How effectively can we test Quantum Gravity with JUNO?

Antonino Marcianò

Fudan University

based on work in collaboration with the VIP and DAMA collaborations

A. Addazi, P. Belli, R. Bernabei & A. Marciano, arXiv:1712.08082, CPC (2018)

A. Addazi & A. Marciano, arXiv:1811.06425

In progress…

TDLI, CosNuMM2019, 28th November 2019

Plan of the talk

Is quantum gravity testable?

The arena of non-commutative space-times

Deformed symmetries and quantum groups in a nutshell

Two main examples of deformed symmetries

Violations of the Pauli exclusion principle and non-commutativity

What Nature has to say on it

Is quantum gravity testable?

Cosmology and astrophysics to test "top-down" models

Amelino-Camelia, Ashtekar, Brandenberger, Bojowald, Vafa, Witten,…

Recent claim about quantum-gravitational microscope

Maselli et al., PRL 120, 081101 (2018) Addazi, Marciano & Yunes, PRL 122, 081301 (2019)

Not all the QG theories fall in one class of universality

Remove ambiguity of the way theories are constructed

We restrict the focus on terrestrial experiments, and consider

$\hbar \to 0$

$G \to 0$

but

$$\sqrt{\frac{\hbar}{G}} \to M_P$$

A shift of paradigm

Quantum gravity phenomenology does not deal only with dispersion relation!

In a mathematical sense, this is deeply related to the underlying structure of quantum field theories endowed with quantum groups

Algebra sector $\longrightarrow$ *Hilbert space and dispersion relations*

Co-algebra sector $\longrightarrow$ *Fock space and statistics*

Curved momentum space and deformed statistics inextricably related

Spin-statistics theorem and NC spacetimes

The Spin statistics theorem of Pauli in QFT is based on Lorentz invariance

But NC spacetimes entail deformation of the Lorentz invariance!
(See e.g. condensed matter instantiations, including anyons et al.)

↓

Effective models of quantum gravity falling in the universality classes of non-commutative spacetimes may entail violations of the Pauli Exclusion

Lorentz symmetry: breakdown vs deformation I

Lorentz invariance breakdown entails Lorentz violating and CPT violating renormalizable operators

↓

Even considering finetuning, this will introduce UV divergent diagrams in the SM sector, affecting basic requirements of unitarity

Possible scenarios also involve dynamical or spontaneous breakdown of Lorentz symmetry at some very high energy scale

↓

Generation of non-renormalizable PEP violating operators at that scale

Lorentz symmetry: breakdown vs deformation II

Deformation of the Lorentz symmetry

↓

CPT is Not violated but deformed, unitarity is still present in most (physically interesting) NC models

An example:

Most studied case in the literature, quantum field theories endowed with θ-Poincare symmetries, dual to a non-commutative spacetime $[x_\mu, x_\nu] = \imath\theta_{\mu\nu}$

$\theta_{0i} = 0$ ⟶ *unitarity preserved*

L. Alvarez-Gaume and M. A. Vazquez-Mozo, Nucl. Phys. B 668, 293 (2003) [hep-th/0305093].

Testing PEP all the ways

The PEP violation induced by (effective) non-commutative models is "democratically" propagating in all the possible PEP forbidden channels.

Constraints can be confronted with all most sensitive experiments: PEP violating atomic or nuclear transitions

Quantum groups in a nutshell: twist I

Non trivial Hopf algebras encode quantum groups obtained by twisting

Introduce the element of the bi-algebra $\mathcal{A}\otimes\mathcal{A}$ *that is called twist element*

$$\mathcal{F}_\theta = e^{\frac{i}{2}\theta^{\mu\nu}P_\mu\otimes P_\nu}$$

such that

$$\mathcal{F}_\theta(\Delta_0\otimes\mathbb{1})\mathcal{F}_\theta = \mathcal{F}_\theta(\mathbb{1}\otimes\Delta_0)\mathcal{F}_\theta$$

Taking into account the element of the θ-Poincare' algebra $Y=\{P_\mu\,, M_{\mu\nu}\}$

$$\Delta_0(Y)\to\Delta_\theta(Y)=\mathcal{F}_\theta\Delta_0(Y)\mathcal{F}_\theta^{-1}$$

Quantum groups in a nutshell: twist II

The algebraic sector is undeformed, yielding the same product rules and the same two Casimir

$$[P_\mu, P_\nu] = 0 \qquad [M_{\mu\nu}, P_\alpha] = -i(\eta_{\mu\alpha}P_\nu - \eta_{\nu\alpha}P_\mu)$$

$$[M_{\mu\nu}, M_{\alpha\beta}] = -i(\eta_{\mu\alpha}M_{\nu\beta} - \eta_{\mu\beta}M_{\nu\alpha} - \eta_{\nu\alpha}M_{\mu\beta} + \eta_{\nu\beta}M_{\mu\alpha})$$

In the co-algebraic sector, deformation involve the coproduct of the Lorentz generators, the others remaining "primitive"

$$\Delta_\theta(P_\alpha) = \Delta_0(P_\alpha) = P_\alpha \otimes 1 + 1 \otimes P_\alpha$$

$$\begin{aligned}\Delta_\theta(M_{\mu\nu}) &= Ade^{(i/2)\theta^{\alpha\beta}P_\alpha \otimes P_\beta}\Delta_0(M_{\mu\nu}) \\ &= M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} - \tfrac{1}{2}\theta^{\alpha\beta}[(\eta_{\alpha\mu}P_\nu - \eta_{\alpha\nu}P_\mu) \\ &\quad \otimes P_\beta + P_\alpha \otimes (\eta_{\beta\mu}P_\nu - \eta_{\beta\nu}P_\mu)]\end{aligned}$$

