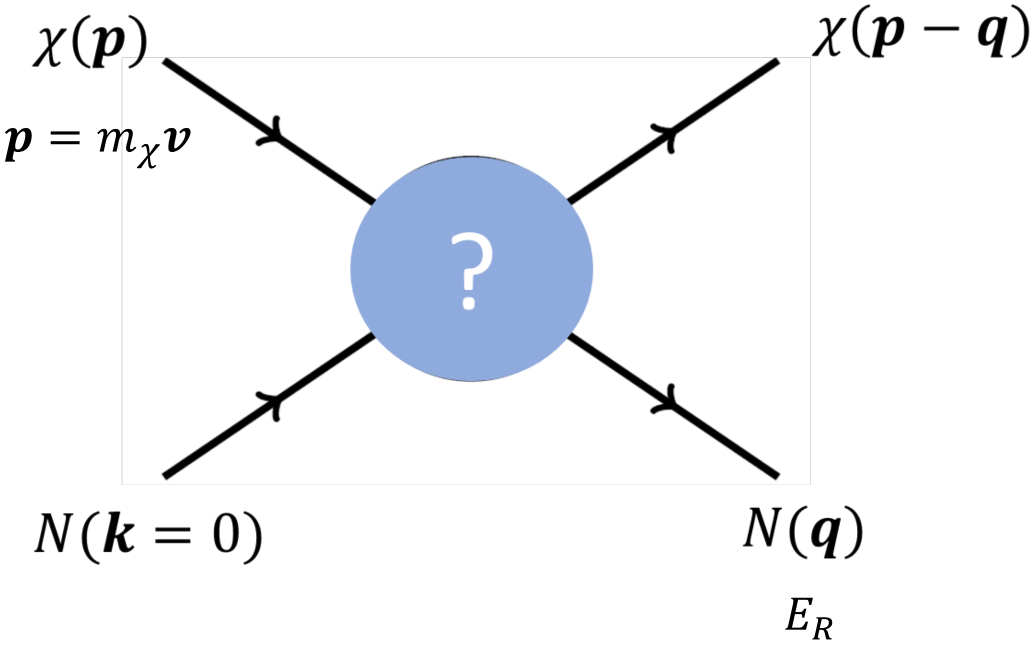
kinematics

Detector @ T $O(300K) \rightarrow$ $K \approx K_B T \sim 0.025\ eV \rightarrow$ In the **lab frame** we can safely neglect the nucleus thermal motion



$$E_i = E_f \rightarrow \frac{p^2}{2m_\chi} = \frac{q^2}{2m_N} + \frac{(\boldsymbol{p}-\boldsymbol{q})^2}{2m_\chi}$$

$\rightarrow \boldsymbol{pq} = \frac{q^2 m_\chi}{2\mu_{\chi N}}$ Where $\mu_{\chi N} = \frac{m_\chi m_N}{m_\chi + m_N}$

Now, $\boldsymbol{pq} = pq\cos\theta_{lab} \rightarrow q = \frac{2\mu_{\chi N}}{m_\chi} p\cos\theta_{lab}$

$$E_R = \frac{q^2}{2m_N} = \frac{2\mu_{\chi N}^2}{m_N} v^2 \cos\theta_{lab}^2$$

Max momentum transfer ($\theta_{lab} = 0$) $\rightarrow$ $q_{max} = \frac{2p\mu_{\chi N}}{m_\chi}$ ➡ $E_R^{max} = \frac{q_{max}^2}{2m_N} = \frac{2\mu_{\chi N}^2}{m_N} v^2$ ➡

For a given Energy threshold E_R there is a minimum velocity of the WIMP to be visible in the detector

$$v_{min} = \sqrt{\frac{E_R m_N}{2\mu_{\chi N}^2}}$$

Expected rate in the detector

The expected WIMP rate in the detector is given by $R \approx N_T \times n_\chi \times \langle v \rangle \times \sigma$

where:

$N_T = M_{det}/m_N$ is the number of target nuclei (NOTE: here $\boldsymbol{M_{det}}$ **is in the same units as** $\boldsymbol{m_N}$)

