Normalization of Partial-Wave CP Asymmetries

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11/26/2025

Normalization problems in angular-distribution CPAs

- Lee-Yang parameters and CPV,

$$A_{CP,\alpha}(\Lambda \to p\pi^-) \equiv \frac{\alpha + \overline{\alpha}}{\alpha - \overline{\alpha}}.$$

- Paritial Wave CPAs

$$A_{CP,l}^{\text{conv}} = \frac{w_l - \bar{w}_l}{w_l + \bar{w}_l}, \qquad |\mathcal{M}|^2 = \sum_l w_l P_l(c_\theta),$$

- neither α nor w_l is positive-definite, hence

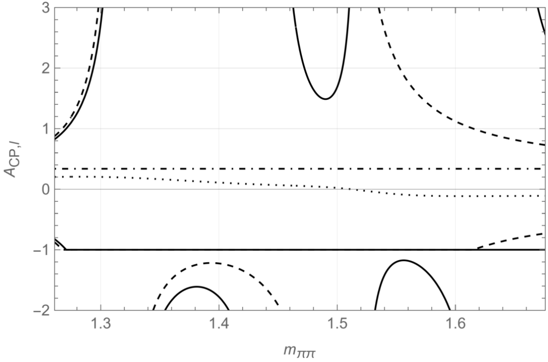
$$-\infty < A_{CP,\alpha}, A_{CP,l}^{\text{conv}} < +\infty$$

The problem to α-CPA in hyperon decays is not serious, since the CPV is small, hence $\alpha \approx -\overline{\alpha}$ and $|\alpha - \overline{\alpha}| << |\alpha|$. But the problem can be much serious for PWCPAs in bottom hadron decays.

How serious?

This is an example: PWCPAs in $B^\pm \to \pi^+\pi^-\pi^\pm$

Figure: $A^{\text{conv}}_{CP,l} = \frac{w_l - \bar{w}_l}{w_l + \bar{w}_l}$, for $l = 1$ (solid), 2 (dotted), 3 (dashed), and 4 (dash-dotted). Also present alternatively-defined PWCPAs: $\tilde{A}^{\text{conv}}_{CP,l} \equiv \frac{w_l - \bar{w}_l}{|w_l| + |\bar{w}_l|}$.



Normalization problems in angular-distribution CPAs

solution to α-CPA

Since α is normalized, we can redefine α-CPA as:

$$A_{CP,\alpha} = \frac{1}{2}(\alpha + \overline{\alpha})$$

This does not work for PWCPAs:

$$A'_l = \frac{w_l}{w_0}, \quad A'_{CP,l} = \frac{1}{2}(A'_l + \overline{A}'_l), \;\text{ or }\; \mathring{A}_{CP,l} = \frac{w_l - \overline{w}_l}{w_0 + \overline{w}_0},$$

extra factor is needed

$$A_l = \eta_l A'_l, \qquad A_{CP,l} = \eta_l \mathring{A}_{CP,l} = \eta_l \frac{w_l - \overline{w}_l}{w_0 + \overline{w}_0},$$

How to chose the normalization factors η_l?

2 The partial wave CP asymmetries and their normalization

normalized PWCPAs

$$A_{CP,l} = \eta_l \mathring{A}_{CP,l} = \eta_l \frac{w_l - \overline{w}_l}{w_0 + \overline{w}_0} = \frac{\eta_l (2l+1) \int \left[|\mathcal{M}|^2 - |\overline{\mathcal{M}}|^2 \right] P_l(c_\theta) \tilde{d} c_\theta}{\int \left[|\mathcal{M}|^2 + |\overline{\mathcal{M}}|^2 \right] \tilde{d} c_\theta},$$

experiment-friendly CPA observables

$$\hat{A}_{CP,l} \equiv \frac{\int \left[|\mathcal{M}|^2 - |\overline{\mathcal{M}}|^2 \right] \mathrm{sgn}(P_l(c_\theta)) \tilde{d} c_\theta}{\int \left[|\mathcal{M}|^2 + |\overline{\mathcal{M}}|^2 \right] \tilde{d} c_\theta} = \frac{\sum_i (N_i - \overline{N}_i) \mathrm{sgn}_{l,i}}{N + \overline{N}},$$