Example I: QFT enjoying θ-Poincare symmetries

We can develop an auxiliary representation in the coordinates space, encoding space-time points' coordinates intrinsic non-commutativity

Star product defined by the twist:

$$f \star g = f(x) e^{\frac{\imath}{2} \overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu} g(y)$$

$$\theta_{\mu\nu} = -\theta_{\nu\mu} = \text{const}$$

Noncommutativity ST coordinates:

$$\hat{x}^\mu(x) = x^\mu$$

$$\hat{x}^\mu \star \hat{x}^\nu - \hat{x}^\nu \star \hat{x}^\mu := [\hat{x}^\mu, \hat{x}^\nu]_\star = \imath\theta^{\mu\nu}$$

Scalar field Fourier expansion:

$$\phi = \int \frac{d^4p}{2p_0} [a(p)\, \mathbf{e}_p + a^\dagger(p)\, \mathbf{e}_{-p}]$$

Example I: QFT enjoying θ-Poincare symmetries

Fourier decomposition : $\phi = \int d\mu(p)\,\tilde{\phi}(p)\mathbf{e}_p\,, \qquad \psi = \int d\mu(q)\,\tilde{\phi}(q)\mathbf{e}_q$

Fields product: $m_\theta(\phi \otimes \psi) = \int d\mu(p)\,d\mu(q)\,\tilde{\phi}(p)\,\tilde{\psi}(q)\,\mathbf{e}_p \star \mathbf{e}_q$

Action of symmetries:

$$\rho(\Lambda)\phi = \int \mu(p)\,\tilde{\phi}(p)\,\mathbf{e}_{\Lambda p} = \int \mu(p)\,\tilde{\phi}(\Lambda^{-1}p)\,\mathbf{e}_p$$

$$\rho(e^{\imath P\cdot\delta})\phi = \int \mu(p)\,e^{\imath p\cdot\delta}\,\tilde{\phi}(p)\,\mathbf{e}_p$$

Deformed statistics induced by the twist element

$$a(p)a^\dagger(q) = \tilde{\eta}'(p,q)\mathcal{F}_\theta(-q,p)a^\dagger(q)a(p) + 2p_0\delta^4(p-q)$$

Example I: QFT enjoying θ-Poincare symmetries

Twisted fermionic states ⟶ *Non-vanishing overlap probability*

Twisted single particle wave-packet created by $\langle a^\dagger, \alpha\rangle = \int \frac{d^4p}{2p_0}\, \alpha(p)\, a^\dagger(p)$

$$|\alpha\rangle = \langle a^\dagger, \alpha\rangle|0\rangle = \langle c^\dagger, \alpha\rangle|0\rangle$$

$$a(p) = e^{\frac{i}{2}p_\mu\theta^{\mu\nu}P_\nu}\, c(p) \qquad c(p) \quad \text{for} \quad \theta^{\mu\nu} = 0$$

Two-particle state, violating the Pauli principle for $\theta^{\mu\nu} \neq 0$

$$|\alpha\,,\alpha\rangle = \langle a^\dagger, \alpha\rangle\,\langle a^\dagger, \alpha\rangle|0\rangle = \int \frac{d^4p_{(1)}}{2p_{0(1)}}\,\frac{d^4p_{(2)}}{2p_{0(2)}}\, e^{-\frac{i}{2}p_{\mu(1)}\theta^{\mu\nu}p_{\nu(2)}}\alpha(p_{(1)})\alpha(p_{(2)})c^\dagger(p_{(1)})c^\dagger(p_{(2)})|0\rangle$$

A. P. Balachandran, T.R. Govindarajan, G. Mangano, A. Pinzul, B.A. Qureshi & S. Vaisya, Phys. Rev. D 75, 045009 (2007)

Example I: QFT enjoying θ-Poincare symmetries

Non-vanishing normalization of the PEP violating state for $\theta^{\mu\nu} \neq 0$

$$N^2(\alpha\,,\alpha) := \langle \alpha\,,\alpha|\alpha\,,\alpha\rangle = \int \frac{d^4p_{(1)}}{2p_{0(1)}}\frac{d^4p_{(2)}}{2p_{0(2)}}\left(\bar{\alpha}(p_{(1)})\alpha(p_{(1)})\right)\left(\bar{\alpha}(p_{(2)})\alpha(p_{(2)})\right)\left[1-\cos(p_{\mu(1)}\theta^{\mu\nu}p_{\nu(2)})\right] \geq 0$$

where the normalization vanishes only on a zero-measure set

Normalized states that are PEP violating: $|\alpha\,,\alpha\rangle' = \frac{1}{N(\alpha\,,\alpha)}\,|\alpha\,,\alpha\rangle$

Given a two-particle state allowed by PEP $|\beta\,,\gamma\rangle = \langle a^\dagger\,,\beta\rangle\,\langle a^\dagger\,,\gamma\rangle|0\rangle\,,\ \beta \neq \gamma$

transitions to PEP violating states can now happen:

$$\langle \beta\,,\gamma|\alpha\,,\alpha\rangle = \int \frac{d^4p_{(1)}}{2p_{0(1)}}\frac{d^4p_{(2)}}{2p_{0(2)}}\left(\bar{\beta}(p_{(1)})\alpha(p_{(1)})\right)\left(\bar{\gamma}(p_{(2)})\alpha(p_{(2)})\right)\left[1-e^{p_{\mu(1)}\theta^{\mu\nu}p_{\nu(2)}}\right]\frac{1}{N(\alpha\,,\alpha)} \geq 0$$

