Reconstructing Laurent expansion of rational functions using p-adic numbers

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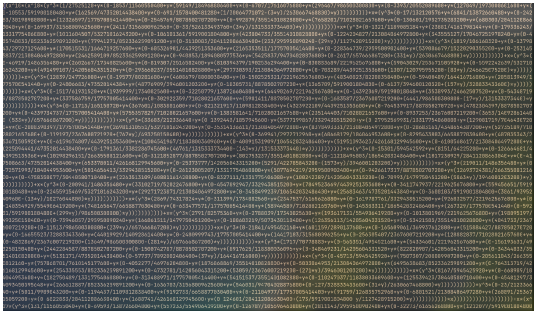
Background



Algebraic manipulation and simplification of polynomial and rational expressions are basic operations encountered in almost all branches of mathematics and theoretical physics. Manipulation of very large expressions can be extremely slow.
A typical situation where large expressions appear is in the perturbative calculation of scattering amplitudes and Feynman integrals.

Background

This leads to the proposal that one may perform the reduction procedure numerically in a finite field $\mathbb{F}_p$, and only reconstruct the final expressions in the end. This approach has been adopted in modern versions of reduction programs such as `Kira`, `FIRE`, `LiteRed` in combination with `FiniteFlow`.
If the final reconstructed expressions remian complicated, the finite-field reconstruction process can still consume substantial computational time and resources.

In practical computations, since we typicallt work within dimensional regularization, where the spacetime dimension is given by $d = 4 - 2\epsilon$, both the master integrals and the amplitudes must be expanded in ϵ up to a certain order. This observation suggests that a full analytic expression is often unnecessary–instead, only the truncated ϵ-expansion is required.

Basics of p-adic numbers

Given a prime number p, a p-adic number x can be defined as a series of the form

$$x = \sum_{i=v}^{\infty} a_i \, p^i \,, \tag{1}$$

where each a_i is an integer in the range $0 \leq a_i < p$, and v is an (possibly negative) integer.

v is called the valuation of x, denoted as $v_p(x)$. In other words, $v_p(x)$ is the exponent of the leading term in the series definition of x. And we assign the valuation of 0 to be $v_p(0) = +\infty$.

Examples

p-adic integers

In base-7, the decimal integer 123 is represented according to

$$123 = (234)_7 = 4 \times 7^0 + 3 \times 7^1 + 2 \times 7^2 \,, \tag{2}$$

As for -123

$$-123 = 3 \times 7^0 + 3 \times 7^1 + 4 \times 7^2 + \sum_{i=3}^{\infty} 6 \times 7^i \,. \tag{3}$$

p-adic rational numbers

$$\begin{aligned} \frac{123}{7} &= 4 \times 7^{-1} + 3 \times 7^0 + 2 \times 7^1 \,, \\ \frac{123}{9} &= 2 \times 7^0 + 4 \times 7^1 + \sum_{i=2}^{\infty} 2 \times 7^i \,. \end{aligned} \tag{4}$$

In general, if $r = n/m$, one has $v_p(r) = v_p(n) - v_p(m)$.

p-adic absolute value

For a general field, an absolute value (Archimedean absolute value) is a map from the field to real numbers satisfying the following properties:
(1) $|x| \geq 0$ **where the equality is achieved if and only if** $x = 0$**; (2)** $|xy| = |x||y|$**; (3)** $|x+y| \leq |x| + |y|$**.**
If an absolute value satisfies the stronger condition (4)$|x+y| \leq \max(|x|, |y|)$ **for all** x **and** y **, it is called a non-Archimedean absolute value.**
There is a trivial possibility that satisfies all three conditions: $|x| = 0$ **when** $x = 0$ **and** $|x| = 1$ **otherwise. This is the only possible absolute value for a finite field.**
On the other hand, the p**-adic absolute value is non-Archimedean, and is defined by**

$$|x|_p = p^{-v_p(x)} . \tag{5}$$

For example, we have $|98|_7 = 1/49$ **and** $|2/7|_7 = 7$**.**

An analogy between a p-adic number and rational function

Here $\boldsymbol{x}$ denotes a collection of variables, while $N(\boldsymbol{x}, \lambda)$ and $D(\boldsymbol{x}, \lambda)$ are polynomials in the variables $\boldsymbol{x}$ and λ. We would like to study the Laurent expansion of $f(\boldsymbol{x}, \lambda)$ with respect to λ near a point in the complex plane. Without loss of generality, we can set the point to be $\lambda = 0$.