$n_\chi = \rho_\chi / m_\chi$ is the number density of WIMPs and m_χ is the WIMP mass

ρ_χ is the DM local density

$\langle v \rangle$ is the mean relative velocity WIMP / detector

σ is the cross section, that in general depends on the transferred moment

If $f(\boldsymbol{v})$ is the WIMP velocity distribution in the **detector's frame**, so $\langle v \rangle = \int_0^\infty v f(\boldsymbol{v}) d^3\boldsymbol{v}$ so:

differential rate per unit of recoil energy:

$$\frac{dR}{dE_R} = \frac{M_{det}\rho_\chi}{m_N m_\chi} \int_{v_{min}}^{\infty} \frac{d\sigma}{dE_R} v f(\boldsymbol{v}) d^3\boldsymbol{v}$$

where $v_{min} = \sqrt{\frac{E_R m_N}{2\mu_{\chi N}^2}}$

The Master formula

nuclear form factor

$$\frac{dR}{dE_R} = \frac{M_{det}\rho_\chi}{m_N m_\chi}\int_{v_{min}}^{v_{esc}} \frac{d\sigma}{dE_R}(q,v) v f(\boldsymbol{v}) d^3\boldsymbol{v}$$

$$\frac{d\sigma}{dE_R}(v,q) = \frac{d\sigma}{dE_R}(v,0) F^2(q)$$

$$\frac{d\sigma}{dE_R}(v,0) = \frac{\sigma^0}{E_R^{max}} = \frac{m_N}{2\mu_{\chi N}^2 v^2}\sigma^0$$

(consider only SI) $\sigma^0 = \sigma_{SI}^0 = \frac{A^2\mu_{\chi N}^2}{\mu_{\chi n}^2}\sigma_{SI}$

DM-nucleon SI cross-section

$$\frac{dR}{dE_R} = \frac{M_{det}\rho_\chi}{2m_\chi \mu_{\chi n}^2} A^2 \sigma_{SI} F_{SI}^2(q) \int_{v_{min}}^{v_{max}} \frac{f(\boldsymbol{v},t)}{v} d^3\boldsymbol{v}$$

WIMP mass

$\eta(v_{min},t) = \int_{v_{min}}^{v_{esc}} \frac{f(v,t)}{v} d^3v$ **is the mean inverse speed function** (note that it depends on time due to the Earth rotation around the Sun) and $v_{min} = \sqrt{\frac{E_R m_N}{2\mu_{\chi N}^2}}$, $\mu_{\chi n} = \frac{m_\chi m_n}{m_\chi + m_n}$

Velocity in the lab reference system: time dependency

$$v_{gal} = v + v_{\odot} + v_{\oplus}(t)$$

Where

v_{gal}: **WIMP velocity in the GALACTIC reference frame**

v: **WIMP velocity in the laboratory reference frame**

v_E: **Earth velocity with respect to the Galaxy frame,** $v_E = v_{\odot} + v_{\oplus}(t)$

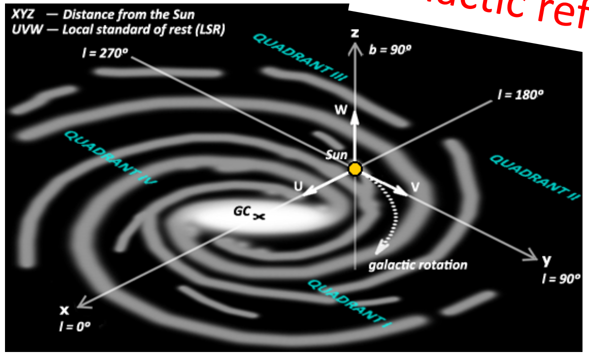
$v_{\odot} = v_0 + v_{pec}$: velocity of the Sun with respect to the galaxy