Example II: QFT enjoying k-Poincare symmetries

The algebraic sector is deformed:

$$[P_0, P_j] = 0 \qquad [M_j, M_k] = i\epsilon_{jkl}M_l \qquad [M_j, N_k] = i\epsilon_{jkl}N_l \qquad [N_j, N_k] = i\epsilon_{jkl}M_l$$

$$[P_0, N_l] = -iP_l \qquad [P_l, N_j] = -i\delta_{lj}\left(\tfrac{\kappa}{2}\left(1 - e^{-\frac{2P_0}{\kappa}}\right) + \tfrac{1}{2\kappa}\vec{P}^2\right) + \tfrac{i}{\kappa}P_l P_j$$

$$[P_0, M_k] = 0 \qquad [P_j, M_k] = i\epsilon_{jkl}P_l$$

In the co-algebraic sector, deformation involve all the coproducts:

$$\begin{aligned} \Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0 \quad \Delta(P_j) = P_j \otimes 1 + e^{-P_0/\kappa} \otimes P_j \\ \Delta(M_j) &= M_j \otimes 1 + 1 \otimes M_j \\ \Delta(N_j) &= N_j \otimes 1 + e^{-P_0/\kappa} \otimes N_j + \frac{\epsilon_{jkl}}{\kappa} P_k \otimes N_l \,. \end{aligned}$$

Example II: QFT enjoying k-Poincare symmetries

The antipode is non-trivial:

$$
\begin{aligned}
S(M_l) &= -M_l \\
S(P_0) &= -P_0 \\
S(P_l) &= -e^{\frac{P_0}{\kappa}} P_l \\
S(N_l) &= -e^{\frac{P_0}{\kappa}} N_l + \frac{1}{\kappa}\epsilon_{ljk} e^{\frac{P_0}{\kappa}} P_j M_k
\end{aligned}
$$

The mass Casimir is deformed:

$$C_\kappa = \left(2\kappa \sinh\left(\frac{P_0}{2\kappa}\right)\right)^2 - \vec{P}^2 e^{\frac{P_0}{\kappa}}$$

Energy-momentum dispersion relations deformed!

Effects linearly suppressed in the Planck energy: $\kappa \propto M_P$

Example II: QFT enjoying k-Poincare symmetries

Ambiguity present in the literature:

i) symplectic geometry approach a la Crnkovic-Witten leads to the deformation of the statistics

M. Arzano & A.M., Phys. Rev. D76 (2007) 125005; M. Arzano & A.M., Phys. Rev. D75 (2007) 081701

ii) 5D differential calculus approach suggests absence of deformation of the statistics

L. Freidel, J. Kowalski-Glikman & S. Nowak, Int.J.Mod.Phys. A23 (2008) 2687-2718

What Nature has to say on it

Parametrization of statistics deformation

To account for all the possible different deformations we use the parametrization

$$a_i a_j^\dagger + \eta\, q(E) a_j^\dagger a_i = \delta_{ij}$$

with q(E) deviation function, and

$$q(E) = -1 + \beta^2(E)\,, \qquad \delta^2(E) = \frac{1}{2}\beta^2(E)$$

We then expand the deviation function, which is assumed to be analytical, in power-series of the ratio between the energy of the system and the deformation energy scale Λ

$$\delta^2(E) = c_k\,\frac{E^k}{\Lambda^k} + O(E^{k+1})$$

Forbidden transition in DAMA and VIP

Two types of experiments to look for PEP violation: i) search for atoms or nuclei in a non-Paulian state; ii) search for the prompt radiation accompanying non-Paulian transitions of electrons or nucleons.

Type i): Novikov et al. '89 and Nolte et al. '91 looked for non-Paulian exotic atoms of ^{20}Ne and ^{36}Ar with 3 electrons on K-shell using mass spectroscopy on fluorine and chlorine samples.

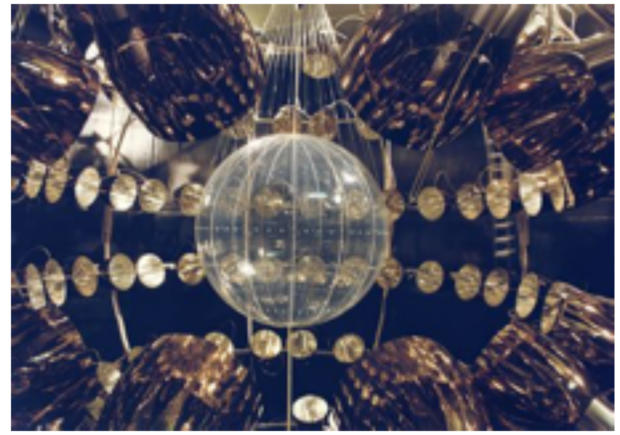
Type ii): Goldhaber '74 pointed out that the same experimental data which were used to set a limit on the lifetime of the electron can be used to test the validity of the PEP for atomic electrons.

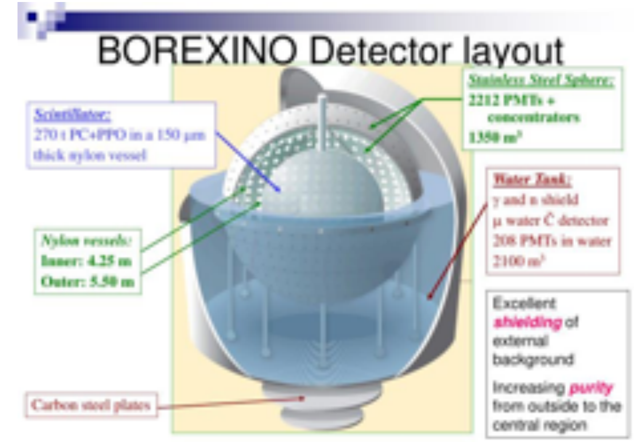
Ramberg and Snow '90 looked for anomalous X-rays emitted by Cu atoms in a conductor. The upper limit on the probability for the 'new' electron passing in the conductor to form a non-Paulian atom with 3 electrons in the K-shell is $1.7 \cdot 10^{-26}$. Improvement of the sensitivity of the method have been achieved by VIP.

