Scaling solutions for gauge invariant flow equations in dilaton quantum gravity

Yadikaer Maitiniyazi ,[1] Christof Wetterich ,[2] and Masatoshi Yamada,[3]

[1]Center for Theoretical Physics and College of Physics, Jilin University, Changchun 130012, China
[2]Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 16, 69120 Heidelberg, Germany
[3]Department of Physics and Astronomy, Kwansei Gakuin University, Sanda, Hyogo, 669−1330, Japan

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VI. Summary

One scalar, two epochs

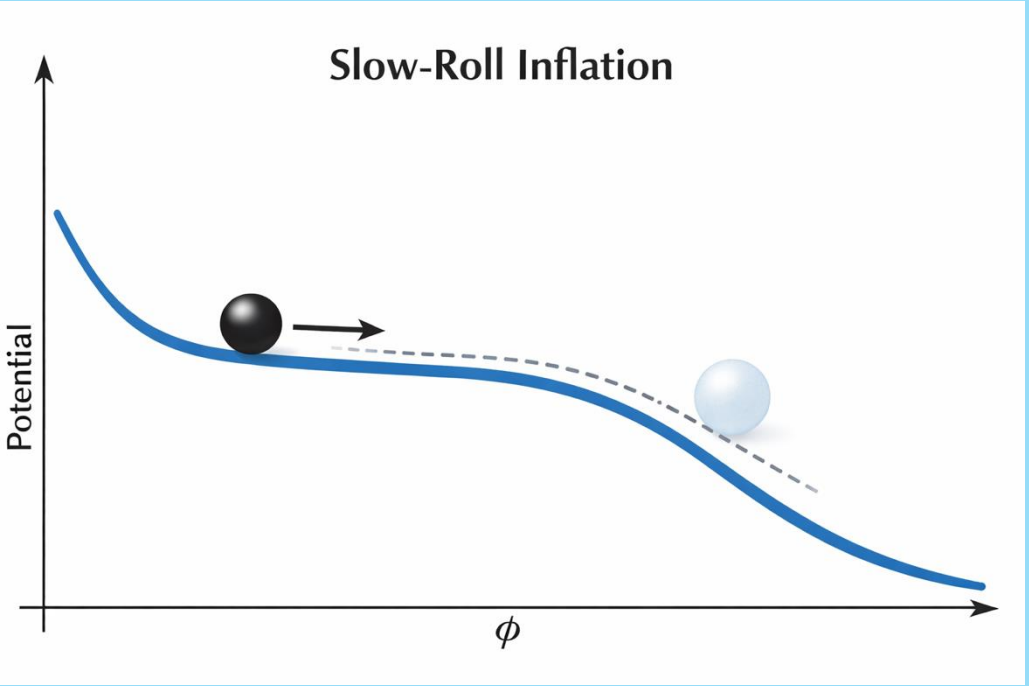
Inflation

$$d_F = e^{H_{inf} T} d_I = e^N d_I$$

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$

$$\frac{1}{2}\dot{\phi}^2 \ll V(\phi), \qquad |\ddot{\phi}| \ll |3H\dot{\phi}|$$

Many possibilities



A. H. Guth, *The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems*, **Phys. Rev. D** 23 (1981) 347.

A. D. Linde, *Chaotic Inflation*, **Phys. Lett. B** 129 (1983) 177.

…

Accelerating Expansion

What drives the late-time accelerated expansion of the Universe?

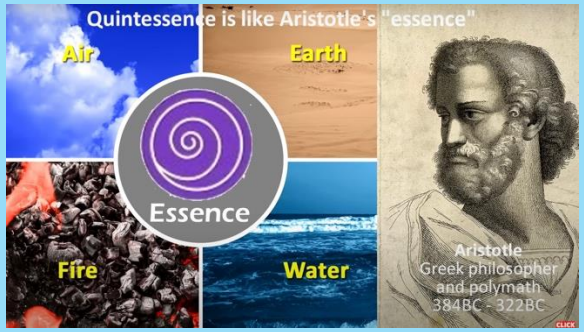
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

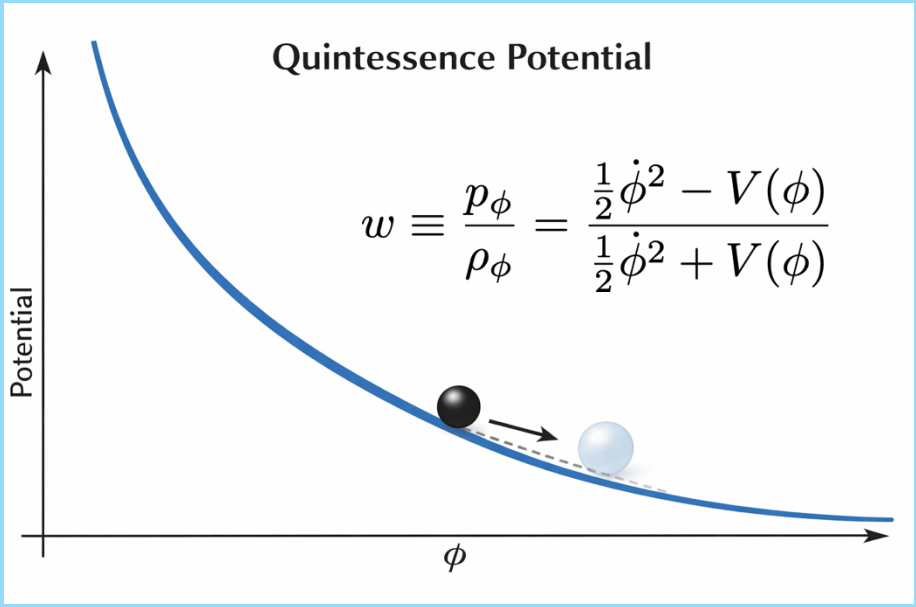
Acceleration requires negative pressure: $\omega < -\frac{1}{3}$