It can be easily seen that

- $\hat{A}_{CP,l}$'s satisfy the normalization condition $-1 \leq \hat{A}_{CP,l} \leq +1$,
- for $l = 0$, $\hat{A}_{CP,l}$ is just the direct CPA,
- $\sigma_{\hat{A}_{CP,l}} \approx \hat{\sigma}$, where the $\hat{\sigma}$ is the statistical error which takes the familiar form $1/\sqrt{N + \overline{N}}$ in the absence of the background.

expressing $\{A_{CP,l}\}$ in terms of $\{\hat{A}_{CP,l}\}$

$$
\begin{aligned}
\hat{A}_{CP,l} &\equiv \frac{\int \left[|\mathcal{M}|^2 - |\overline{\mathcal{M}}|^2\right] \mathrm{sgn}(P_l(c_\theta))\tilde{d}c_\theta}{\int \left[|\mathcal{M}|^2 + |\overline{\mathcal{M}}|^2\right] \tilde{d}c_\theta} \\
&= \frac{\int \left[\sum_k (w_k - \overline{w}_k) P_k(c_\theta)\right] \mathrm{sgn}(P_l(c_\theta))\tilde{d}c_\theta}{\int \left[\sum_k (w_k + \overline{w}_k) P_k(c_\theta)\right] \tilde{d}c_\theta} \\
&= \frac{\int \left[\sum_k (w_k - \overline{w}_k) P_k(c_\theta)\right] \mathrm{sgn}(P_l(c_\theta))\tilde{d}c_\theta}{w_0 + \overline{w}_0} \\
&= \sum_k \frac{w_k - \overline{w}_k}{w_0 + \overline{w}_0} \int P_k(c_\theta)\mathrm{sgn}(P_l(c_\theta))\tilde{d}c_\theta \\
&= \sum_k A_{CP,k} \frac{1}{\eta_k} \int P_k(c_\theta)\mathrm{sgn}(P_l(c_\theta))\tilde{d}c_\theta = \sum_k \Omega_{lk} A_{CP,k}
\end{aligned}
$$

$$
\Omega_{lk} = \frac{\omega_{lk}}{\eta_k}, \quad \omega_{lk} = \frac{1}{2}\int_{-1}^{+1} \mathrm{sgn}\left(P_l(c_\theta)\right) P_k(c_\theta) dc_\theta.
$$

Let the diagonal elements of Ω equal 1

$$\hat{A}_{CP,l} = \sum_k \Omega_{lk} A_{CP,k} \quad \rightarrow \quad \hat{A}_{CP,l} = A_{CP,l} + \sum_{k \neq l} \Omega_{lk} A_{CP,k}.$$

The advantages for this choice is clear, the elements of Ω_{lk} are quite smaller than 1 for $k \neq l$ $(\Omega_{lk} < \Omega_{ll})$, hence one can see transparently that $\hat{A}_{CP,l}$ will get the most important contribution from $A_{CP,l}$. Since $\hat{A}_{CP,l}$ are clearly normalized, we conclude that $A_{CP,l}$ are now *roughly* normalized.

This implies that the factors η_l are chosen as

$$\tilde{\eta}_l = \omega_{ll} = \frac{1}{2} \int_{-1}^{+1} \text{sgn}\left(P_l(c_\theta)\right) P_l(c_\theta) dc_\theta = \frac{1}{2} \int_{-1}^{+1} \left|P_l(c_\theta)\right| dc_\theta.$$

More "precise" way to obtain η_l

$$\hat{A}_{CP,l} = \sum_k \Omega_{lk} A_{CP,k},$$

Truncate the Legendre expansion at some finite number L, this allows us to express $A_{CP,l}$ inversely in terms of $\hat{A}_{CP,k}$:

$$A_{CP,l} = \sum_{k=0}^{L} (\Omega_L^{-1})_{lk} \hat{A}_{CP,k} = \eta_l \sum_{k=0}^{L} (\omega_L^{-1})_{lk} \hat{A}_{CP,k},$$

where Ω_L^{-1} and ω_L^{-1} are the inverse matrices of Ω_L and ω_L, respectively, and the relation $(\Omega_L^{-1})_{lk} = \eta_l (\omega_L^{-1})_{lk}$ has been used.