Laser atomic and molecular spectroscopy to search for anomalous PEP- forbidden spectral lines of ^{4}He atoms (Deilamian et al.) and molecules of O_2 (Hilborn et al., Angelis et al.,) and CO_2 (Modugno et al.).

The violation of PEP in the nucleon system searching for non-Paulian transitions with γ- emission (Kamiokande '93, NEMO-II '99), p-emission (Elegant-V '93, DAMA/LIBRA '97) and n-emission (Koshomoto et al.'92), non-Paulian β^+- and β^-- decays (LSD, Kekez et al. '90, NEMO-II '99).

BOREXINO collaboration I





γ, $\beta^{\pm}$, n, p from nucleons PEP violating transitions $1P_{3/2} \to 1S_{1/2}$

$$\tau(^{12}C \to ^{12}\widetilde{C} + \gamma) \geq 5.0 \times 10^{31} yr, \quad \longrightarrow \quad \delta^2_\gamma \leq 2.2 \times 10^{-57}$$

$$\tau(^{12}C \to ^{12}\widetilde{N} + e^- + \bar{\nu}_e) \geq 3.1 \times 10^{30} yr$$

$$\tau(^{12}C \to ^{12}\widetilde{B} + e^+ + \nu_e) \geq 2.1 \times 10^{30} yr \quad \longrightarrow \quad \delta^2_\beta \leq 2.1 \times 10^{-35}$$

$$\tau(^{12}C \to ^{11}\widetilde{B} + p) \geq 8.9 \times 10^{29} yr$$

$$\tau(^{12}C \to ^{11}\widetilde{C} + n) \geq 3.4 \times 10^{30} yr \quad \longrightarrow \quad \delta^2_N \leq 4.1 \times 10^{-60}$$

BOREXINO collaboration II

Extremely low background level

(200 times lower than in CTF at 2 MeV)

$$\tau \geq \varepsilon(\Delta E)\frac{N_N N_n}{S_{lim}}T$$

Candidate events: (1) have a unique cluster of PMT hits; (2) should not be flagged as muons by the outer Cherenkov detector; (3) should not follow a muon within a time window of 2 ms; (4) should not be followed by another event within a time window of 2 ms except in case of neutron emission; (5) must be reconstructed within the detector volume.