v_0: **local standard of rest velocity (**v_0**)**

v_{pec}: solar peculiar velocity

$v_{\oplus}(t)$: velocity of the Earth with respect to the Sun

Galactic reference system



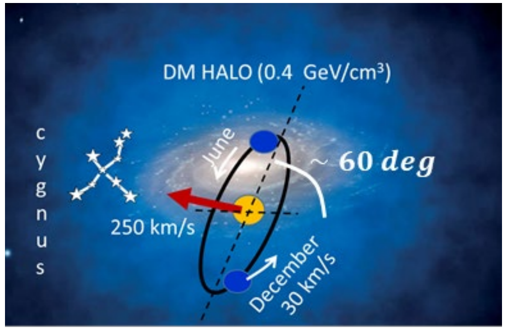


Table 1 Suggested Standard Halo Model parameters. Vectors are given as (v_r, v_ϕ, v_θ) with r pointing radially inward and ϕ in the direction of the Milky Way's rotation. Analyses insensitive to annular modulation can approximate $\mathbf{v}_\oplus(t)$ with Eq. 12

Parameter	Description	Value	References
ρ_χ	Local dark matter density	$0.3\text{GeV}/c^2/\text{cm}^3$	[9]
v_{esc}	Galactic escape speed	544 km/s	[45]
$\langle\lvert\mathbf{v}_\oplus\rvert\rangle$	Average galactocentric Earth speed	29.8 km/s	[41]
$\mathbf{v}_\circledast$	Solar peculiar velocity	(11.1, 12.2, 7.3) km/s	[46]
$\mathbf{v}_0$	Local standard of rest velocity	(0, 238, 0) km/s	[47,48]

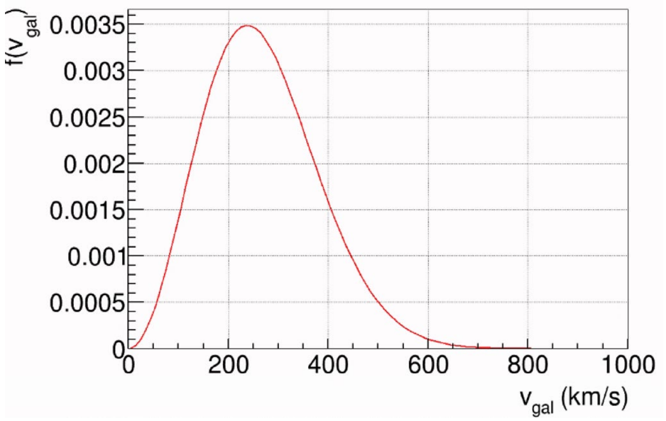
$$\mathbf{v}_\oplus(t) = \langle\lvert\mathbf{v}_\oplus\rvert\rangle \times \begin{pmatrix} 0.9941\cos(\omega\Delta t) - 0.0504\sin(\omega\Delta t) \\ 0.1088\cos(\omega\Delta t) + 0.4946\sin(\omega\Delta t) \\ 0.0042\cos(\omega\Delta t) - 0.8677\sin(\omega\Delta t) \end{pmatrix}$$

where $\omega = 0.0172\ d^{-1}$ and t is the number of days since March 22, 2018

D. Baxter et al. "Recommended conventions for reporting results from direct dark matter searches" Eur. Phys. J. C (2021) 81:907 [2105.00599]

Standard halo model (SHM)

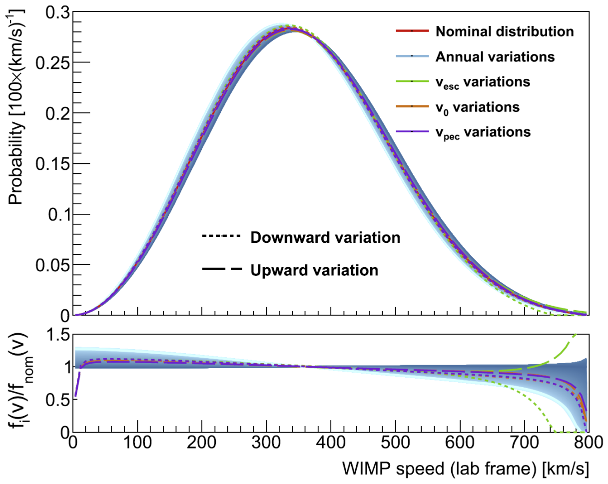
The standard halo model is an isotropic, isothermal sphere **in the galatic frame**, with density profile $\rho(r) \propto r^{-2}$. In this case the solution to the collisionless Botzmann equation is a Maxwellian velocity distribution (**$\boldsymbol{v_{gal}}$:DM $\boldsymbol{v}$ in the galactic reference system**)



