\lambda}_1 = \frac{(1-x)(1+x)}{x}, \quad \bar{\lambda}_2 = \frac{(1-x)^2(1-z)}{x(1+z)}, \quad \bar{\lambda}_3 = \frac{(1-x)^2 z + xy^2}{xy(1+z)}$$

Integral families of VWW

Symbol letters

$$d\mathbf{I}(x, y, z, \epsilon) = \epsilon \left[\sum_{i=1}^{20} \mathbb{M}_i d \log \eta_i(x, y, z) \right] \mathbf{I}(x, y, z, \epsilon),$$

♣ 20 rational letters

$$\begin{aligned} \eta_1 &= x, & \eta_2 &= y \\ \eta_3 &= z, & \eta_4 &= 1 - x \\ \eta_5 &= 1 + x, & \eta_6 &= 1 + y \\ \eta_7 &= 1 - z, & \eta_8 &= 1 + z \\ \eta_9 &= x + z, & \eta_{10} &= y + z \\ \eta_{11} &= 1 + xz, & \eta_{12} &= 1 - x + y \\ \eta_{13} &= -1 + x + xy, & \eta_{14} &= 1 + y + z \\ \eta_{15} &= y + z - xz, & \eta_{16} &= xy + xz - z \\ \eta_{17} &= xy - (1 - x)^2, & \eta_{18} &= xy - (1 - x)^2 z \\ \eta_{19} &= xy^2 + (1 - x)^2 z, & \eta_{20} &= xy(1 + z) - (1 - x)^2 z \end{aligned}$$

Integral families of VWW

Multiple polylogarithms

Rational letter

♣ MPLs [Goncharov, '98, '01]

$$G(a_1, \dots, a_n; z) = \int_0^z \frac{1}{t - a_1} G(a_2, \dots, a_n; t) dt,$$

with

$$G(a_1; z) = \int_0^z \frac{1}{t - a_1} dt \quad a_1 \neq 0, \quad G(\underbrace{0, \dots, 0}_{n \text{ times}}; z) = \frac{\log^n(z)}{n!}$$

- Transcendental weight: $\mathcal{T}(G(a_1, \dots, a_n; z)) = n$
- Scaling relation: $G(a_1, \dots, a_n; z) = G(\lambda a_1, \dots, \lambda a_n; \lambda z)$
 $\forall \lambda \in \mathbb{C} \setminus \{0\}, a_n \neq 0$
- Shuffle algebra: $G(\mathbf{a}; z)G(\mathbf{b}; z) = \sum_{\mathbf{c} \in \mathbf{a} \sqcup \mathbf{b}} G(\mathbf{c}; z)$

- HPL, HyperInt
- PolyLogTools
- GiNac, HandG, NumPolyLog, FastGPL...

♣ eMPLs [J. Broedel, '17, '18] Unrationalizable square root

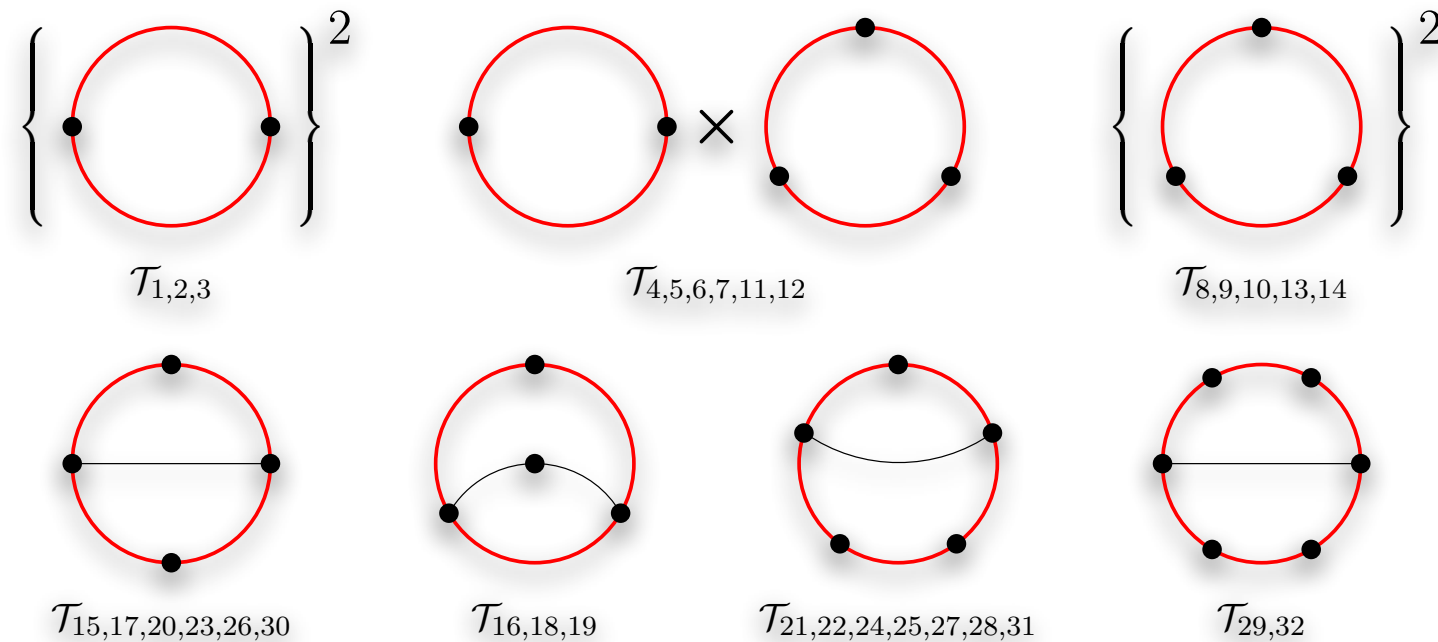
$$E_4 \left(\begin{matrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{matrix}; x, \vec{a} \right) = \int_0^x dt \psi_{n_1}(c_1, t, \vec{a}) E_4 \left(\begin{matrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{matrix}; t, \vec{a} \right)$$

Integral families of VWW

Boundary conditions

✿ Boundary ($s = m_W^2 = 0, m_b^2 = m_t^2$) $I_i(\vec{x}_0, \epsilon) = \begin{cases} 1, & i = 1, 2, 3, \\ 0, & i \neq 1, 2, 3. \end{cases}$

✿ Vacuums



✿ Weights

GPL	Weight
$G(a_1, \dots; x)$	$-1, \quad 0, \quad 1,$
$G(a_1, \dots; y)$	$-1, \quad 0, \quad -(1-x), \quad (1-x)/x, \quad (1-x)^2/x,$
$G(a_1, \dots; z)$	$-1, \quad 0, \quad 1, \quad -x, \quad -y, \quad -(1+y), \quad -1/x, \quad -y/(1-x),$
	$xy/(1-x), \quad xy/(1-x)^2, \quad -xy^2/(1-x)^2, \quad xy/[(1-x)^2 - xy]$

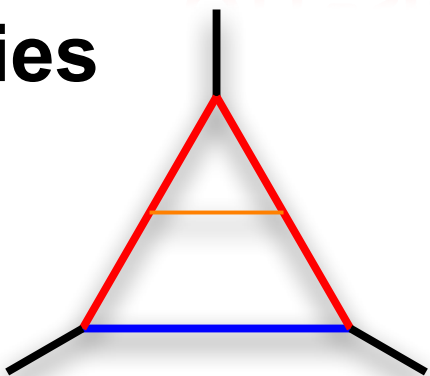
Integral families of VW^*W^*

[paper on preparing]

