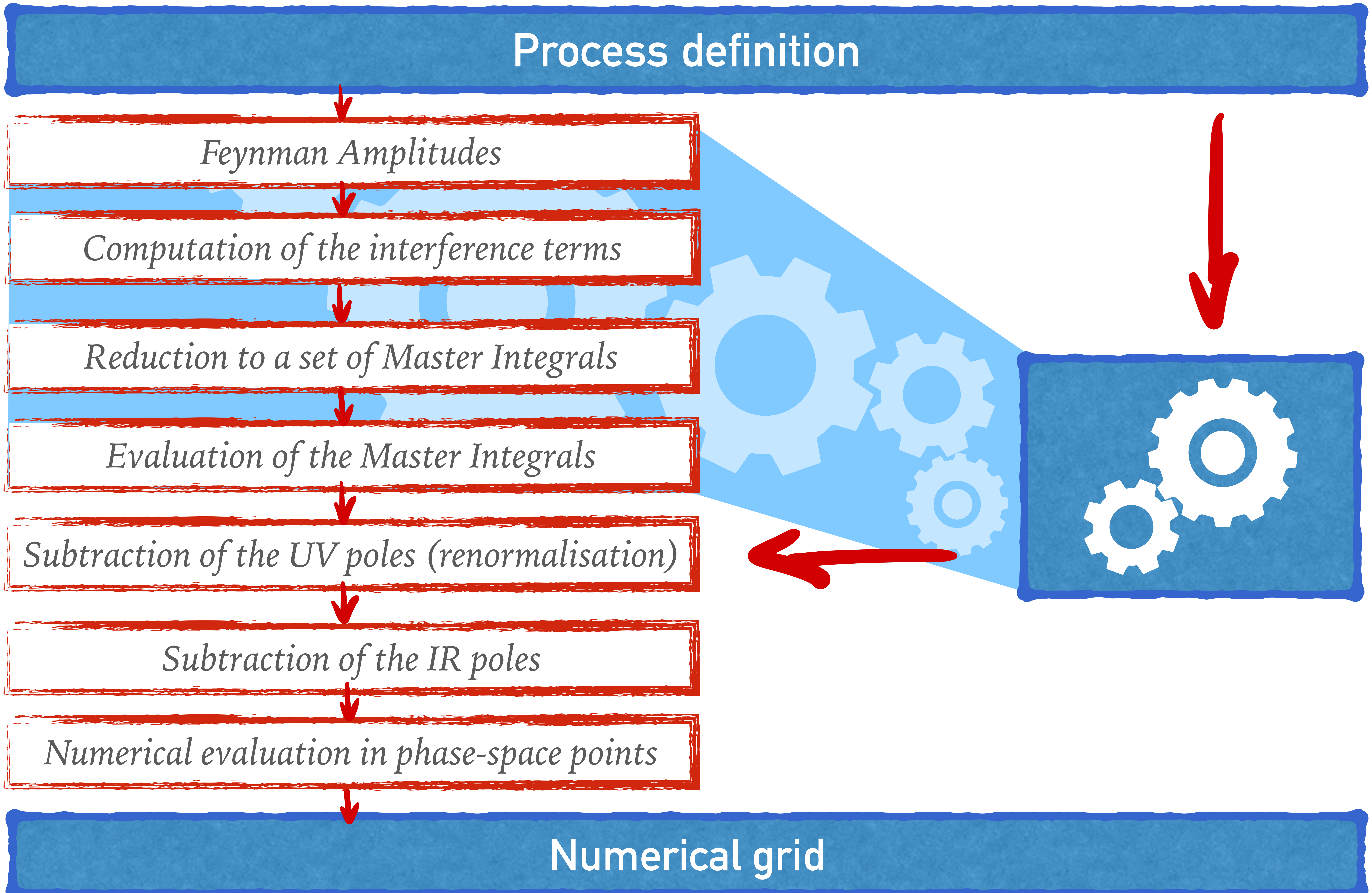
STRUCTURE OF A LOOP COMPUTATION

courtesy of Simone Devoto



Solving Feynman integrals

$$I[\alpha_0, \alpha_1, \ldots, \alpha_l] = \int \frac{d^n k_1}{(2\pi)^n} \int \frac{d^n k_2}{(2\pi)^n} \frac{1}{[k_1^2 - m_0^2]^{\alpha_0} \, [(k_1+p_1)^2 - m_1^2]^{\alpha_1} \, \ldots \, [(k_1+k_2+p_j)^2 - m_j^2]^{\alpha_j} \, \ldots \, [(k_2+p_l)^2 - m_l^2]^{\alpha_l}}$$

where I is the name of the "integral family" → a specific value of momentum and mass of the k-th denominator is understood

The solution of a Feynman integral can be achieved by different techniques:

- direct numerical integration (naive, sector decomposition,…)
- integration of auxiliary parameters (Feynman, Schwinger,…)
- differential equations

A Feynman integral can be considered as an unknown function $I = I(s, t, u, \ldots, m_0, m_1, \ldots, \varepsilon)$

which satisfies a (system of) linear first-order differential equations, with respect to kinematical invariants and masses, e.g. $\frac{\partial \vec{I}}{\partial s} = \mathbf{A} \cdot \vec{I}$

How can we write the matrix $\mathbf{A}$? How do we solve the system ?