$$f_{gal}(\boldsymbol{v_{gal}})d^3\boldsymbol{v_{gal}} = \frac{1}{v_0^3\pi^{3/2}}e^{-v_{gal}^2/v_0^2}d^3\boldsymbol{v_{gal}}$$

Move to lab. frame

$$f(\boldsymbol{v},t) = f_{gal}(\boldsymbol{v} + \boldsymbol{v}_\odot + \boldsymbol{v}_\oplus(t))$$



The mean inverse speed function

$$\eta(v_{min}, t) = \int_{v_{min}}^{\infty} \frac{f(v,t)}{v} d^3v$$

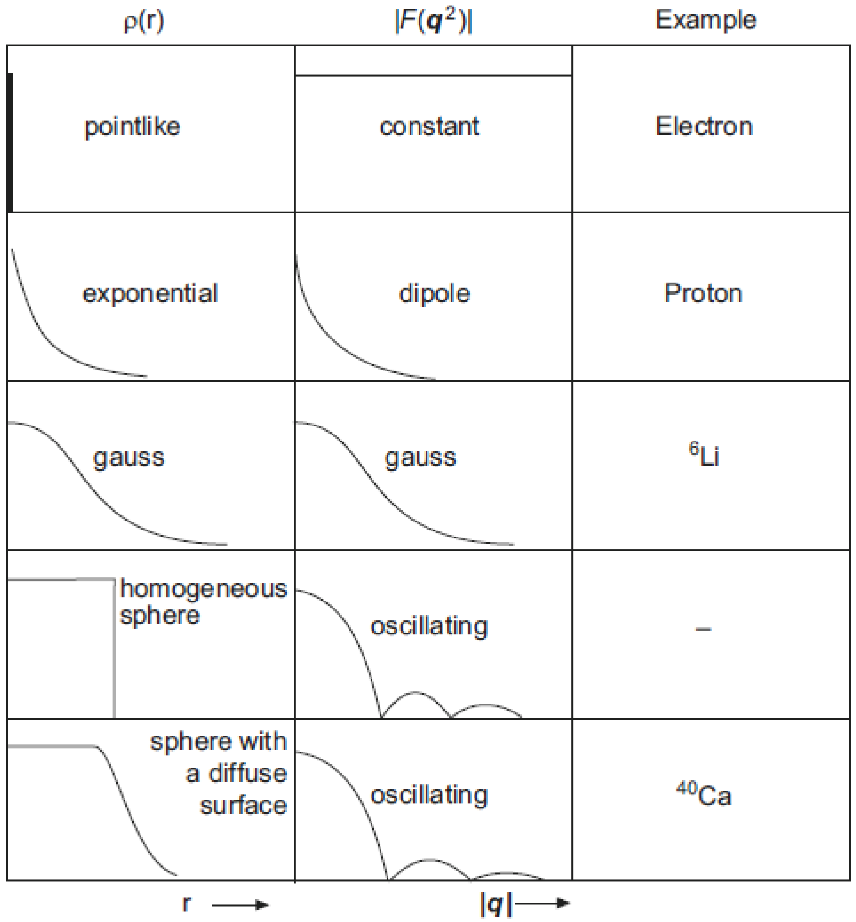
This is the integral to be solved for the DM halo model considered. For the SHM, this case the integral can be solved analytically. Defining:

$$x = \frac{v_{min}}{v_o}, y = \frac{v_{\odot}+v_{\oplus}(t)}{v_o}, z = \frac{v_{esc}}{v_o}$$

$$\eta(v_{min}, t) = \frac{1}{2yv_0}\frac{1}{N}\begin{cases} \mathrm{erf}(x+y) - \mathrm{erf}(x-y) - \frac{4}{\sqrt{\pi}} y e^{-z^2} & 0 \le x \le z-y \\ \mathrm{erf}(z) - \mathrm{erf}(x-y) - \frac{2}{\sqrt{\pi}}(z+y-x)e^{-z^2} & z-y < x \le z+y \\ 0 & x > z+y \end{cases}$$