$$f(\boldsymbol{x}, \lambda) = \frac{N(\boldsymbol{x}, \lambda)}{D(\boldsymbol{x}, \lambda)} = \sum_{i=k}^{\infty} b_i(\boldsymbol{x})\, \lambda^i \,, \tag{6}$$

$$x = \sum_{i=v}^{\infty} a_i\, p^i \,, \tag{7}$$

Comparing Eq. (7) and Eq. (6), we see that p-adic numbers provide a natural framework for studying the Laurent expansion of rational functions. In Eq. (7), each coefficient a_i (vaguely) corresponds to an element in the finite field $\mathbb{F}_p$. This offers the possibility for reconstructing the coefficients $b_i(\boldsymbol{x})$ in Eq. (6) using finite field techniques.

Finite field reconstruction of rational functions

To proceed, we assume that the polynomial is of degree-n, where n could be unknown before the reconstruction. We choose a sequence of distinct interpolation points $y_1, \cdots, y_{n+1}$, and write down an ansatz for the function in the form of Newton polynomials:

$$f(x) = c_0 + \sum_{i=1}^{n} c_i \prod_{j=1}^{i} (x - y_j)\,. \tag{8}$$

By evaluating $f(x)$ at these interpolation points in the finite field $\mathbb{F}_p$, we can obtain the coefficients c_i, also as an element in $\mathbb{F}_p$. In other words, what we obtain are actually c_i modulo p. The maximal exponent n can be determined when a couple of sequential coefficients are zero.

Here c_i can be reconstructed by using Chinese Remainder Theorem (CRT). We preferred to choose as large prime numbers p as possible to reduce the probes.

Benefit of large p in p-adic numbers

An additional benefit of using a large p: As an illustrative example, consider a simple linear function

$$f(x, \lambda) = b_0(x) + \lambda\,, \tag{9}$$

where b_0 is a nonzero polynomial of x.

For $f(y, p)$ to have an overall factor of p with a randomly chosen $y \in \mathbb{Q}_p$, we will need $b_0(y) = 0 \pmod{p}$. The probability for this to happen becomes extremely low for large enough p. Similar considerations apply for higher degree polynomials and for rational functions.

This allows us to determine the minimal order of the expansion using only one probe, without performing an analytical expansion.

Reconstruction of Laurent expansion of rational functions

Consider a function which depends on variables x in addition to the variable λ used for Laurent expansion:

$$f(\boldsymbol{x}, \lambda) = \sum_{i=k}^{\infty} b_i(\boldsymbol{x})\, \lambda^i \,. \tag{10}$$

By evaluating the function with $\lambda = p$ and $\boldsymbol{x} = \boldsymbol{y}$, where elements of $\boldsymbol{y}$ belong to $\mathbb{Z}_{>0}$, we obtain a p-adic number

$$f(\boldsymbol{y}, p) = \sum_{i=k}^{\infty} b_i(\boldsymbol{y})\, p^i = \sum_{i=k}^{\infty} c_i(\boldsymbol{y})\, p^i \,. \tag{11}$$

b_i admits the expansion (start from p^0 based on previous discssions)

$$b_i(\boldsymbol{y}) = \sum_{j=0}^{\infty} a_{i,j}(\boldsymbol{y})\, p^j \,, \tag{12}$$

where $a_{i,j}(\boldsymbol{y})$ are integers in the range $[0, p-1]$.