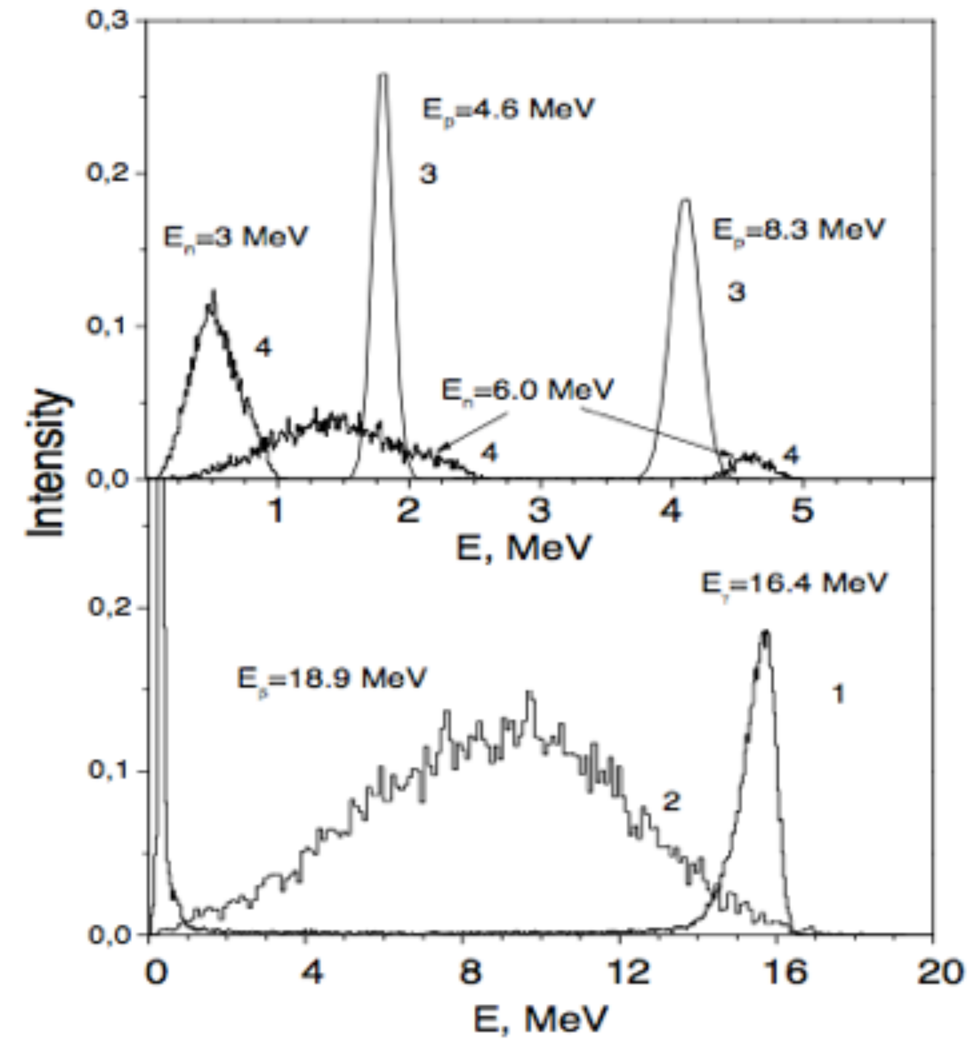
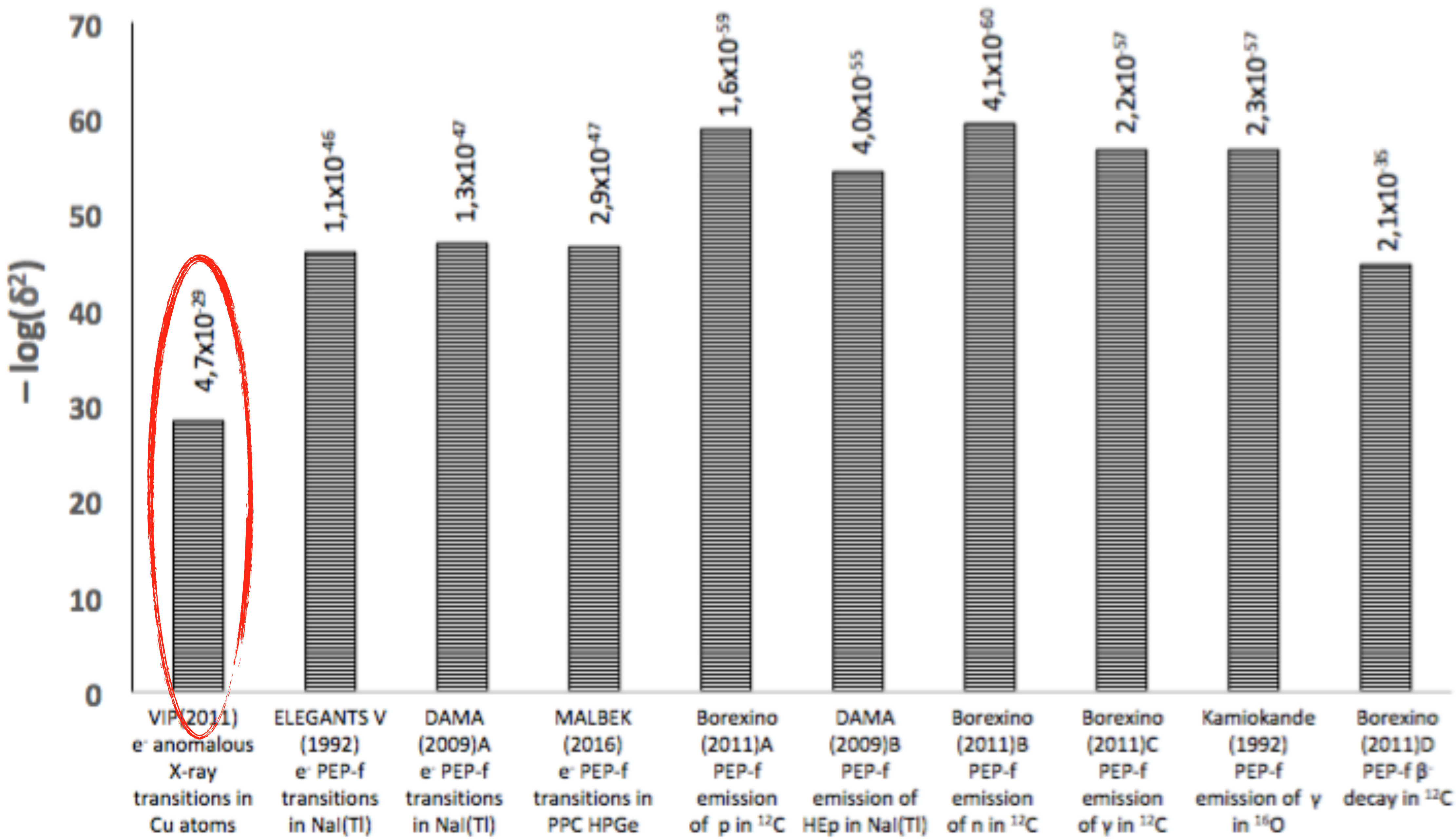