The statistical error of $A_{CP,l}$ is related to that of $\hat{A}_{CP,l}$ according to

$$\sigma_l^2 = \left[\Omega_L^{-1} \hat{\rho} \left(\Omega_L^{-1}\right)^T\right]_{ll} \hat{\sigma}^2 = \eta_l^2 \left[\omega_L^{-1} \hat{\rho} \left(\omega_L^{-1}\right)^T\right]_{ll} \hat{\sigma}^2,$$

where $\hat{\rho}_{kk'}$ are the correlation coefficients between $\hat{A}_{CP,k}$ and $\hat{A}_{CP,k'}$, which, as will be shown below, can be well estimated as

$$\hat{\rho}_{kk'} \approx \frac{1}{2} \int_{-1}^{+1} \text{sgn}(P_k(c_\theta)) \text{sgn}(P_{k'}(c_\theta)) dc_\theta.$$

quasi-normalized PWCPAs: $A_{CP,l}$

Instead of the requirement of normalization, it would be practically more useful to choose the η_l's in such a way that *the statistical error of $A_{CP,l}$ also equals to $\hat{\sigma}$, i.e., $\sigma_l = \hat{\sigma}$.* This means that η_l should be chosen as

$$\eta_l^{(L)} = \frac{1}{\sqrt{\left[\omega_L^{-1}\hat{\rho}(\omega_L^{-1})^T\right]_{ll}}} = \frac{1}{\sqrt{\sum_{k,k'=0}^{L}(\omega_L^{-1})_{lk}\omega_L^{-1})_{lk'}\hat{\rho}_{kk'}}}, \quad l = 0, 1, \cdots, L.$$

The expression of the correlation coefficients $\hat{\rho}_{kk'}$

The correlation matrix is defined according to

$$\hat{\sigma}^2 \hat{\rho}_{lk} = \sum_i \left(\frac{\partial \hat{A}_{CP,k}}{\partial N_i} \frac{\partial \hat{A}_{CP,l}}{\partial N_i} \sigma^2_{N_i} + \frac{\partial \hat{A}_{CP,l}}{\partial \overline{N}_i} \frac{\partial \hat{A}_{CP,k}}{\partial \overline{N}_i} \sigma^2_{\overline{N}_i} \right).$$

Hence $\hat{\rho}_{lk}$ takes the form

$$\hat{\rho}_{lk} = \frac{1}{(N+\overline{N})^2} \sum_i \left\{ \frac{\sigma^2_{N_i} + \sigma^2_{\overline{N}_i}}{\hat{\sigma}^2} \left[\mathrm{sgn}_{l,i}\mathrm{sgn}_{k,i} + \frac{\sum_j (N_j - \overline{N}_j)\mathrm{sgn}_{l,j} \sum_{j'} (N_{j'} - \overline{N}_{j'})\mathrm{sgn}_{k,j'}}{(N+\overline{N})^2} \right] - \frac{\sigma^2_{N_i} - \sigma^2_{\overline{N}_i}}{\hat{\sigma}^2} \left[\frac{\mathrm{sgn}_{l,i} \sum_j (N_j - \overline{N}_j)\mathrm{sgn}_{k,j} + \mathrm{sgn}_{k,i} \sum_j (N_j - \overline{N}_j)\mathrm{sgn}_{l,j}}{N+\overline{N}} \right] \right\}.$$

In the limit of CP symmetry, we obtain

$$\begin{aligned} \hat{\rho}_{lk} &\approx \frac{1}{(N+\overline{N})^2} \sum_i \left(\frac{\sigma^2_{N_i} + \sigma^2_{\overline{N}_i}}{\hat{\sigma}^2} \right) \mathrm{sgn}_{l,i}\mathrm{sgn}_{k,i} = \frac{\sum_i \left(N_i + \overline{N}_i \right) \mathrm{sgn}_{l,i}\mathrm{sgn}_{k,i}}{N+\overline{N}} \\ &\approx \frac{1}{2} \int_{-1}^{+1} \mathrm{sgn}(P_k(c_\theta))\mathrm{sgn}(P_{k'}(c_\theta)) dc_\theta. \end{aligned}$$