Integral family

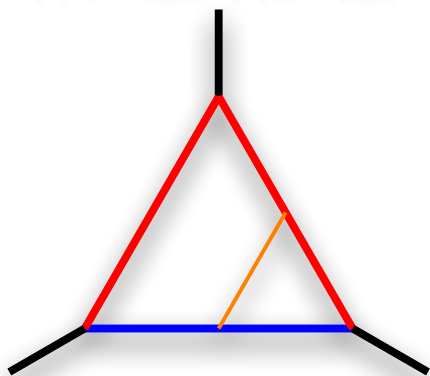
Off-shell W : $p_1^2 \neq p_2^2 \neq p_3^2 \neq m_W^2$

✿ Topologies



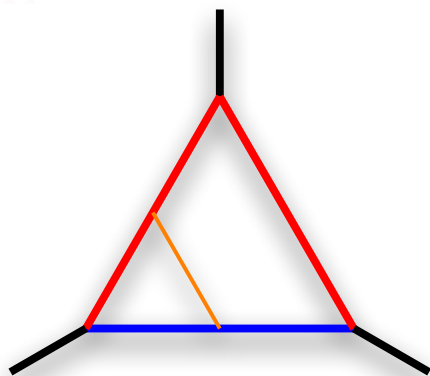
$\mathcal{T}_1$

$[1,1,0,1,1,1,1]$



$\mathcal{T}_2$

$[1,1,1,1,1,1,0]$



$\mathcal{T}_3$

$[1,1,1,1,1,0,1]$

✿ Propagators

Full mass dependence!

$$D_1 = (k_1 - k_2)^2,$$

$$D_2 = k_1^2 - m_b^2,$$

$$D_3 = k_2^2 - m_b^2.$$

$$D_4 = (k_1 + p_2)^2 - m_t^2,$$

$$D_5 = (k_1 - p_3)^2 - m_t^2,$$

$$D_6 = (k_2 + p_2)^2 - m_t^2,$$

$$D_7 = (k_2 - p_3)^2 - m_t^2,$$

✿ Dimensionless kinematic variables

5 scales!!!

$$w = -\frac{p_1^2}{m_t^2},$$

$$x = -\frac{p_2^2}{m_t^2},$$

$$y = -\frac{p_3^2}{m_t^2},$$

$$z = \frac{m_b^2}{m_t^2}.$$

Integral families of VW^*W^*

Canonical basis

❖ Four square roots

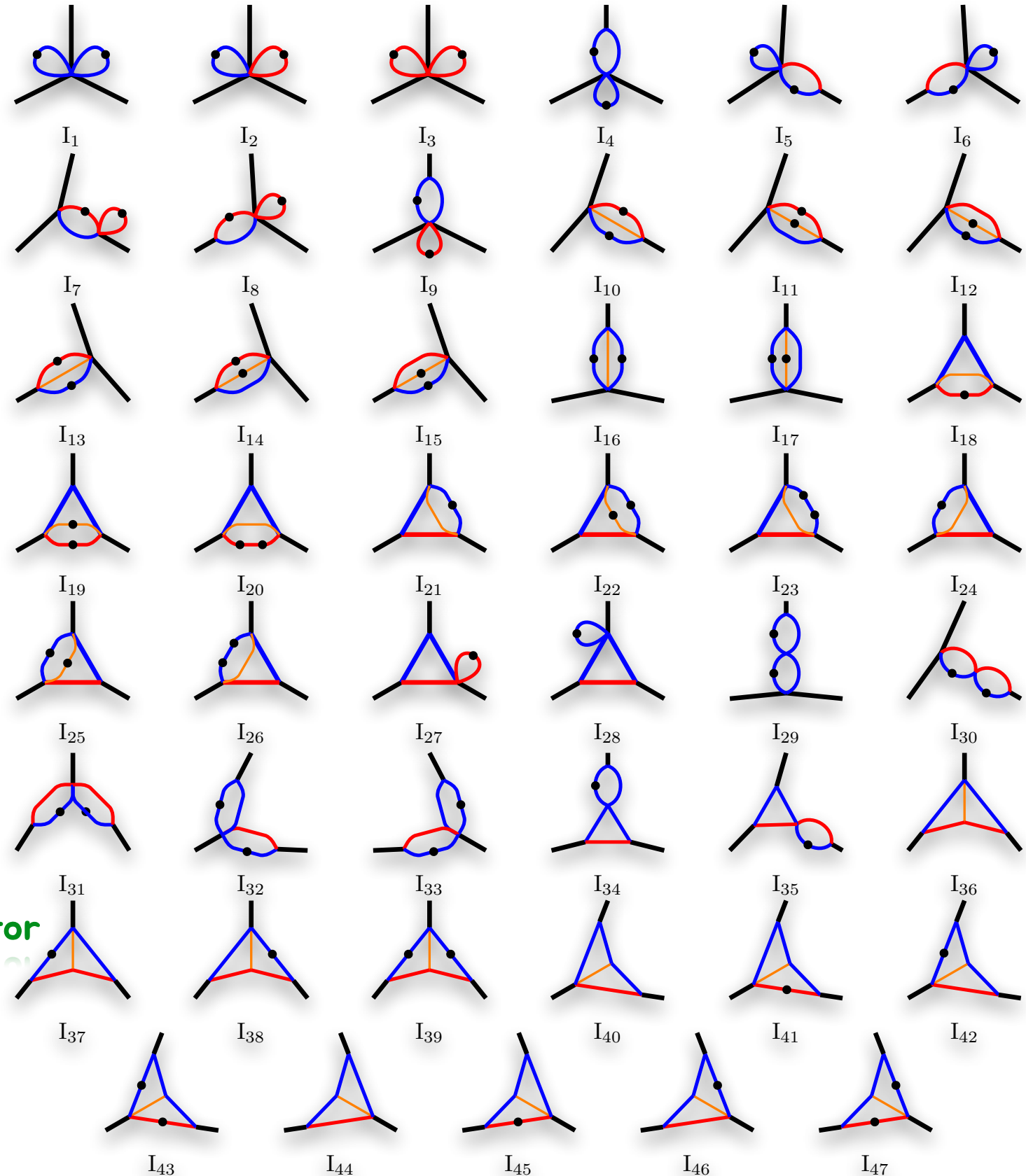
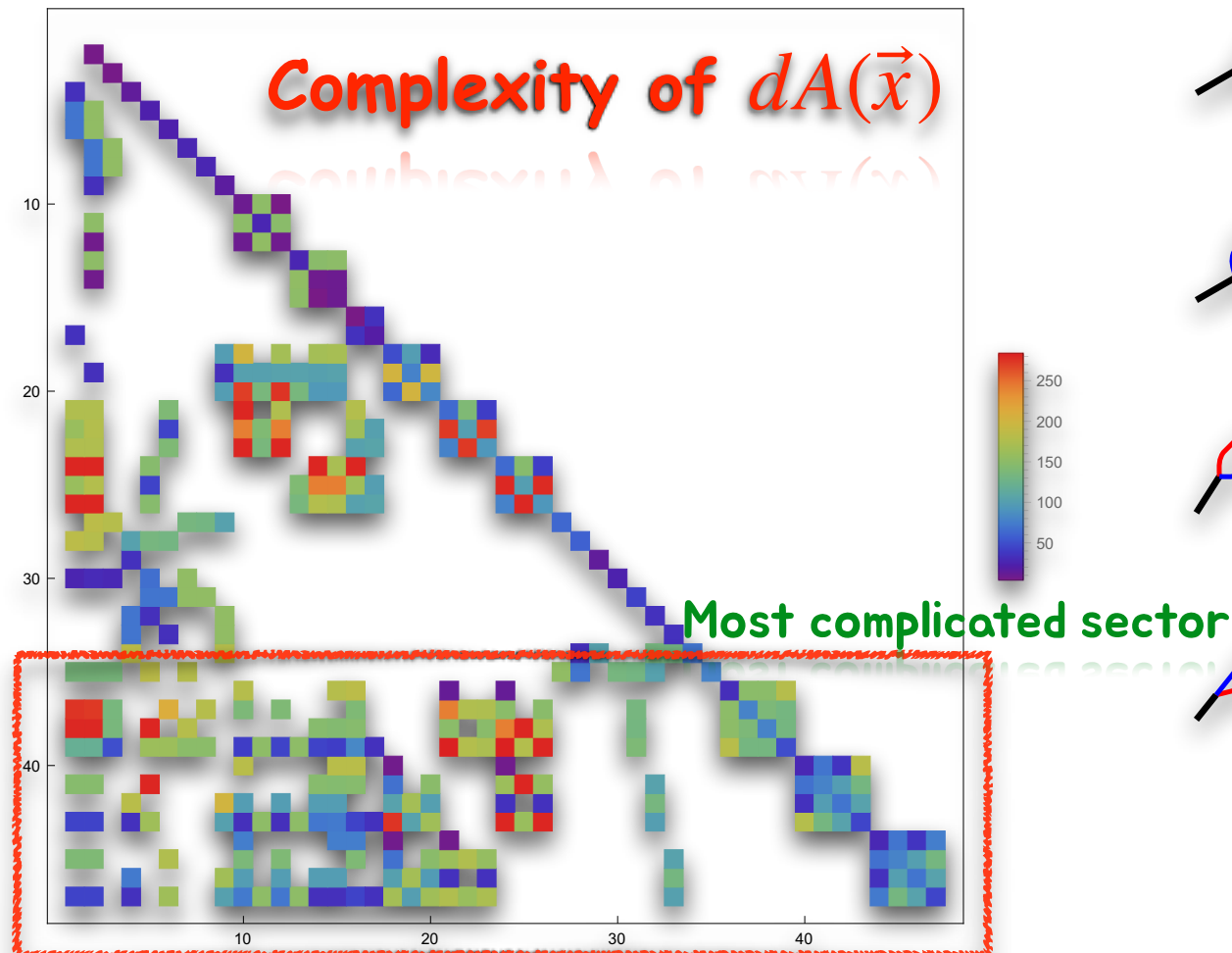
$$R_1 = \sqrt{w(w+4)},$$