The simplest explanation is a cosmological constant

Quintessence replaces CC by a slowly evolving scalar field

Quintessence as dynamical dark energy





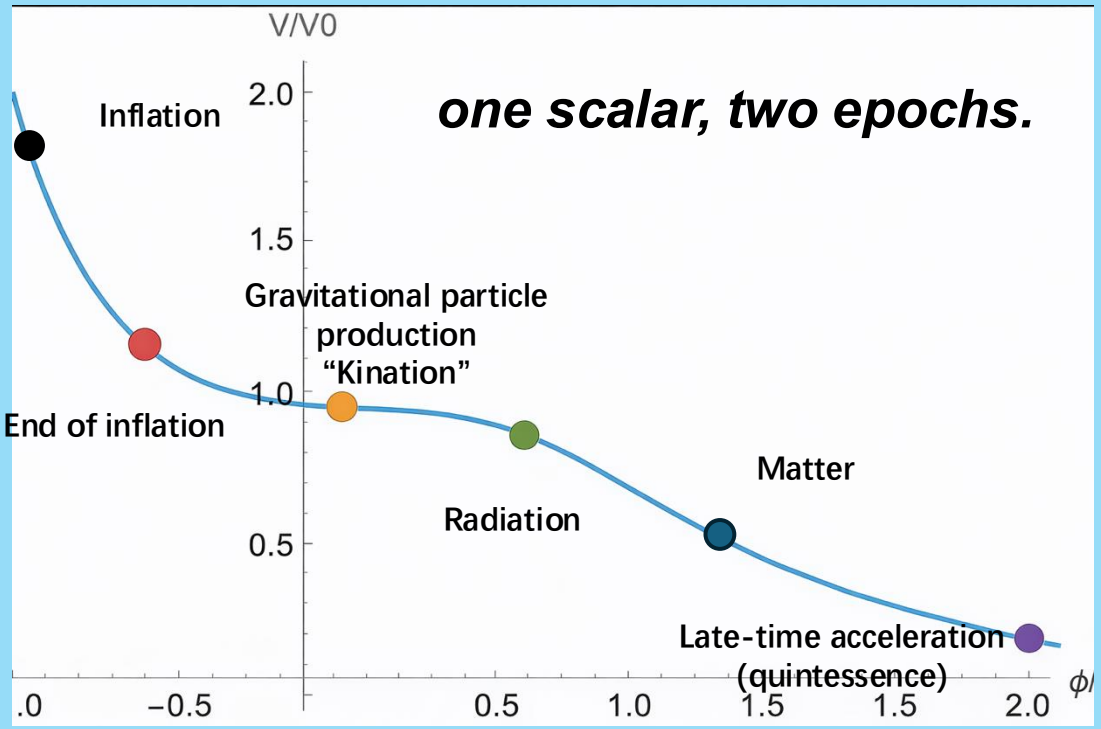
C. Wetterich, Cosmology and the Fate of Dilatation Symmetry, Nucl. Phys. B 302 (1988) 668 [1711.03844]

B. Ratra and P. J. E. Peebles, Cosmological Consequences of a Rolling Homogeneous Scalar Field, Phys. Rev. D 37 (1988) 3406.

…

Quintessential inflation (unification)

$$V(\phi) = \begin{cases} \frac{\lambda}{4!} M^4 \left(\frac{|\phi|^4}{M^4} + 1 \right), & \phi \leq 0, \\ \frac{\lambda}{4!} M^4 \left(\frac{1}{\frac{|\phi|^4}{M^4} + 1} \right), & \phi > 0. \end{cases}$$



P. J. E. Peebles and A. Vilenkin, "Quintessential inflation," Physical Review D 59 (1999) 063505.

Scale symmetry

Inflation and dark energy may be the consequence of the emergence scale symmetry at non-trivial fixed points

Variable gravity

$$\Gamma = \int_x \sqrt{g}\left[-\frac{1}{2}\chi^2 R + \mu^2\chi^2 + \frac{1}{2}K(\chi)\partial^\mu\chi\partial_\mu\chi\right] \qquad K(\chi) = B(\chi/\mu) - 6$$

- χ^2 sets the effective Planck mass: $M_{Pl,eff}^2 \sim \chi^2$
- Dimensionless running kinetial B controlling the crossover
- Near fixed points, approximate scale symmetry can emerge
- Inflation, standard cosmological evolution, and late-time acceleration may then be interpreted as different stages of one flow