Differential equations and IBPs

M.Argeri and P.Mastrolia, " Feynman Diagrams and Differential Equations "
Int.J.Mod.Phys.A 22 (2007) 4375-4436 • e-Print: 0707.4037

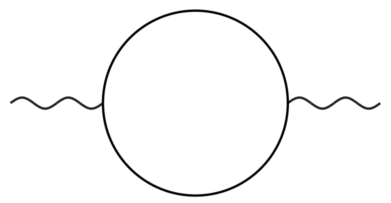
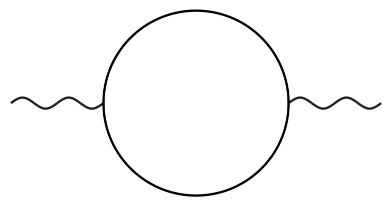
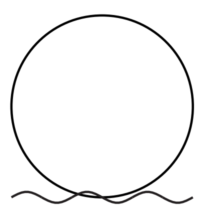
- Not all the Feynman integrals in one amplitude are independent
  → exploit Integration-by-parts (IBP) and Lorentz identities to reduce to a basis of independent Master Integrals

$$\int \frac{d^n k_1}{(2\pi)^n} \int \frac{d^n k_2}{(2\pi)^n} \frac{\partial}{\partial k_1^\mu} \frac{(k_1^\mu,\, k_2^\mu,\, p_r^\mu)}{[k_1^2 - m_0^2]^{\alpha_0}\, [(k_1+p_1)^2 - m_1^2]^{\alpha_1} \;\ldots\; [(k_1+k_2+p_j)^2 - m_j^2]^{\alpha_j} \;\ldots\; [(k_2+p_l)^2 - m_l^2]^{\alpha_l}} = 0$$

$$\int \frac{d^n k_1}{(2\pi)^n} \int \frac{d^n k_2}{(2\pi)^n} \frac{\partial}{\partial k_2^\mu} \frac{(k_1^\mu,\, k_2^\mu,\, p_r^\mu)}{[k_1^2 - m_0^2]^{\alpha_0}\, [(k_1+p_1)^2 - m_1^2]^{\alpha_1} \;\ldots\; [(k_1+k_2+p_j)^2 - m_j^2]^{\alpha_j} \;\ldots\; [(k_2+p_l)^2 - m_l^2]^{\alpha_l}} = 0$$

- The independent Master Integrals (MIs) satisfy a system of first-order linear differential equations
  with respect to each of the kinematical invariants / internal masses

  When considering the complete set of MIs, the system can be cast in homogeneous form: $d\vec{I}(\vec{s};\varepsilon) = \mathbf{A}(\vec{s};\varepsilon) \cdot \vec{I}(\vec{s};\varepsilon)$

$\frac{d}{dk^2}$  $+ \frac{1}{2}\left[\frac{1}{k^2} - \frac{(D-3)}{(k^2+4m^2)}\right]$  $= -\frac{(D-2)}{4m^2}\left[\frac{1}{k^2} - \frac{1}{(k^2+4m^2)}\right]$ 

- Henn's conjecture (2013): if a change of basis exists which leads to $d\vec{J}(\vec{s};\varepsilon) = \varepsilon \tilde{\mathbf{A}}(\vec{s}) \cdot \vec{J}(\vec{s};\varepsilon)$
  then the solution is expressed in terms of iterated integrals (Chen integral representation)
  depending only on the results at previous orders in the ε expansion

SeaSyde

https://github.com/TommasoArmadillo/SeaSyde

T.Armadillo, R.Bonciani, S.Devoto, N.Rana, A.Vicini,
"Evaluation of Feynman integrals with arbitrary complex masses via series expansions",
Comput.Phys.Commun. 282 (2023) 108545 • e-Print: 2205.03345

Tommaso Armadillo,
"Evaluating Feynman Integrals through differential equations and series expansions"
2502.14742

A Simple Example

$$\begin{cases} f'(x) + \frac{1}{x^2 - 4x + 5} f(x) = \frac{1}{x+2} \\ f(0) = 1 \end{cases}$$

$$f_{hom}(x) = x^r \sum_{k=0}^{\infty} c_k x^k$$

$$f'_{hom}(x) = \sum_{k=0}^{\infty} (k+r)\, c_k\, x^{(k+r-1)}$$

$$\begin{cases} r c_0 = 0 \\ \frac{1}{5} c_0 + c_1 (r+1) = 0 \\ \frac{4}{25} c_0 + \frac{1}{5} c_1 + c_2 (2+r) = 0 \\ \dots \end{cases}$$

$$f_{hom}(x) = 5 - x - \frac{3}{10} x^2 + \frac{11}{150} x^3 + \dots$$

$$f_{part}(x) = f_{hom}(x) \int_0^x dx' \underbrace{\frac{1}{(x'+2)} f_{hom}^{-1}(x')}_{\text{Expanded around } x'=0}$$

$$= \frac{1}{2} x - \frac{7}{40} x^2 + \frac{2}{75} x^3 + \dots$$

$$f(x) = f_{part}(x) + C\, f_{hom}(x)$$

$$f(0) = 1 \rightarrow C = \frac{1}{5}$$

Evaluation of the Master Integrals by series expansions

T.Armadillo, R.Bonciani, S.Devoto, N.Rana, AV, 2205.03345

- **Taylor expansion**: **avoids** the singularities;
- **Logarithmic expansion**: uses the singularities as **expansion points**.
- Logarithmic expansion has larger convergence radius but requires longer evaluation time. **We use Taylor expansion as default.**