$$R_2 = \sqrt{(x+z-1)^2 + 4x},$$

$$R_3 = \sqrt{(y+z-1)^2 + 4y},$$

$$R_4 = \sqrt{(w-x-y)^2 - 4xy}.$$

Complexity of $dA(\vec{x})$



Integral families of VW^*W^*

Symbol letters

$$\begin{aligned}
 \alpha_1 &= w, & \alpha_2 &= x, & \alpha_3 &= y, & \alpha_4 &= z \\
 \alpha_5 &= R_1^2, & \alpha_6 &= R_2^2, & \alpha_7 &= R_3^2, & \alpha_8 &= R_4^2 \\
 \alpha_9 &= w(x+1)(y+1) - (x-y)^2, & \alpha_{10} &= -A_{19,19}, & \alpha_{11} &= \alpha_{12}(x \leftrightarrow y), \\
 \alpha_{12} &= w[(x+z)(y+z) - (w+1)z + x] + (y-x)(x+z) + x.
 \end{aligned}$$

Even Letter

$$\begin{aligned}
 \alpha_{13} &= \frac{R_1 - w}{R_1 + w}, & \alpha_{14} &= \frac{R_2 - A_{7,2}(x)}{R_2 + A_{7,2}(x)}, & \alpha_{15} &= \alpha_{14}(x \leftrightarrow y), & \alpha_{16} &= \frac{R_1 R_4 + A_{19,18}}{R_1 R_4 - A_{19,18}}, \\
 \alpha_{17} &= \frac{R_4 - (w - x + y)}{R_4 + (w - x + y)}, & \alpha_{18} &= \alpha_{17}(x \leftrightarrow y), & \alpha_{19} &= \frac{R_1 R_2 - A_{19,10}(x, y)}{R_1 R_2 + A_{19,10}(x, y)}, & \alpha_{20} &= \alpha_{19}(x \leftrightarrow y), \\
 \alpha_{21} &= \frac{R_2 R_4 + A_{22,21}(x, y)}{R_2 R_4 - A_{22,21}(x, y)}, & \alpha_{22} &= \alpha_{21}(x \leftrightarrow y), & \alpha_{23} &= \frac{R_2 R_4 + [A_{22,21}(x, y) - 2x]}{R_2 R_4 - [A_{22,21}(x, y) - 2x]}, & \alpha_{24} &= \alpha_{23}(x \leftrightarrow y), \\
 \alpha_{25} &= \frac{R_2 R_3 - A_{22,10}}{R_2 R_3 + A_{22,10}}, & \alpha_{26} &= \frac{(x+1)R_2 + A_{30,2}(-z, -x)}{(x+1)R_2 - A_{30,2}(-z, -x)}, & \alpha_{27} &= \alpha_{26}(x \leftrightarrow y), \\
 \alpha_{28} &= \frac{R_1 R_4 + (A_{19,18} + 2wz)}{R_1 R_4 - (A_{19,18} + 2wz)}, & \alpha_{29} &= \frac{(w+1)R_4 + (A_{19,18} - w - x + y)}{(w+1)R_4 - (A_{19,18} - w - x + y)}, & \alpha_{30} &= \alpha_{28}(x \leftrightarrow y), \\
 \alpha_{31} &= \frac{(y+1)R_1 - [w(y+1) + 2(y-x)]}{(y+1)R_1 + [w(y+1) + 2(y-x)]}, & \alpha_{32} &= \frac{(w+1)R_2 - [A_{19,10}(x, y) - A_{7,2}(x)]}{(w+1)R_2 + [A_{19,10}(x, y) - A_{7,2}(x)]}, & \alpha_{33} &= \alpha_{31}(x \leftrightarrow y), \\
 \alpha_{34} &= \frac{(y+1)R_4 - [w(y+1) + (y-1)(x-y)]}{(y+1)R_4 + [w(y+1) + (y-1)(x-y)]}.
 \end{aligned}$$

Odd letter

Integral families of VW^*W^*

Rationalization

❖ Parameter transformations

$$\gamma_t : [0, 1] \rightarrow \mathbb{C}^4 \quad \rho_i(t) = \rho_i t, \quad z(t) = (z - 1)t + 1, \quad \rho = (w, x, y, z),$$

❖ Square roots

$$R_1(t) = \sqrt{w^2 t^2 + 4 w t},$$

$$R_2(t) = \sqrt{(x + z - 1)^2 t^2 + 4 x t},$$

$$R_3(t) = \sqrt{(y + z - 1)^2 t^2 + 4 y t},$$

$$R_4(t) = R_4 t. \quad \text{Rational}$$

❖ Rationalizable square roots

• One root: $t = \frac{\beta_1^2}{(\beta_1 + 1)w}, \quad t = \frac{\beta_2^2}{\beta_2(x + z - 1) + x}, \quad t = \frac{\beta_3^2}{\beta_3(y + z - 1) + y}.$

• Two roots: $t = \frac{(\sigma_1^2 - wx)^2}{w\sigma_1[\sigma_1 + c_{7,2}(x)][\sigma_1 c_{7,2}(x) + wx]}, \quad t = \frac{(\sigma_3^2 - xy)^2}{\sigma_3[c_{7,2}(x)\sigma_3 + xc_{7,2}(y)][c_{7,2}(y)\sigma_3 + yc_{7,2}(x)]}.$

$t = \frac{(\sigma_2^2 - wy)^2}{w\sigma_2[\sigma_2 + c_{7,2}(y)][\sigma_2 c_{7,2}(y) + wy]},$