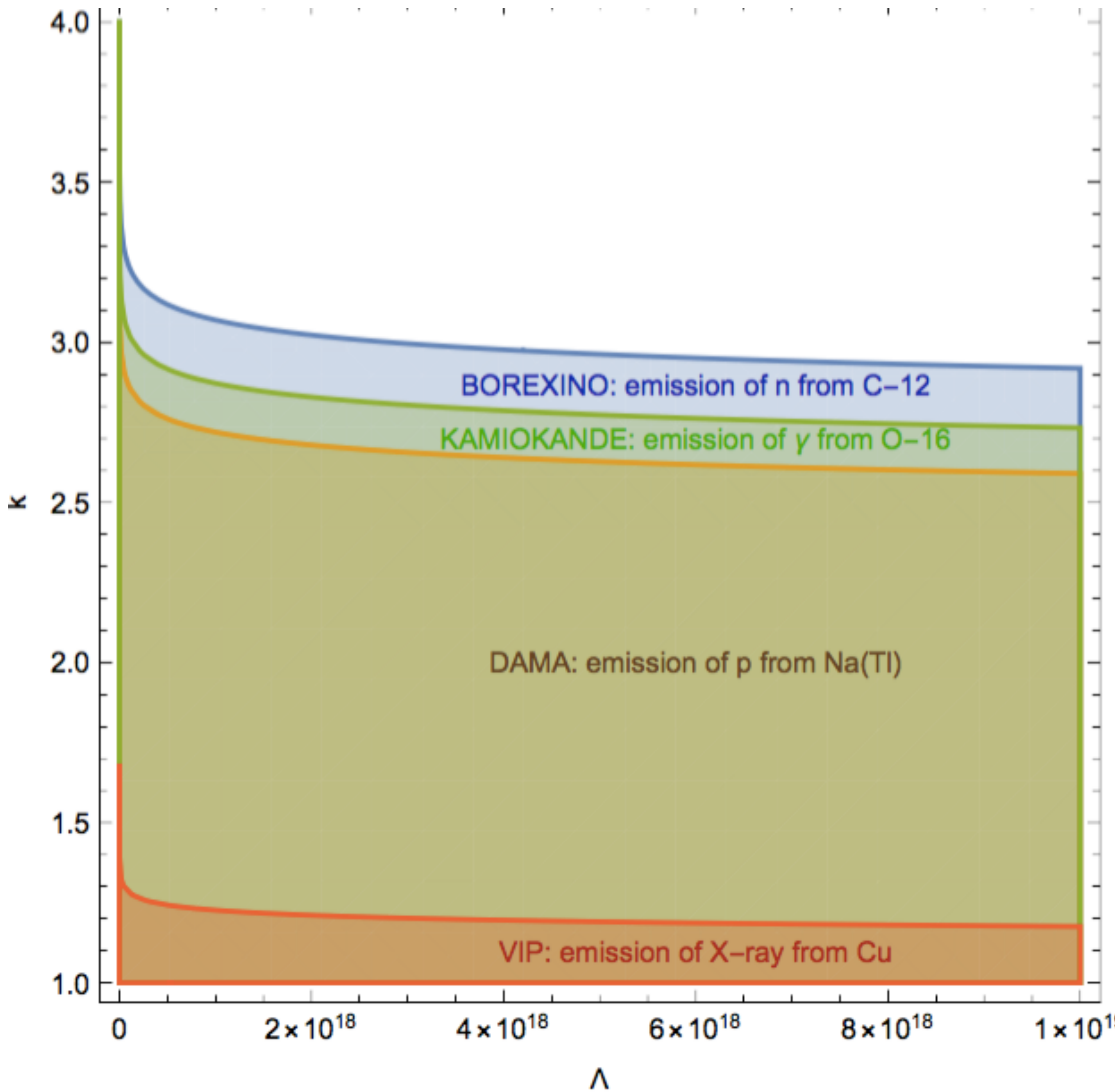