The expression of the correlation coefficients $\hat{\rho}_{kk'}$

There is no need for CP symmetry assumption at all

Obtain ρ_{lk} by an alternative way

Normalized Legendre weights:

$$A_l = \eta_l \frac{w_l}{w_0}.$$

Experiment-friendly observables

$$\hat{A}_l \equiv \frac{\int \left[|\mathcal{M}|^2\right] \mathrm{sgn}(P_l(c_\theta))\tilde{d}c_\theta}{\int \left[|\mathcal{M}|^2\right] \tilde{d}c_\theta} = \frac{\sum_i N_i \mathrm{sgn}_{l,i}}{N}.$$

Perform almost exactly the same procedure one obtain exactly the same expressions for $\tilde{\eta}_l$ and $\eta_l^{(L)}$.
The PWCPAs can be alternatively defined as

$$\tilde{A}_{CP,l} = (A_l - \overline{A}_l)/2.$$

Example: $L = 4$

One first calculates the numerical values of the matrix ω and $\hat{\rho}$, which read

$$\omega = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & -0.1250 & 0 \\ -0.1547 & 0 & 0.3849 & 0 & -0.0642 \\ 0 & -0.1 & 0 & 0.3250 & 0 \\ -0.0400 & 0 & -0.0768 & 0 & 0.2866 \end{pmatrix},$$

and

$$\hat{\rho} = \begin{pmatrix} 1 & 0 & -0.1547 & 0 & -0.0423 \\ 0 & 1 & 0 & -0.5492 & 0 \\ -0.1547 & 0 & 1 & 0 & -0.2475 \\ 0 & -0.5492 & 0 & 1 & 0 \\ -0.0423 & 0 & -0.2475 & 0 & 1 \end{pmatrix},$$

Example: $L = 4$

Then substituting these results into Eq. (1), one obtains

$$\eta_0^{(4)} = 1, \quad \eta_1^{(4)} = 0.5418, \quad \eta_2^{(4)} = 0.3847, \quad \eta_3^{(4)} = 0.3312, \quad \eta_4^{(4)} = 0.2829.$$

For comparison, the factors η_l for the uncorrelated limit read

$$\eta_0^{(4)}\Big|_{\hat{\rho}=\mathbb{1}} = 1, \; \eta_1^{(4)}\Big|_{\hat{\rho}=\mathbb{1}} = 0.4308, \; \eta_2^{(4)}\Big|_{\hat{\rho}=\mathbb{1}} = 0.3540, \; \eta_3^{(4)}\Big|_{\hat{\rho}=\mathbb{1}} = 0.2942, \; \eta_4^{(4)}\Big|_{\hat{\rho}=\mathbb{1}} = 0.2674,$$

and the first few $\tilde{\eta}_l$'s based on the rough estimation in Eq. (1) take the following values:

$$\tilde{\eta}_0 = 1, \quad \tilde{\eta}_1 = 0.5, \quad \tilde{\eta}_2 = 0.3849, \quad \tilde{\eta}_3 = 0.3250, \quad \tilde{\eta}_4 = 0.2866.$$

The comparison shows that

- 1) all the above factors η_l are universal.
- 2)the contributions to the factors η_l from the correlation between the statistical errors of different CPA observables $A_{CP,l}$ are small,
- 3) the rough estimation of the η_l based on Eq. (1), i.e., $\tilde{\eta}_l$, works pretty well.
- 4) the truncation-dependence of $\eta_l^{(L)}$ and $\eta_l^{(L)}\Big|_{\hat{\rho}=\mathbb{1}}$ is small.

Table: The numerical values of $\eta_l^{(L)}$, $\eta_l^{(L)}\Big|_{\hat{\rho}=\mathbb{1}}$ (for truncation $L \leq 8$), and $\tilde{\eta}_l$ (for $l \leq 8$).

	$L \backslash l$	0	1	2	3	4	5	6	7	8
$\eta_l^{(L)}$	0	1								
	1	1	0.5							
	2	1	0.5	0.3896						
	3	1	0.5418	0.3896	0.3312					
	4	1	0.5418	0.3847	0.3312	0.2829				
	5	1	0.5552	0.3847	0.3332	0.2829	0.2615			
	6	1	0.5552	0.3899	0.3332	0.2807	0.2615	0.2345		
	7	1	0.5656	0.3899	0.3312	0.2807	0.2630	0.2345	0.2205	
	8	1	0.5656	0.3967	0.3312	0.2793	0.2630	0.2322	0.2205	0.2064
$\eta_l^{(L)}\Big\|_{\hat{\rho}=\mathbb{1}}$	0	1								
	1	1	0.5							
	2	1	0.5	0.3804						
	3	1	0.4308	0.3804	0.2942					
	4	1	0.4308	0.3540	0.2942	0.2674				
	5	1	0.4383	0.3540	0.2787	0.2674	0.2441			
	6	1	0.4383	0.3200	0.2787	0.2423	0.2441	0.2131		
	7	1	0.4256	0.3200	0.2668	0.2423	0.2232	0.2131	0.1998	
	8	1	0.4256	0.3252	0.2668	0.2378	0.2232	0.1986	0.1998	0.1915
$\tilde{\eta}_l$		1	0.5	0.3849	0.3250	0.2866	0.2592	0.2385	0.2220	0.2086