❖ Unrationalizable square roots

$$\gamma_{\sigma_j}^* \gamma_t^* \alpha_i(\rho) = \frac{b_1(\sigma_j) - b_2(\sigma_j)}{b_1(\sigma_j) + b_2(\sigma_j)} \frac{\sqrt{P_j(\sigma_j)}}{\sqrt{P_j(\sigma_j)}},$$

$$\begin{aligned} H_e &= \int_{\gamma_{\sigma_j} \gamma_t} \gamma_{\sigma_j}^* \gamma_t^* [\mathrm{d} \log \alpha_i(\rho) G(q'_n, \dots, q'_1; \sigma'_j(\rho; 1))] \\ &= \int_{\sigma_j(\rho; 0)}^{\sigma_j(\rho; 1)} \frac{F(\sigma_j)}{\sqrt{P_j(\sigma_j)}} G(q'_n, \dots, q'_1; \sigma'_j) \mathrm{d} \sigma_j. \end{aligned}$$

3rd residual root

Degree 4

Relate to elliptic curves

Elliptic multiple polylogarithms

[J. Broedel et al., '17, '18]

❖ Elliptic curve

$$\zeta^2(\xi) = P(\xi) = (\xi - a_1)(\xi - a_2)(\xi - a_3)(\xi - a_4)$$

❖ Branches

$$\sqrt{P(\xi)} \equiv \sqrt{|P(\xi)|} \times \begin{cases} -1, & \xi \leq a_1 \text{ or } \xi > a_4 \\ -i, & a_1 < \xi \leq a_2 \\ 1, & a_2 < \xi \leq a_3 \\ i, & a_3 < \xi \leq a_4 \end{cases}$$

branch points are real, distinct
ordered as $a_1 < a_2 < a_3 < a_4$

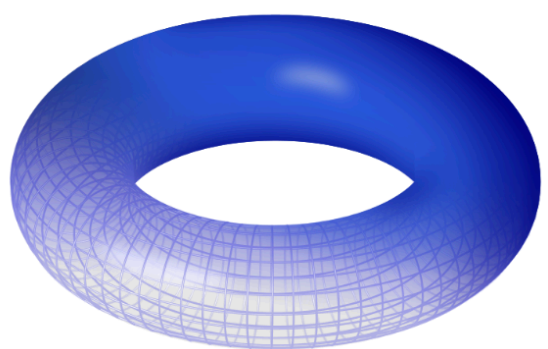
❖ Periods

$$\Psi_1 = 2\kappa \int_{a_2}^{a_3} \frac{d\xi}{\zeta} = 2K(\lambda), \quad \Psi_2 = 2\kappa \int_{a_1}^{a_2} \frac{d\xi}{\zeta} = 2iK(1-\lambda),$$

with $\lambda = \frac{a_{14} a_{23}}{a_{13} a_{24}}, \quad \kappa = \frac{1}{2} \sqrt{a_{13} a_{24}} \quad \text{and} \quad a_{ij} = a_i - a_j.$

❖ Geometry

genus 1



Riemann torus

$$\tau = \frac{\Psi_2}{\Psi_1}$$

Elliptic multiple polylogarithms

❖ Rational functions on the elliptic curve [J. Broedel et al., '17, '18]

A rational function on the elliptic curve is a function $R(x, y)$,
subject to the constraint $y = \sqrt{P(x)}$

$$R(x, y) = \frac{p_1(x) + p_2(x)y}{q_1(x) + q_2(x)y} = \frac{p_1(x) + p_2(x)\sqrt{P(x)}}{q_1(x) + q_2(x)\sqrt{P(x)}} = R_1(x) + \frac{1}{\sqrt{P(x)}} R_2(x)$$

❖ Iterated integrals of rational functions on the curve

$$\int dx \left(R_1(x) + \frac{1}{\sqrt{P(x)}} R_2(x) \right) = ?$$

❖ Partial fractioning

$$\begin{array}{ll} \int \frac{dx}{(x - c_i)^k} & \text{from } R_1(x) \\ \int \frac{dx}{y} x^k, \quad \int \frac{dx}{y(x - c_i)^k}, & \text{from } \frac{1}{y} R_2(x) \end{array}$$

❖ 4 integration kernels

Double pole @ ∞

$$\int \frac{dx}{(x - c_i)}, \quad \int \frac{dx}{y}, \quad \boxed{\int \frac{x dx}{y}}, \quad \int \frac{dx}{y(x - c_i)}$$

Elliptic multiple polylogarithms

❖ eMPLs

[J. Broedel et al., '17, '18]

$\vec{a} = (a_1, a_2, a_3, a_4)$: Zeros of $P(\xi)$

$$E_4\left(\begin{smallmatrix} n_k & \cdots & n_1 \\ \omega_k & \cdots & \omega_1 \end{smallmatrix}; \xi, \vec{a}\right) = \int_0^\xi dt \psi_{n_k}(\omega_k, t) E_4\left(\begin{smallmatrix} n_{k-1} & \cdots & n_1 \\ \omega_{k-1} & \cdots & \omega_1 \end{smallmatrix}; t, \vec{a}\right), \quad E_4(; x, \xi) = 1$$

$n_i \in \mathbb{Z}, \omega_i \in \mathbb{C} \cup \{\infty\}$

Length (number of integrations): k
Weight: $\sum_i |n_i|$

❖ Integration kernels

$$\psi_0(0, x) = \frac{c_4}{y},$$

$$\psi_1(c, x) = \frac{1}{x - c},$$

$$\psi_{-1}(c, x) = \frac{y_c}{y(x - c)},$$

$$\psi_1(\infty, x) = \frac{c_4}{y} Z_4(x), \quad \psi_{-1}(\infty, x) = \frac{x}{y},$$

$$\psi_{-n}(\infty, x) = \frac{x}{y} Z_4^{(n-1)}(x) - \frac{\delta_{n2}}{c_4},$$

$$\psi_n(c, x) = \frac{1}{x - c} Z_4^{(n-1)}(x) - \delta_{n2} \Phi_4(x),$$

$$\psi_n(\infty, x) = \frac{c_4}{y} Z_4^{(n)}(x), \quad \psi_{-n}(c, x) = \frac{y_c}{y(x - c)} Z_4^{(n-1)}(x),$$

Sufficient
for our work
LOL ONL MOLK

❖ MPLs

$$E_4\left(\begin{smallmatrix} 1 & \cdots & 1 \\ \omega_k & \cdots & \omega_1 \end{smallmatrix}; \xi, \vec{a}\right) = G(\omega_k, \dots, \omega_1; \xi).$$

rational simple pole

Elliptic multiple polylogarithms

❖ eMPLs

[J. Broedel et al., '17, '18]

$\vec{a} = (a_1, a_2, a_3, a_4)$: Zeros of $P(\xi)$

$$E_4\left(\begin{smallmatrix} n_k & \dots & n_1 \\ \omega_k & \dots & \omega_1 \end{smallmatrix}; \xi, \vec{a}\right) = \int_0^\xi dt \psi_{n_k}(\omega_k, t) E_4\left(\begin{smallmatrix} n_{k-1} & \dots & n_1 \\ \omega_{k-1} & \dots & \omega_1 \end{smallmatrix}; t, \vec{a}\right), \quad E_4(; x, \xi) = 1$$