Fig. 6. The response functions of Borexino: 1) $^{12}C \to {}^{12}\widetilde{C}+\gamma$ (16.4 MeV) decays in IV and 1 m thick layer of buffer; 2) $^{12}C \to {}^{12}\widetilde{N} + e^- + \overline{\nu}$ (18.9 MeV); 3) $^{12}C \to^{11} \widetilde{B} + p$ (4.6 and 8.3 MeV); 4)$^{12}C \to {}^{11}\widetilde{C} + n$ (3.0 and 6.0 MeV);

Underground experiments combined



Constraints on non-commutative spacetimes



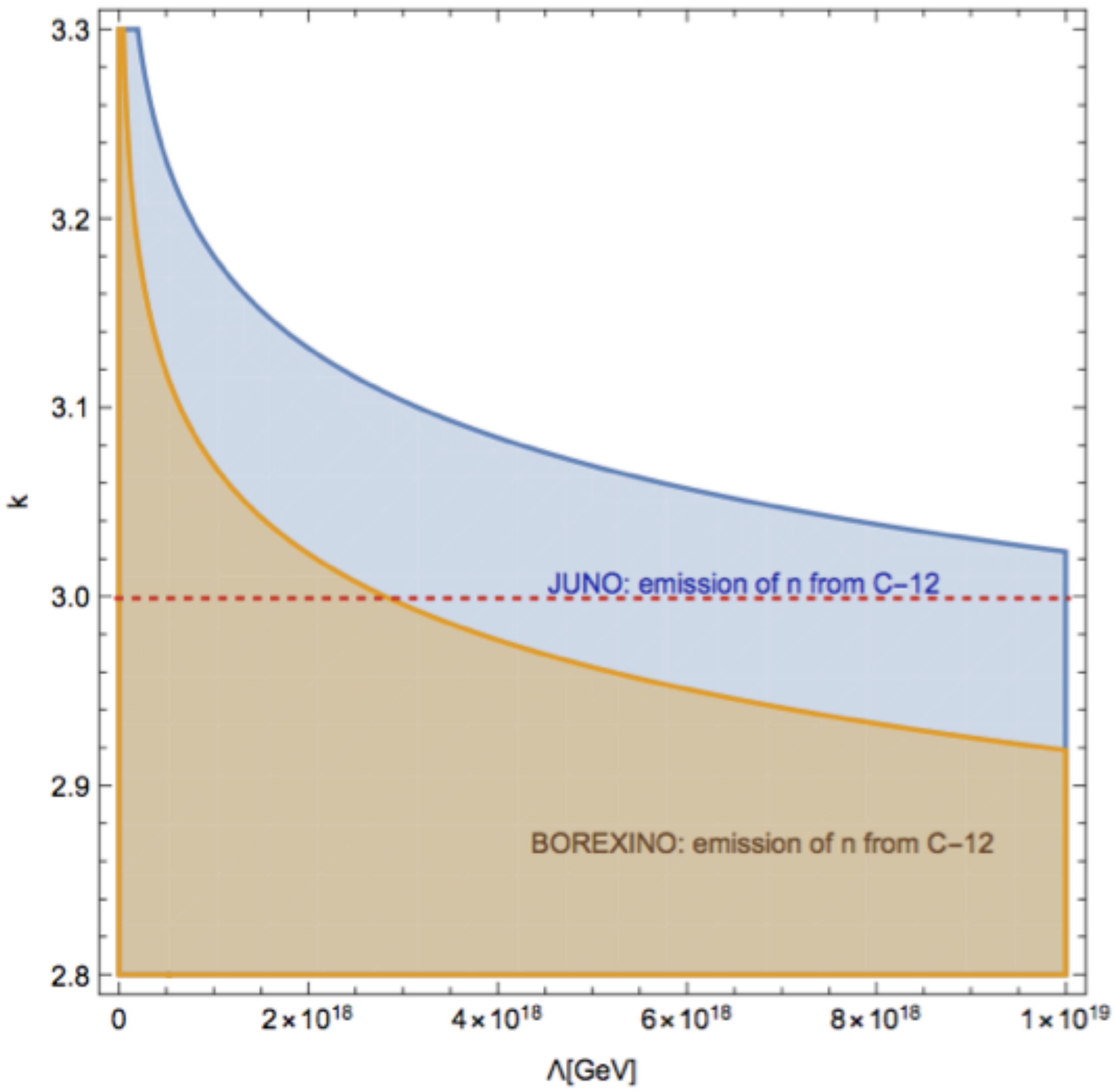
Future improvement: JUNO

(JUNO) Jiangmen Underground Neutrino Experiment, underground reactor antineutrino experiment under construction near Kaiping, China

Experiment	Day Bay	Borexino	KamLAND	JUNO
Liquid Scintillator mass	20 ton	$\sim$ 300 ton	$\sim$1 kton	20 kton
Coverage	$\sim$ 12%	$\sim$ 34%	$\sim$ 34%	$\sim$ 80%
Energy Resolution	$\frac{7.5\%}{\sqrt{E}}$	$\frac{\sim 5\%}{\sqrt{E}}$	$\frac{\sim 6\%}{\sqrt{E}}$	$\frac{\sim 3\%}{\sqrt{E}}$
Light Yield	$\sim 160 \, \frac{\text{p.e.}}{\text{MeV}}$	$\sim 500 \, \frac{\text{p.e.}}{\text{MeV}}$	$\sim 250 \, \frac{\text{p.e.}}{\text{MeV}}$	$\sim 1200 \, \frac{\text{p.e.}}{\text{MeV}}$

Liang Zhan, Yifang Wang, Jun Cao, Liangjian Wen, Phys. Rev. D 78, 111103 (2008)

A present (for free) from JUNO



Conclusions

A new path to provide quantum gravity studies with experimental guidance

Thanks to underground experiments, we can test with high accuracy energy-dependent violations of PEP induced from (effective) non-commutative models of quantum gravity

θ-Poincare is ruled out up to 10^7 the Planck scale by BOREXINO and DAMA. JUNO might even improve the constraints.

In some instantiations, k-Poincare is also ruled out up to 10^{35} the Planck scale.

谢谢

Thank you! Grazie!

X-ray transitions and PEP violations

Searches for characteristic X-rays due to electron decay inside an atomic shell are often indistinguishable from the PEP-violating transition. Nonetheless, according to Amado and Primakoff [PRC '80], such kind of electron decay transitions does not take place even in presence of PEP violation.

A caveat should be considered: the above limitation does not hold when transitions also encode a change of the number of identical fermions (for instance, the non-Paulian $\beta^{\pm}$- transitions). Furthermore, the arguments can be evaded while considering composite models of electron or models including extra dimensions [Greenberg & Mohapatra, PRL '87, Akama, Terazawa & Yasue, PRL '92]

Borexino Background

Expected solar neutrino rate in 100 tons of scintillator ~ 50 counts/day (**~ 5 10^{-9} Bq/Kg**)

Just for comparison:

Natural water	~ 10 Bq/kg in ^{238}U, ^{232}Th and ^{40}K
Air	~ 10 Bq/m^3 in ^{39}Ar, ^{85}Kr and ^{222}Rn
Typical rock	~ 100-1000 Bq/kg in ^{238}U, ^{232}Th and ^{40}K

BX scintillator must be **9/10 order of magnitude less** radioactive than anything on earth!

- **Low background nylon vessel** fabricated in hermetically sealed low radon clean room (~1 yr)
- **Rapid transport** of scintillator solvent (PC) from production plant to underground lab to avoid cosmogenic production of radioactivity (^7Be)
- Underground **purification plant** to distill scintillator components.
- **Gas stripping** of scintlllator with special nitrogen free of radioactive ^{85}Kr and ^{39}Ar from air
- All materials **electropolished SS or teflon**, precision cleaned with a dedicated cleaning module

DAMA collaboration (2009)