Here, $\mathrm{erf}()$ is the error function and $N = \mathrm{erf}(z) - \frac{2z}{\sqrt{\pi}} e^{-z^2}$ is a normalization factor

Spin independent form factor



An analytic expression for the FF (and the most commonly used in DD calculations) is that from Helm:

$$F_{\mathrm{SI}}^2(q) = \left(\frac{3j_1(qR_1)}{qR_1}\right)^2 e^{-q^2s^2}$$

where $j_1(x) = \frac{sinx}{x^2} - \frac{cosx}{x}$

is a spherical Bessel function of the first kind, $\boldsymbol{R_1}$ **is an effective nuclear radius** and $\boldsymbol{s}$ **is the nuclear skin thickness**, These parameters that need to be fit separately for each nucleus, but good results are obtained with

$$R_1 = \sqrt{R^2 - 5s^2}, \qquad R \approx 1.2A^{1/3}\ \mathrm{fm}, \qquad s = 1\mathrm{fm}$$

Other parametrizations can give more precise results for high q, see for example:
G. Duda, A. Kemper and P. Gondolo, “Model Independent Form Factors for Spin Independent Neutralino-Nucleon Scattering from Elastic Electron Scattering Data”, JCAP 0704:012,2007 [arXiv:hep-ph/0608035]

Functions in the python code

`vearth(t)`: Earth velocity

`eta(E,t,A,mW)`: mean inverse speed function

`FF(E,A)`: SI form factor

`rate(E,t,A,mW,sigmaSI)`: differential rate $\frac{dR}{dE} = \frac{M_{det}\,\rho_\chi}{2\,m_\chi\,\mu_{\chi n}^2}\,A^2\,\sigma_{SI}\,F_{SI}^2\,\eta$

`totalRate(Ei,Ef,t,A,mW,sigmaSI)`: $R = \int_{E_{min}}^{E_{max}} \frac{dR}{dE}\,dE$

```
t in days from March 22
E in keV
mW in GeV/c2
sigmaSI in cm2
A is the mass number of the target
```

+ examples to plot Earth velocity, form factor, meas inverse speed function and differential rate

Exercises

- Plot the differential rate vs recoil energy for several targets: Ar(40), Ge(72), Xe(132)(*)
- Plot the differential rate vs recoil energy for several Wimp masses: 10, 100, 1000 GeV/c^2
- Plot the differential rate vs recoil energy for 2nd June (timeMax) and 1st December (timeMin)
- For Xenon target and a WIMP of $m_\chi = 70\ GeV$, $\sigma_{SI} = 10^{-41} cm^2$ and two different energy intervals ([0-60] keV and [10-60] keV):

  1) Implement a function to plot the TOTAL rate in (E_i, E_f): `ratevsTime(Ei,Ef,A,mW,sigmaSI)`
  2) Plot rate total vs time for 1 year
  3) Compute the maximum rate (R_{max}), the minimum rate (R_{min}), the average rate (R_0) and the day corresponding to the maximum (t_{max})
  4) In the same figure as before, plot the following approximation for the rate:

$$R_{approx} = R_0 + R_{mod} \cos(\omega(t - t_{max}))$$ where R_0 is the average rate and

$$R_{mod} = \frac{1}{2}(R_{max} - R_{min})$$ and $\omega = 2\pi/365\ d^{-1}$

4) Calculate the error (in %) of the approximation in t_{max} as:

$$err(t_{max}) = \frac{R(t_{max}) - R_{approx}(t_{max})}{R(t_{max})} \times 100$$

(*) in the python code you will find an example, just modify it