$n_i \in \mathbb{Z}, \quad \omega_i \in \mathbb{C} \cup \{\infty\}$

Length (number of integrations): k
Weight: $\sum_i |n_i|$

❖ Integration kernel

$$\begin{aligned} \psi_0(0, x) &= \frac{c_4}{y}, \\ \psi_1(c, x) &= \frac{1}{x - c}, & \psi_{-1}(c, x) &= \frac{y_c}{y(x - c)}, \\ \psi_1(\infty, x) &= \frac{c_4}{y} Z_4(x), & \psi_{-1}(\infty, x) &= \frac{x}{y}, \end{aligned}$$

$$\psi_{-n}(\infty, x) = \frac{x}{y} Z_4^{(n-1)}(x) - \frac{\delta_{n2}}{c_4},$$

Sufficient
for our work

• Example:

$$\begin{aligned} H_e = & C_1 E_4\left(\begin{smallmatrix} -1 & 1 & \dots & 1 \\ \infty & q'_n & \dots & q'_1 \end{smallmatrix}; \sigma'_j(\boldsymbol{\rho}, 1), \vec{a}'_j\right) + \sum_k \frac{C_{-1,k}}{\zeta'_{j,q_k}} E_4\left(\begin{smallmatrix} -1 & 1 & \dots & 1 \\ q'_k & q'_n & \dots & q'_1 \end{smallmatrix}; \sigma'_j(\boldsymbol{\rho}, 1), \vec{a}'_j\right) \\ & + \frac{C_0 - C_1 \sigma_j(\boldsymbol{\rho}, 0)}{\kappa'_j} E_4\left(\begin{smallmatrix} 0 & 1 & \dots & 1 \\ 0 & q'_n & \dots & q'_1 \end{smallmatrix}; \sigma'_j(\boldsymbol{\rho}, 1), \vec{a}'_j\right), \end{aligned}$$

Integral families of VW^*W^*

❖ Simplification with shuffle algebra:

$$G(a; z)G(b; z) = \sum_{c \in a \sqcup b} G(c; z)$$

$$\begin{aligned} H_{31}^{(2)} = & [2G(\omega_{16}, \omega_{37}, \beta_3) - G(\omega_{16}, \omega_{38}, \beta_3) + G(\omega_{16}, \omega_{39}, \beta_3) + G(\omega_{16}, \omega_{40}, \beta_3) - 2G(\omega_{16}, \omega_{84}, \beta_3) \\ & - 2G(\omega_{16}, \omega_{86}, \beta_3) + 2G(\omega_{16}, \omega_{89}, \beta_3) + 2G(\omega_{16}, \omega_{90}, \beta_3) - G(\omega_{16}, \omega_{92}, \beta_3) - G(\omega_{16}, \omega_{94}, \beta_3) \\ & \dots \\ & + G(\omega_{98}, \omega_{31}, \beta_3) - G(\omega_{98}, \omega_{36}, \beta_3) + G(\omega_{98}, \omega_{41}, \beta_3) - G(\omega_{98}, \omega_{42}, \beta_3) + G(\omega_{98}, \omega_{43}, \beta_3) \\ & + G(\omega_{98}, \omega_{51}, \beta_3) - G(\omega_{98}, \omega_{52}, \beta_3)] / 8 \end{aligned}$$

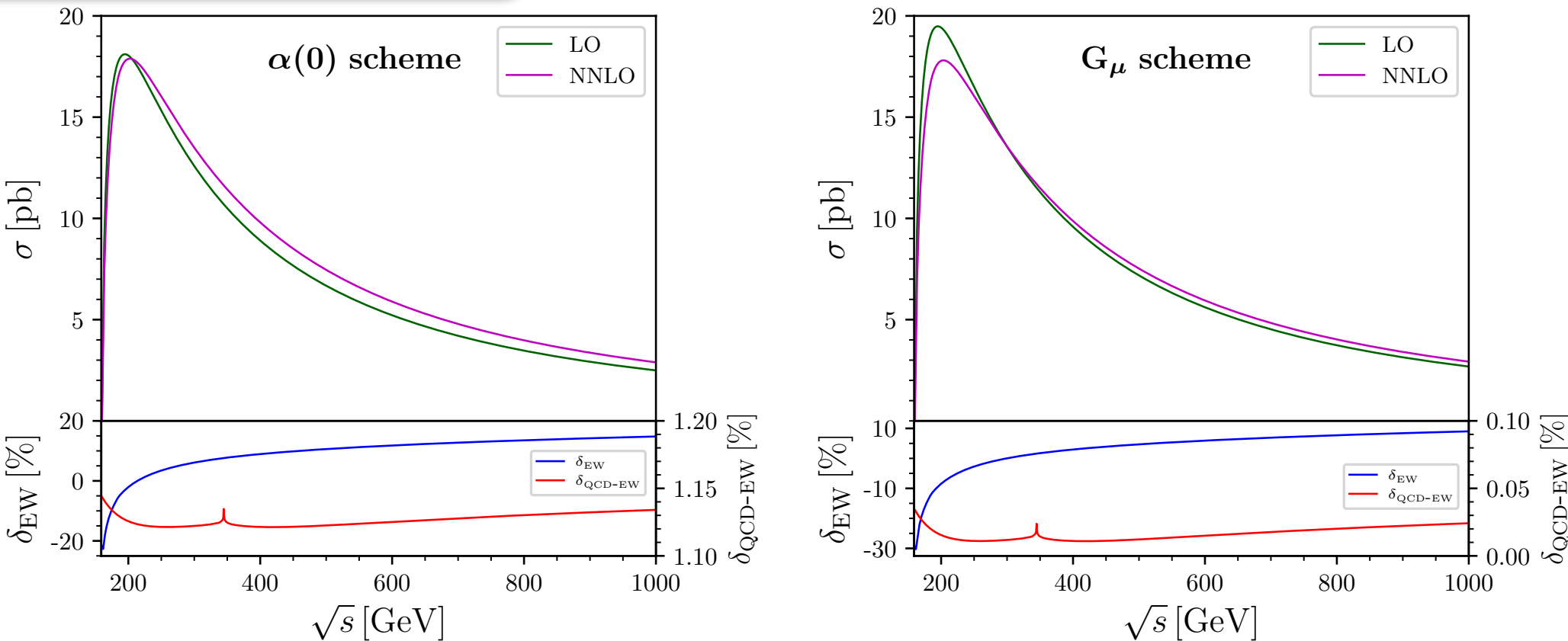
192 terms

shuffle

algebra

$$\begin{aligned} H_{31}^{(2)} = & [G(\omega_{16}, \beta_3) - G(\omega_{31}, \beta_3) + G(\omega_{36}, \beta_3) - G(\omega_{41}, \beta_3) + G(\omega_{42}, \beta_3) - G(\omega_{43}, \beta_3) - G(\omega_{51}, \beta_3) \\ & + G(\omega_{52}, \beta_3)] \cdot [G(\omega_{37}, \beta_3) + G(\omega_{38}, \beta_3) + G(\omega_{39}, \beta_3) + G(\omega_{40}, \beta_3) - 2G(\omega_{84}, \beta_3) \\ & - 2G(\omega_{86}, \beta_3) + 2G(\omega_{89}, \beta_3) + 2G(\omega_{90}, \beta_3) - G(\omega_{92}, \beta_3) - G(\omega_{94}, \beta_3) - G(\omega_{95}, \beta_3) \\ & - G(\omega_{98}, \beta_3)] / 8 \end{aligned}$$

Total cross section



$\sqrt{s}$ [GeV]	Scheme	σ_{LO} [pb]	σ_{NLO} [pb]	δ_{EW} [%]	σ_{NNLO} [pb]	$\delta_{\text{QCD-EW}}$ [%]
240	$\alpha(0)$	15.93577	16.3653	2.6952	16.5440	1.1216
	G_μ	17.14888	16.5484	-3.5015	16.5503	0.0113
500	$\alpha(0)$	6.673608	7.38135	10.6051	7.45626	1.1225
	G_μ	7.181637	7.51822	4.6867	7.51909	0.0122