Eur. Phys. J. C (2009) 62: 327–332

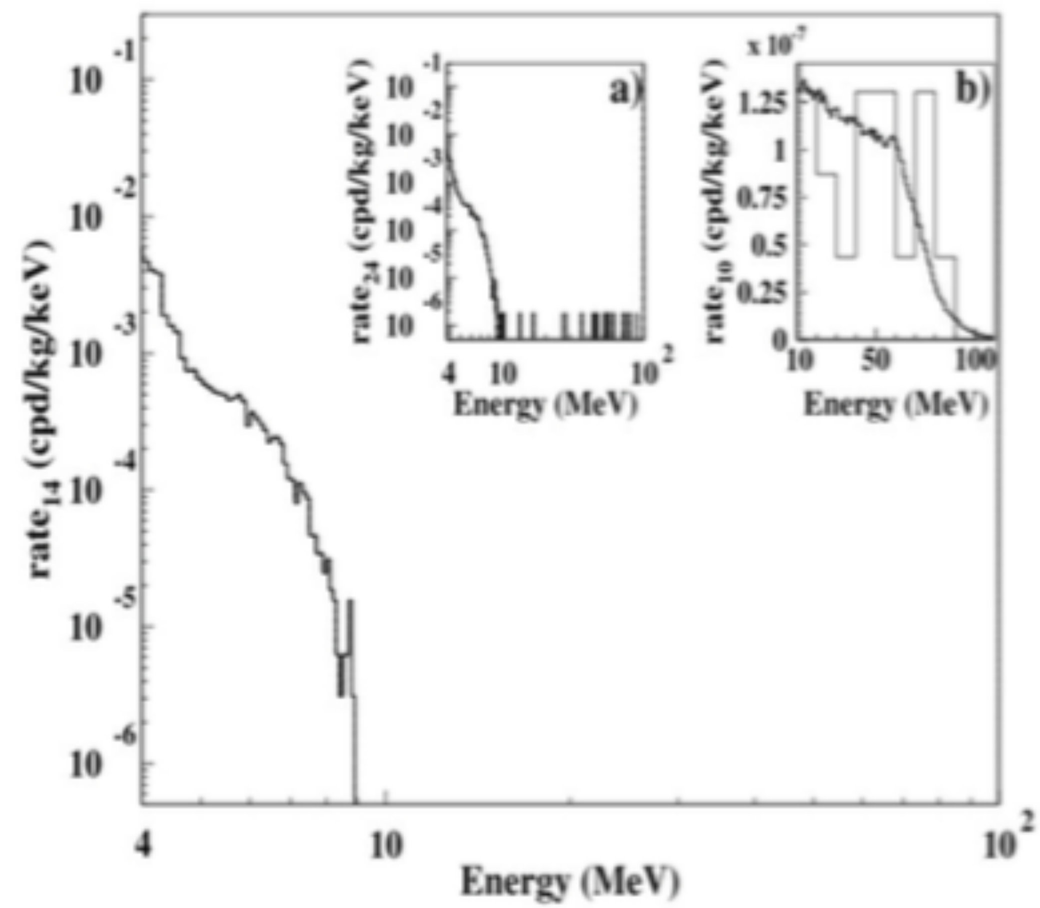


Fig. 1 Counting rate ($rate_{14}$) of the events measured by the 14 highly radiopure NaI(Tl) detectors in operation in the three central rows of the DAMA/LIBRA detectors matrix. The events in the 4–10 MeV energy region are essentially due to α particles from internal contaminants in the detectors (detailed studies are available in [34]). In inset (**a**) the counting rate measured by all the 24 working detectors ($rate_{24}$) is shown. Events with $E > 10$ MeV are present only in detectors belonging to the upper or to the lower rows in the detectors matrix. In inset (**b**) the same events as in (**a**)—with different binning—are shown above 10 MeV (histogram) with superimposed a solid line, which corresponds to the background events expected from the vertical muon intensity distribution and the Gran Sasso rock overburden map of [37]. See text

Nuclear models in DAMA

Two main models used in the momentum distribution of nucleons:

i) Fermi momentum distribution with 255 MeV/c

i) realistic functions taking into account correlation effects.

Bernabei, Belli et al (DAMA collaboration) EPJC (2009)

Quantum groups in a nutshell: Hopf algebras I

Consider the infinite dimensional representation of the translation algebra $\mathcal{A}$ *on 4D Minkowski spacetime*

$$P_\mu \blacktriangleright m(f(x) \otimes g(x)) = m(P_\mu \blacktriangleright f(x) \otimes g(x) + f(x) \otimes P_\mu \blacktriangleright g(x))$$

We then associate the (trivial) "coproduct" $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$

$$\Delta(P_\mu) = P_\mu \otimes \mathbb{1} + \mathbb{1} \otimes P_\mu$$

an element of the co-algebra, forming together with the algebra a bi-algebra when specific axioms are taken into account.

Quantum groups in a nutshell: Hopf algebras II

Introduce now:

$$\epsilon : \mathcal{A} \to \mathbb{C} \quad \text{such that, for } a \in \mathcal{A}, \quad \int d^4x \, a f(x) = \epsilon(a) \int d^4x f(x)$$

$$m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$$

$$S : \mathcal{A} \to \mathcal{A} \quad \text{recreating, for } a \in \mathcal{A}, \text{ the inverse element of } a$$

For the trivial case under scrutiny:

$$\epsilon(\mathbb{1}) = \mathbb{1} \qquad S(\mathbb{1}) = \mathbb{1}$$

$$\epsilon(P_\mu) = 0 \qquad S(P_\mu) = -P_\mu$$

We can then extend the structure of a Lie algebra to a Hopf algebra

Quantum groups in a nutshell: Hopf algebras III

Algebra axioms

$$m(m \otimes 1) = m(1 \otimes m) \qquad \text{(associativity)},$$

$$m(1 \otimes \eta) = m(\eta \otimes 1) = 1 \qquad \text{(unit)},$$

Co-algebra axioms

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta \qquad \text{(coassociativity)},$$

$$(1 \otimes \varepsilon)\Delta = (\varepsilon \otimes 1)\Delta = 1 \qquad \text{(counit)},$$

Bialgebra

Antipode axioms

$$m(S \otimes 1)\Delta = m(1 \otimes S)\Delta = \eta \circ \varepsilon.$$

Hopf algebra