Reconstruction of Laurent expansion of rational functions

Most importantly, this means that $a_{i,0}(\boldsymbol{y}) = b_i(\boldsymbol{y}) \pmod p$, and we can employ the finite field techniques to reconstruct $b_i(\boldsymbol{x})$ from the knowledge of $a_{i,0}(\boldsymbol{y})$. Substituting Eq. (12) into Eq. (11), and comparing the coefficients, we can extract the following relations

$$\begin{aligned} c_k(\boldsymbol{y}) &= a_{k,0}(\boldsymbol{y}) \,, \\ c_{k+1}(\boldsymbol{y}) &= a_{k,1}(\boldsymbol{y}) + a_{k+1,0}(\boldsymbol{y}) \pmod p \,, \\ c_{k+2}(\boldsymbol{y}) &= a_{k,2}(\boldsymbol{y}) + a_{k+1,1}(\boldsymbol{y}) + a_{k+2,0}(\boldsymbol{y}) + q_{k+1}(\boldsymbol{y}) \pmod p \,, \end{aligned} \tag{13}$$

where $q_{k+1}(\boldsymbol{y})$ denotes the possible "carry" from the previous order.

All desired coefficients $b_i(\boldsymbol{x})$ is contained in Eq. (11). This is the major benefit of using p-adic numbers. The required number of probes depends on the most complicated function in the list $\{b_k(\boldsymbol{x}), \cdots, b_{i_{\max}}(\boldsymbol{x})\}$.

Precision of p-adic numbers in computer systems

If we want to calculate $3\sqrt{2}$ in a computer, the computer may calculate it as

$$3 \times 1.414 = 4.242 \tag{14}$$

However the true value should be 4.243 . The point here is that the ignored digits may accumulate giving rise to a carry, which is not accounted for.
Such round-off error is not a problem for p-adic numbers, since the "carry" from the last digit is added "to the right" which can be discarded.

$$x_1 = p^{v_1}\left(\sum_{i=0}^{h-1} a_{1,i}\, p^i + \mathcal{O}(p^h)\right), \quad x_2 = p^{v_2}\left(\sum_{i=0}^{h-1} a_{2,i}\, p^i + \mathcal{O}(p^h)\right), \tag{15}$$

Without loss of generality, we assume that $v_1 \leq v_2$, or $\delta_v \equiv v_2 - v_1 \geq 0$. We now consider the sum $x_1 + x_2$:

$$x_1 + x_2 = p^{v_1}\left(\sum_{i=0}^{h-1} \left[a_{1,i} + \theta(i-\delta_v)\, a_{2,i-\delta_v}\right]\, p^i + \mathcal{O}(p^h)\right) = p^{v_1}\left(\sum_{i=0}^{h-1} c_i\, p^i + \mathcal{O}(p^h)\right), \tag{16}$$

Precision of p-adic numbers in computer systems

There is one situation that may lead to a precision loss. Suppose we have $v_1 = v_2 = v$ in Eq. (16), and it happens that $a_{1,0} + a_{2,0} = p$. In this case the valuation of $x_1 + x_2$ becomes $v + 1$, and we have

$$x_1 + x_2 = p^{v+1} \left(\sum_{i=0}^{h-2} c_i \, p^i + \mathcal{O}(p^{h-1}) \right). \tag{17}$$

Hence, the precision of the result becomes $h - 1$ instead of h. The probability for this to happen can be reduced by choose a large prime number p. When this is unavoidable, we can also simply increase the precision by one or two digits in the evaluation.

(The same consideration can be applied to multiplication and division. These two operations do not cause precision loss.)