QCD-EW
corrections

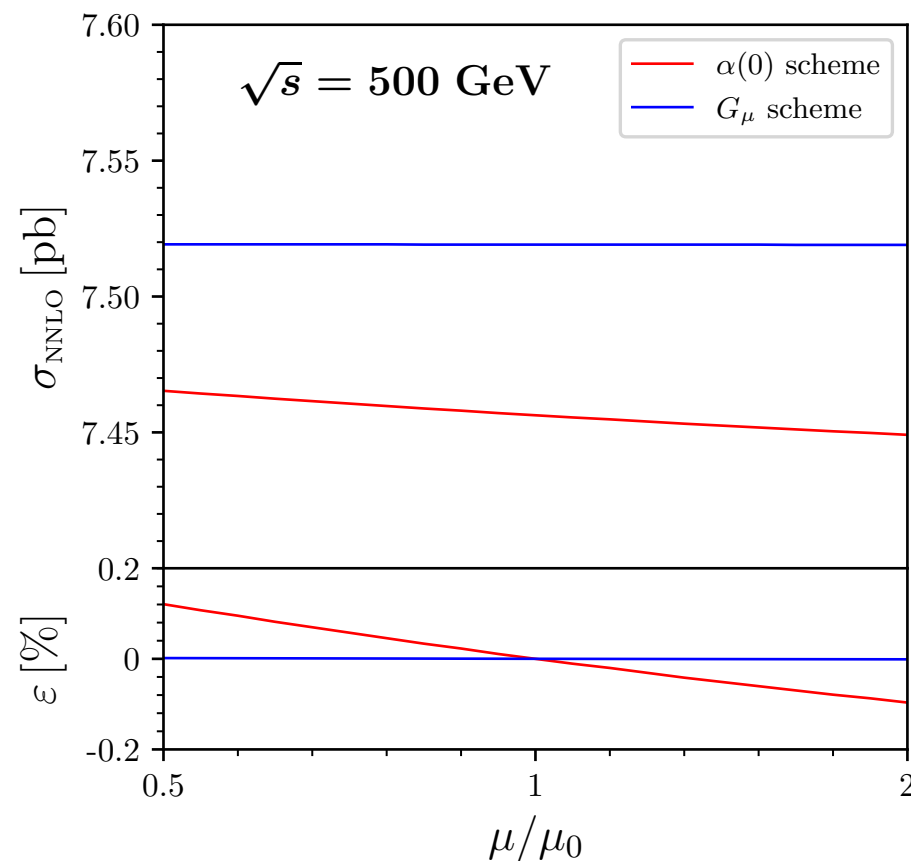
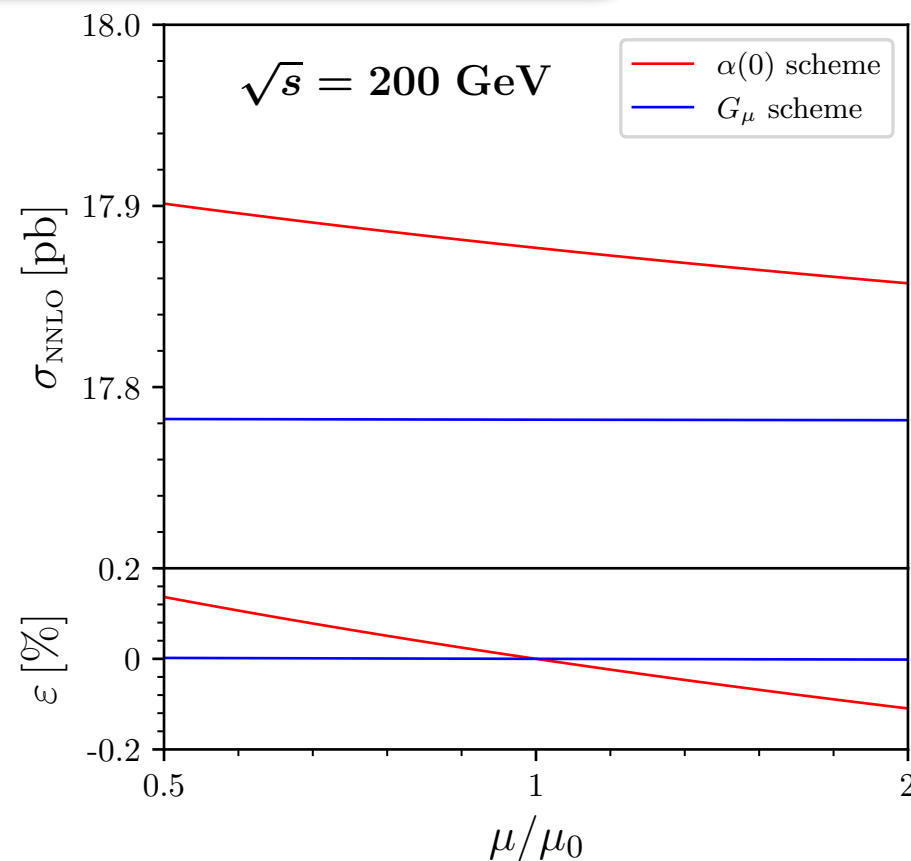
COLLECTION2

~1%

Suppressed by
 G_μ scheme

Phenomenological results

Scale dependence



$\sqrt{s}$ [GeV]	Scheme	$\sigma(\mu_0/2)$ [pb]	$\sigma(\mu_0)$ [pb]	$\sigma(2\mu_0)$ [pb]	$\varepsilon_{\text{scale}}$ [%]
200	$\alpha(0)$	17.9012	17.8768	17.8572	0.25
	G_μ	17.7823	17.7820	17.7817	0.003
500	$\alpha(0)$	7.46526	7.45626	7.44904	0.22
	G_μ	7.51920	7.51909	7.51901	0.003