3 Application to $B^{\pm} \to \pi^{\pm}\pi^{+}\pi^{-}$ decays

Analysis of the PWCPAs for the decays $B^\pm \to \pi^\pm\pi^+\pi^-$

Focus on $m_{\pi\pi} \sim (1.25, 1.7)$ GeV: $\rho^0(1450)$, $f_0(1370)$, $f_0(1500)$, $f_2(1270)$, $f_2(1430)$, $f_2'(1525)$, etc..

Legendre expansion of the decay amplitude squared (truncated up to $l = 4$):

$$|\mathcal{M}^\pm|^2 = \sum_{l=0}^{4} w_l^\pm P_l(c_\theta),$$

$$w_0^\pm = \left(|\mathcal{M}_{f_0}^\pm|^2 + \frac{1}{5}|\mathcal{M}_{f_2}^\pm|^2 + \frac{1}{3}|\mathcal{M}_{\rho^0}^\pm|^2\right), \quad w_1^\pm = \left(2\mathrm{Re}[\mathcal{M}_{\rho^0}^\pm \mathcal{M}_{f_0}^{\pm *}] + \frac{4}{5}\mathrm{Re}[\mathcal{M}_{\rho^0}^\pm \mathcal{M}_{f_2}^{\pm *}]\right),$$

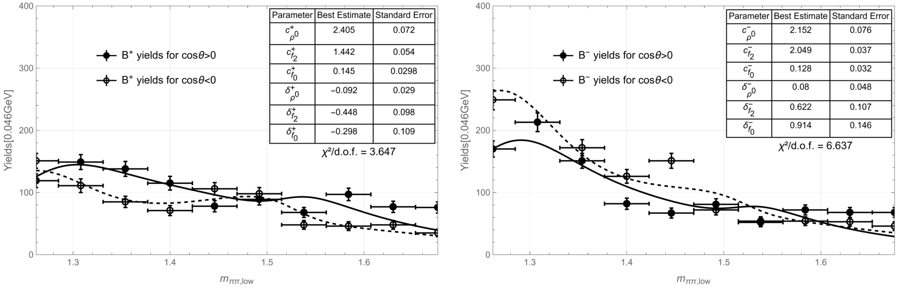
$$w_2^\pm = \left(\frac{2}{3}|\mathcal{M}_{\rho^0}^\pm|^2 + \frac{2}{7}|\mathcal{M}_{f_2}^\pm|^2 + 2\mathrm{Re}[\mathcal{M}_{f_0}^\pm \mathcal{M}_{f_2}^{\pm *}]\right), \quad w_3^\pm = \frac{6}{5}\mathrm{Re}[\mathcal{M}_{\rho^0}^\pm \mathcal{M}_{f_2}^{\pm *}],$$

$$w_4^\pm = \frac{18}{35}|\mathcal{M}_{f_2}^\pm|^2,$$

The decay amplitudes are parameterized as

$$\mathcal{M}_R^\pm = c_R^\pm F_R^{BW} e^{i\delta_R^\pm} P_l(c_\theta).$$

Data fitting



(a) (b)

Figure: The fitting results of the parameters (shown in the upper right corner of each figure) from the event yields of $B^+ \to \pi^+\pi^-\pi^+$ (a) and $B^- \to \pi^+\pi^-\pi^-$ (b) for $\cos\theta > 0$ and $\cos\theta < 0$.