Recycling information in the reconstruction of different orders

Consider a rational function $f(\boldsymbol{x}, \lambda)$ in the form

$$f(\boldsymbol{x}, \lambda) = \frac{a_0 + a_1\lambda + a_2\lambda^2 + \mathcal{O}(\lambda^3)}{b_0 + b_1\lambda + b_2\lambda^2 + \mathcal{O}(\lambda^3)}\,, \tag{18}$$

where a_i and b_i are polynomials of x (whose dependence is suppressed for simplicity). We assume that a_0 and b_0 are coprime polynomials, and therefore the series expansion with respect to λ can be written as

$$f(\boldsymbol{x}, \lambda) = \frac{a_0}{b_0} + \frac{a_1 b_0 - a_0 b_1}{b_0^2}\lambda + \frac{a_2 b_0^2 - a_1 b_0 b_1 + a_0 b_1^2 - a_0 b_0 b_2}{b_0^3}\lambda^2 + \mathcal{O}(\lambda^3)\,. \tag{19}$$

More complicated:

$$\begin{aligned} f(\boldsymbol{x}, \lambda) &= \frac{a_0 c_0 e_0 + a_1 e_0 \lambda + a_2\lambda^2 + \mathcal{O}(\lambda^3)}{d_0 e_0\,(b_0 c_0 + b_1\lambda + b_2\lambda^2 + \mathcal{O}(\lambda^3))} \\ &= \frac{a_0}{b_0 d_0} + \frac{a_1 b_0 - a_0 b_1}{b_0^2 c_0 d_0}\lambda + \frac{a_2 b_0^2 c_0 - a_1 b_0 b_1 e_0 + a_0 b_1^2 e_0 - a_0 b_0 b_2 c_0 e_0}{b_0^3 c_0^2 d_0 e_0}\lambda^2 + \mathcal{O}(\lambda^3)\,. \end{aligned} \tag{20}$$

Benchmarks

Number of probes required for the reconstruction of ϵ-expansion and m^2-expansion coefficients for a 400 KB expression. As a comparison, the full reconstruction of the unexpanded expression requires 313483 probes.

Order in ϵ	Without recycling	With recycling
0	1334	1334
1	18853	5313
2	140124	5771
3	389795	13653

Order in m^2	Without recycling	With recycling
0	277	277
1	792	572
2	1760	1214
3	8749	4566

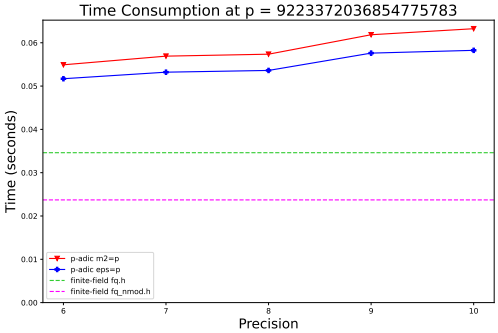
Benchmarks

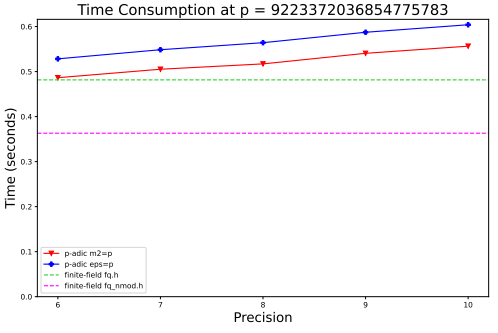
Degrees of denominators in the reconstruction of ϵ-expansion coefficients for a 400 KB expression. For reference, the degree of the denominator in the unexpanded expression is 31 .

Order in ϵ	Without recycling	With recycling
0	12	12
1	22	6
2	32	0
3	41	0

Order in m^2	Without recycling	With recycling
0	5	5
1	9	3
2	13	2
3	20	4

The time consumption per probe in the finite field and in the p-adic field. The dashed lines show the time consumption in the finite field using two different interfaces `fq.h` and `fq_nmod.h` in `FLINT`. The solid lines represent the time consumption in the p-adic field as a function of the precision, with different variables set to p. The upper and lower plots correspond to the evaluation of a 400 KB expression and a 1.6 MB expression, respectively





Summary

We propose a novel method for reconstructing Laurent expansion of rational functions from numerical evaluations in p-adic fields. Our method can be applied if there exists an algorithm using p-adic numbers for obtaining numerical values of the unknown rational functions.

In particular, this applies to solutions of linear equations, and a typical situation in high-energy physics is the IBP reduction of Feynman integrals and scattering amplitudes.

Thanks for listening !