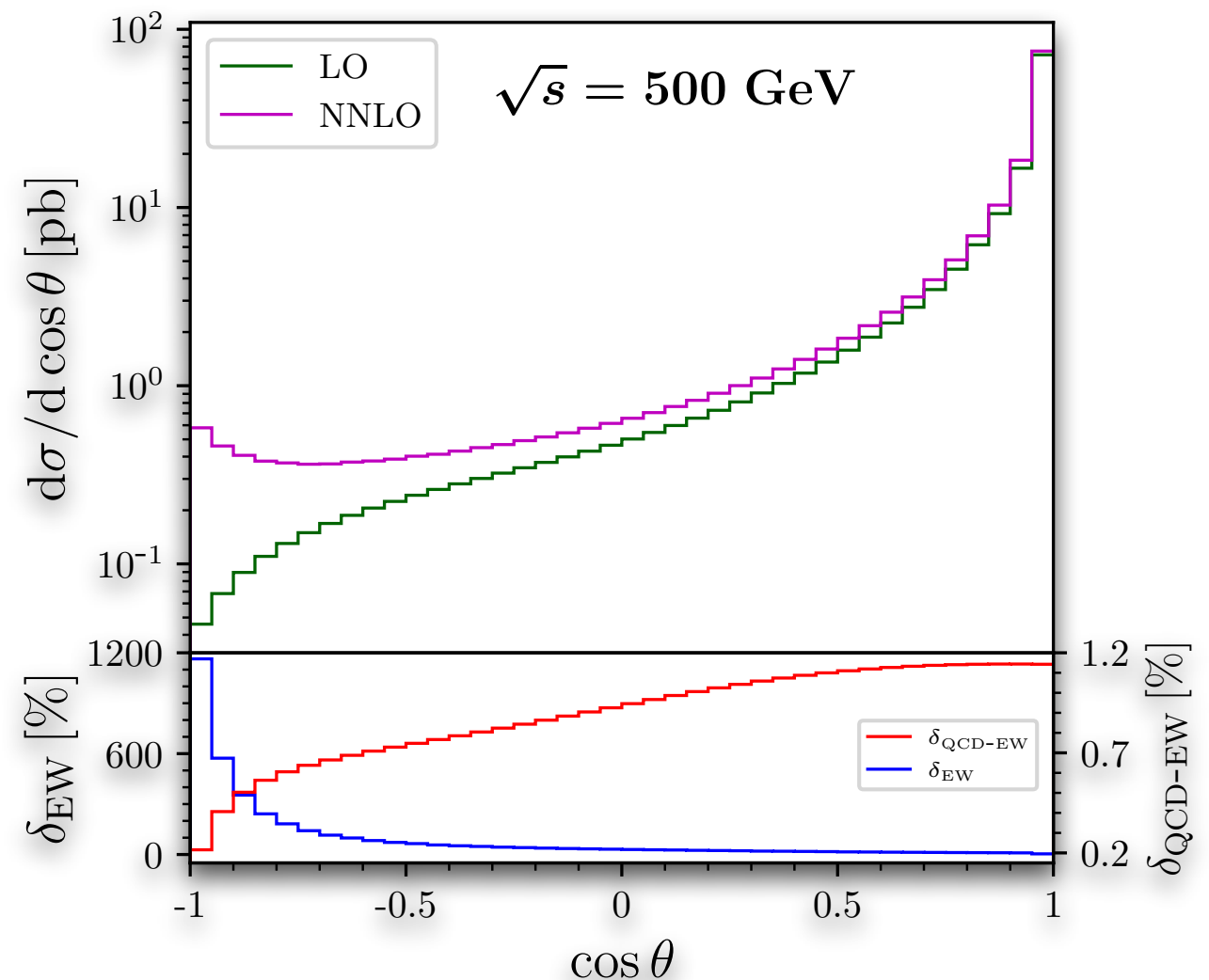
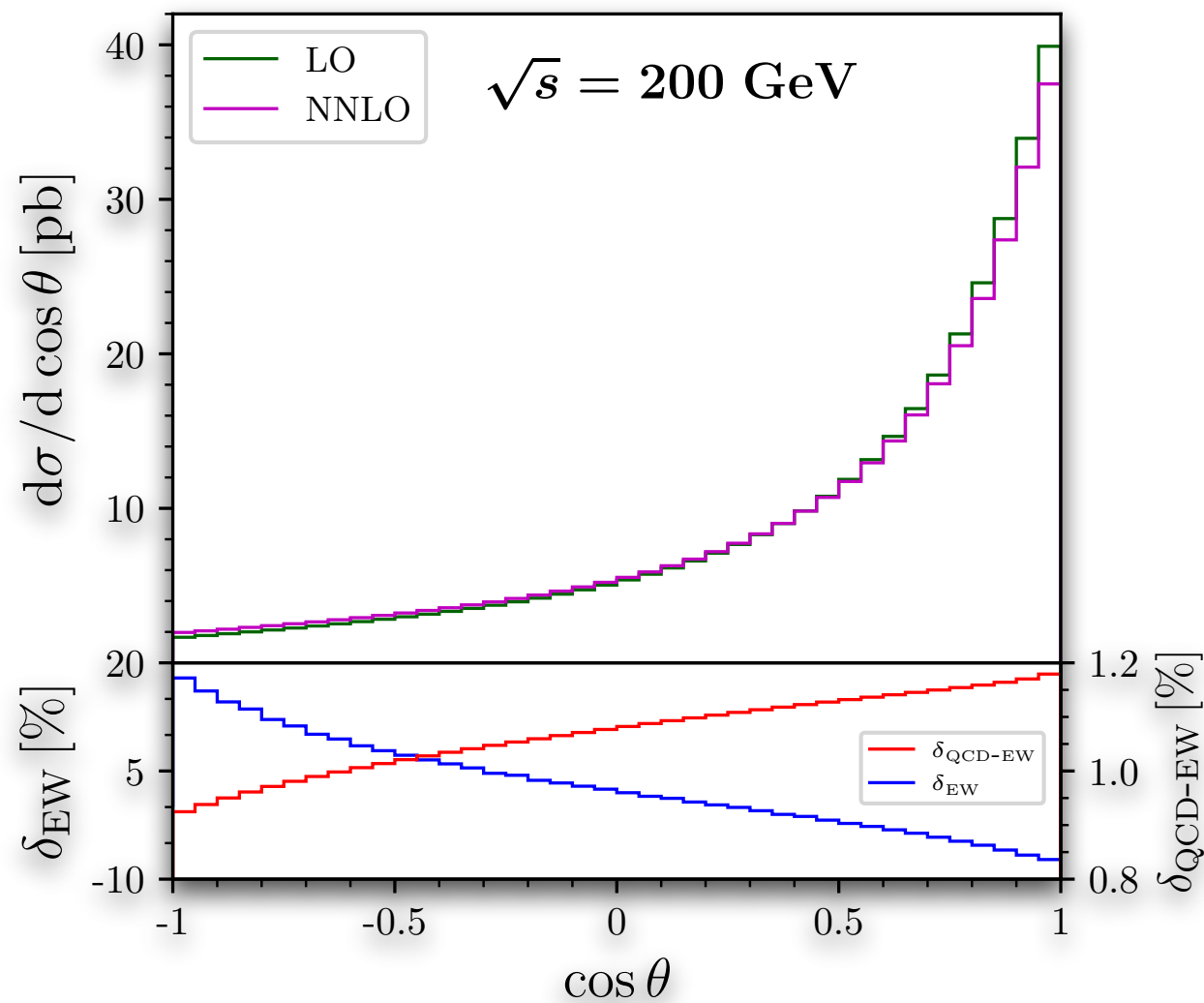
Scale uncertainties

0.2-0.3%

$\sim \mathcal{O}(\alpha^2)$

Phenomenological results

Scattering angle distributions



❖ Symmetry relation

$$\frac{d\sigma}{d\cos\theta_{W^-}} = \frac{d\sigma}{d\cos\theta_{W^+}} \Big|_{\theta \rightarrow \pi - \theta},$$

Strongly peak in the forward direction!!!

- ❖ Two-loop MIs for NNLO mixed QCD-EW corrections to VWW vertex (**MPLs**)
- ❖ Two-loop MIs for NNLO mixed QCD-EW corrections to VW^*W^* vertex (**elliptic MPLs**)
- ❖ NNLO mixed QCD-EW corrections to W-pair production at lepton colliders (**$\sim 1\%$**)

Thanks for your attention

Backup

Analytic continuation

Euclidean region
 $s < 0 \wedge m_W^2 < 0$

$$\begin{array}{c} s + i0^+, m_W^2 + i0^+ \\ \xrightarrow{\hspace{1cm}} \\ m_t^2 - i0^+, m_b^2 - i0^+ \end{array}$$