Predictions of PWCPAs based on the extracted parameters

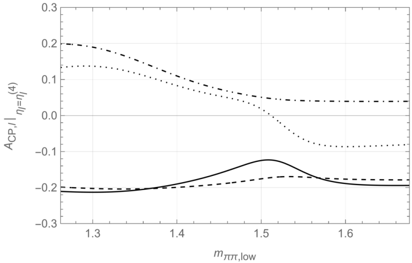


Figure: The quasi-normalized PWCPAs $A_{CP,l}|_{\eta_l=\eta_l^{(4)}}$ of $B^{\pm} \to \pi^{+}\pi^{-}\pi^{\pm}$ for $l = 1$ (solid), 2 (dotted), 3 (dashed), and 4 (dash-dotted).

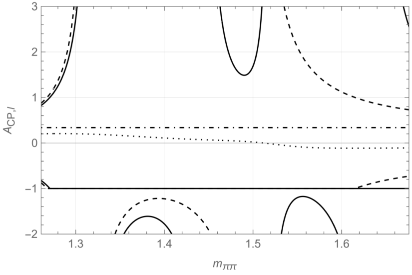
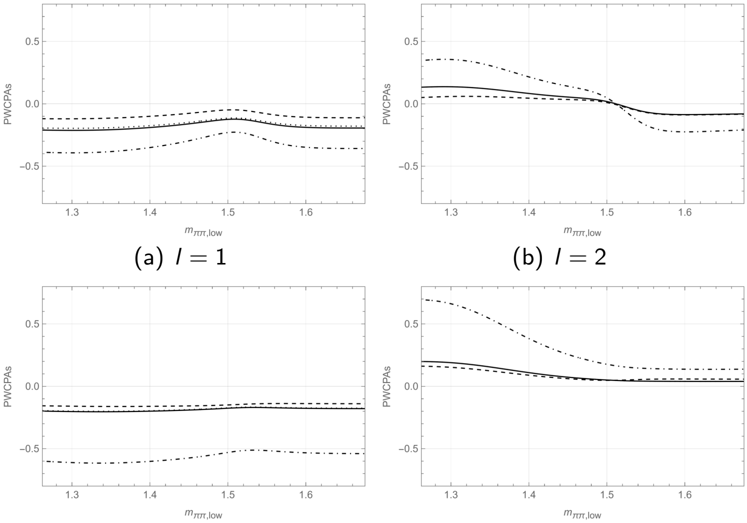


Figure: The conventionally defined PWCPAs, $A_{CP,l}^{\mathrm{conv}}$, for $l = 1$ (solid), 2 (dotted), 3 (dashed), and 4 (dash-dotted). Also present alternatively-defined PWCPAs: $\tilde{A}_{CP,l}^{\mathrm{conv}} \equiv \frac{w_l - \bar{w}_l}{|w_l| + |\bar{w}_l|}$.



(a) $l = 1$ (b) $l = 2$ (c) $l = 3$ (d) $l = 4$

Figure: Comparisons of various definitions of PWCPAs, $A_{CP,l}\big|_{\eta_l=\eta_l^{(4)}}$ (solid), $A_{CP,l}\big|_{\eta_l=\tilde{\eta}_l}$ (dotted), $\hat{A}_{CP,l}$ (dashed), and $\mathring{A}_{CP,l}$ (dot-dashed).

Evidences indicating we are on the right path for the choices of η_l

- 1) $A_{CP,l}\big|_{\eta_l=\eta_l^{(4)}}$ and $A_{CP,l}\big|_{\eta_l=\tilde{\eta}_l}$ are very close to $\hat{A}_{CP,l}$;
- 2) $A_{CP,l}\big|_{\eta_l=\eta_l^{(4)}}$ and $A_{CP,l}\big|_{\eta_l=\tilde{\eta}_l}$ are very close to each other;
- 3) $\mathring{A}_{CP,l}$ are quite different form $A_{CP,l}\big|_{\eta_l=\eta_l^{(4)}}$, $A_{CP,l}\big|_{\eta_l=\tilde{\eta}_l}$, and $\hat{A}_{CP,l}$.

4 Summary

summary

- The problem of the normalization of the PWCPAs in multi-body decay processes of heavy hadrons.
- Introduce the quasi-normalized PWCPAs.
- Analysis of the quasi-normalized PWCPAs in the decay processes $B^{\pm} \to \pi^{+}\pi^{-}\pi^{\pm}$.
- Generalized PWCPAs in multibody decays (such as 4-body decays of heavy hadrons).

Thanks for the invitation!

Backups