Physical region
 $s > 4 m_W^2 > 0$

❖ Inverse transform

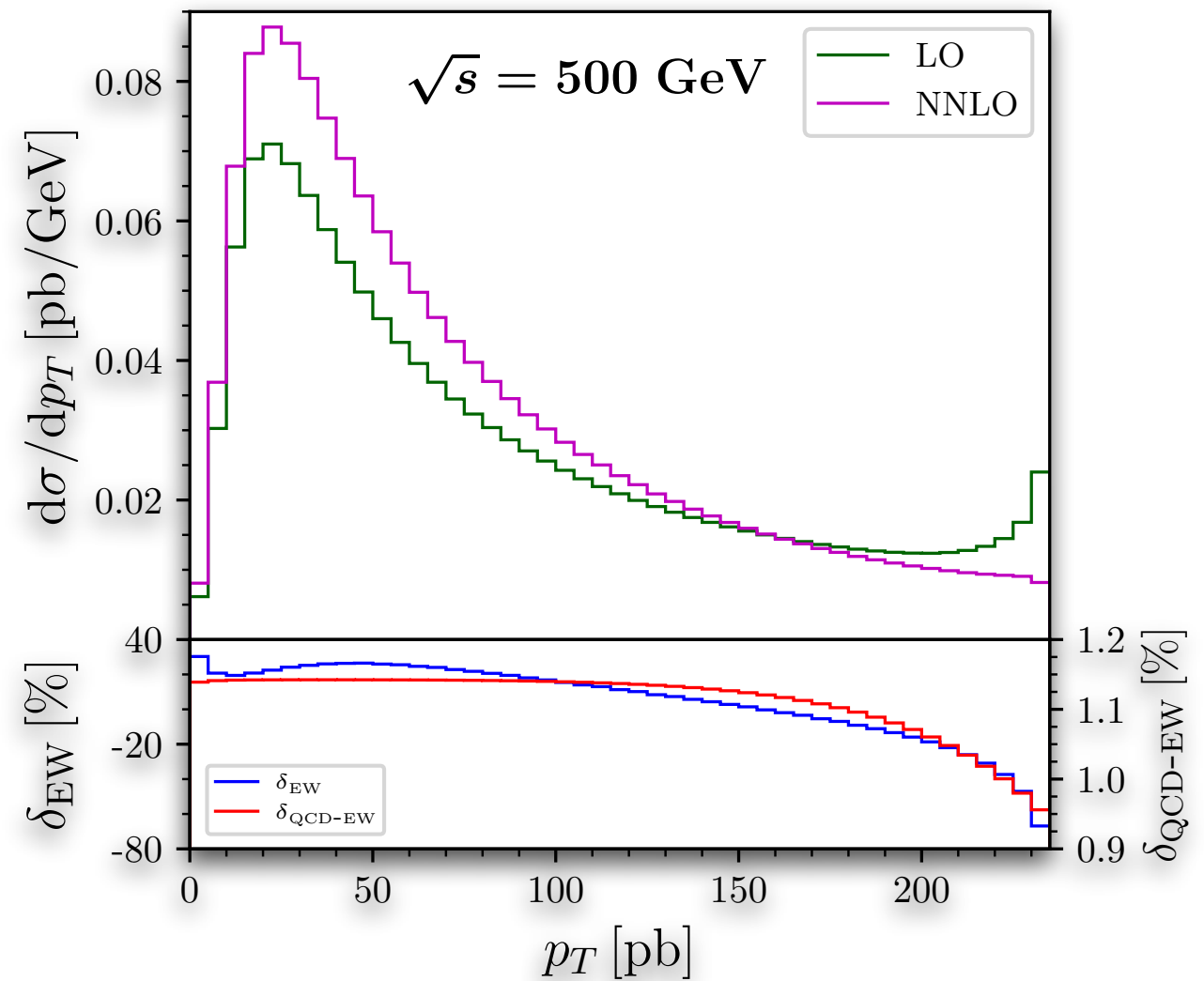
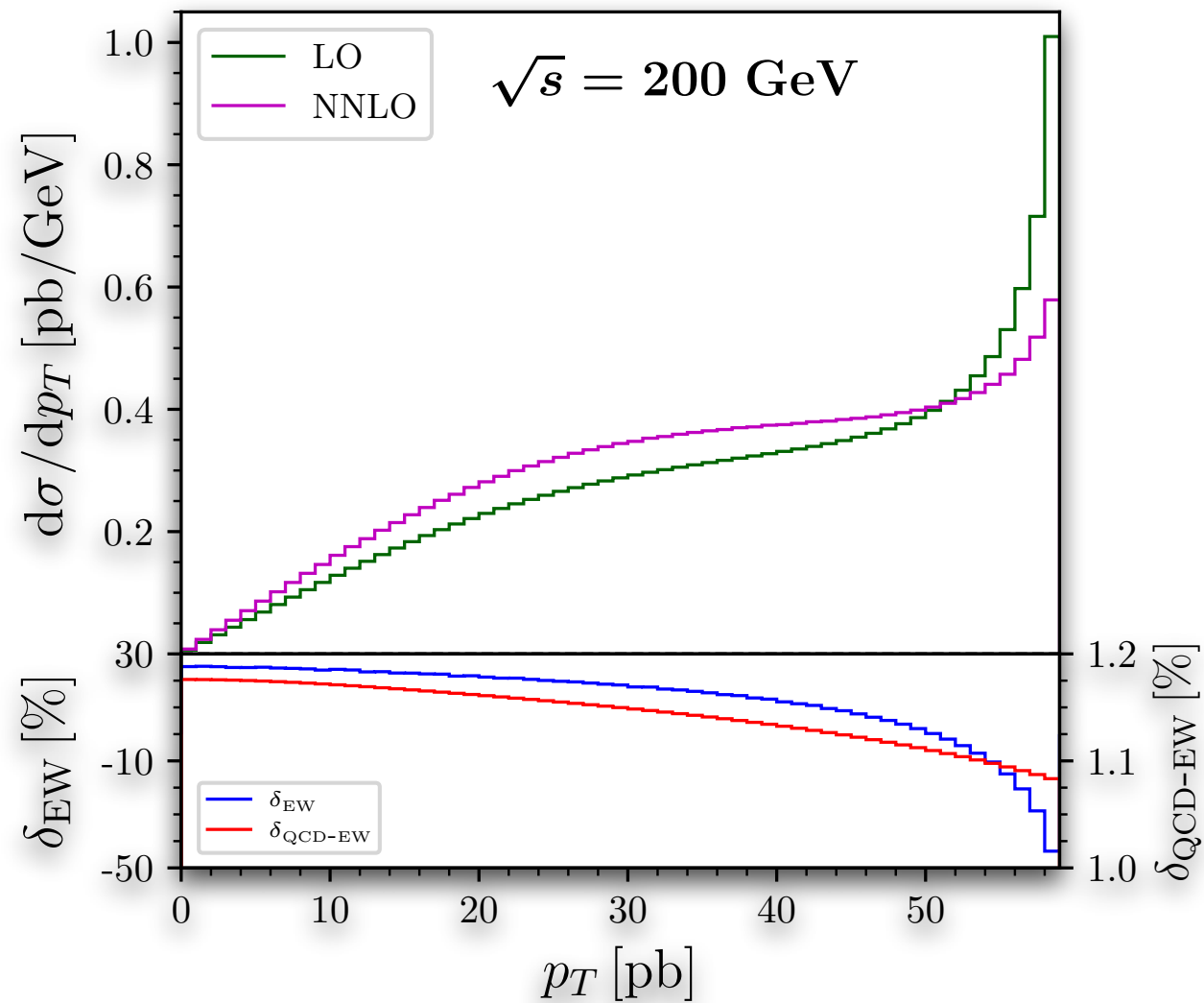
$$x = \frac{\sqrt{4 m_t^2 - s} - \sqrt{-s}}{\sqrt{4 m_t^2 - s} + \sqrt{-s}}, \quad z = \frac{\sqrt{-s} - \sqrt{4 m_W^2 - s}}{\sqrt{-s} + \sqrt{4 m_W^2 - s}},$$

$$y = (1 + z) \frac{1}{2 m_t^2} \left[m_b^2 - m_t^2 - m_W^2 + \sqrt{\lambda(m_W^2, m_t^2, m_b^2)} \right].$$

Family	Kinematic configuration	x	z	y
$\mathcal{F}$	$2 m_W < \sqrt{s} < 2 m_t$	$e^{i\vartheta}$	z	$y - i 0^+$
	$2 m_t < \sqrt{s}$	$x + i 0^+$	z	$y - i 0^+$
$\mathcal{F}^*$	$2 m_W < \sqrt{s}$	$x + i 0^+$	z	$y - i 0^+$

Backup

Kinematic distributions



❖ Symmetry relation

$$\frac{d\sigma}{dp_{T,W^-}} = \frac{d\sigma}{dp_{T,W^+}}$$