C. Wetterich, Variable gravity Universe, Phys. Rev. D 89 (2014) 024005 [1308.1019].
C. Wetterich, Inflation, quintessence, and the origin of mass, Nucl. Phys. B897 (2015) 111 [1408.0156].
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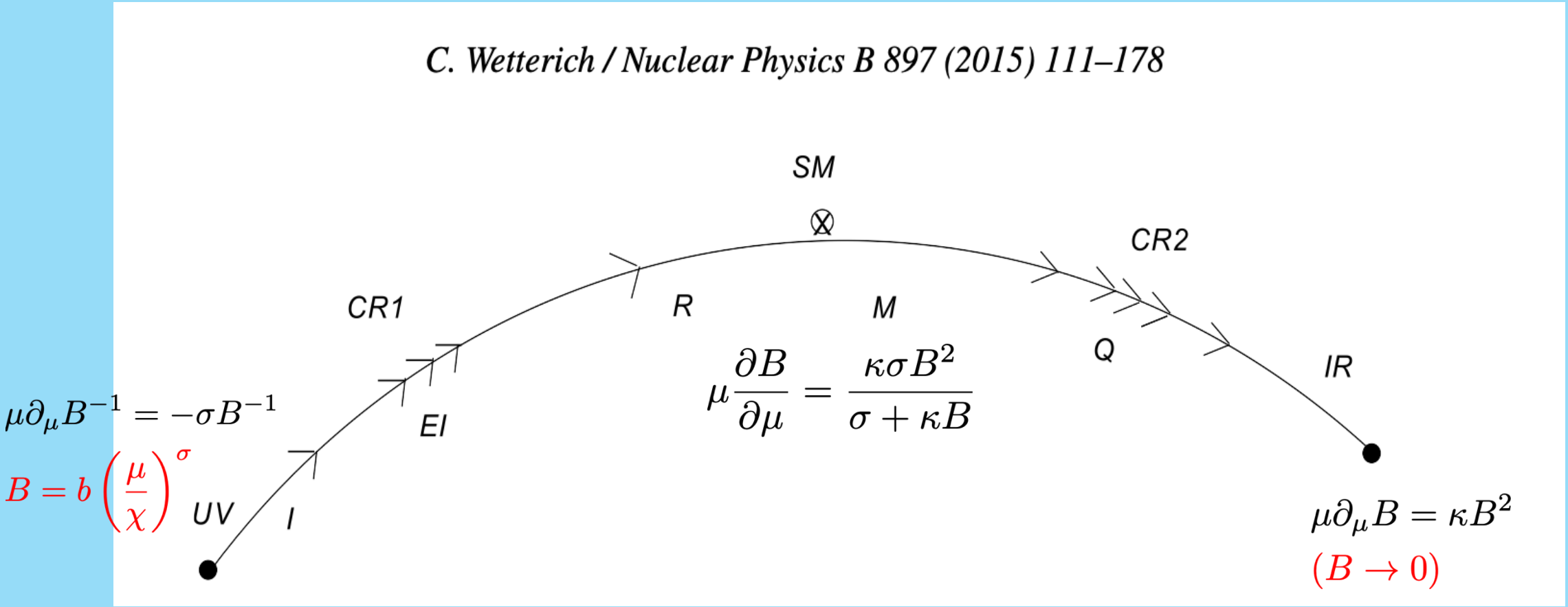

C. Wetterich, Variable gravity Universe, Phys. Rev. D 89 (2014) 024005 [1308.1019].
C. Wetterich, Inflation, quintessence, and the origin of mass, Nucl. Phys. B897 (2015) 111 [1408.0156].
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Asymptotically safe quantum gravity

How we quantize gravity?

$$Z[J] = \int \mathcal{D}g_{\mu\nu} \exp\left(-S[g_{\mu\nu}] + \int_x J^{\mu\nu}(x) g_{\mu\nu}(x)\right)$$

Diffeomorphism invariance is the fundamental symmetry of spacetime.

The metric carries the fundamental degrees of freedom.

$$\begin{aligned}
&\mathcal{O}(R^0): \quad \Lambda, \\
&\mathcal{O}(R^1): \quad R, \\
&\mathcal{O}(R^2): \quad R^2, \quad R_{\mu\nu}R^{\mu\nu}, \quad C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}, \\
&\mathcal{O}(R^3): \quad \cdots .
\end{aligned}$$

The perturbative quantum gravity in 4 dim.

$$S_{\rm EH} = \frac{1}{\kappa^2}\int d^4x\sqrt{g}R.$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \qquad \kappa^2 = 32\pi G_N .$$

$$S_{\rm EH} \sim \int d^4x \left[h\,\partial^2 h + \kappa h^2\partial^2 h + \kappa^2 h^3\partial^2 h + \kappa^3 h^4\partial^2 h + \cdots\right].$$

However, this conclusion assumes that the ultraviolet theory is controlled by the Gaussian fixed point.

The asymptotically safe quantum gravity

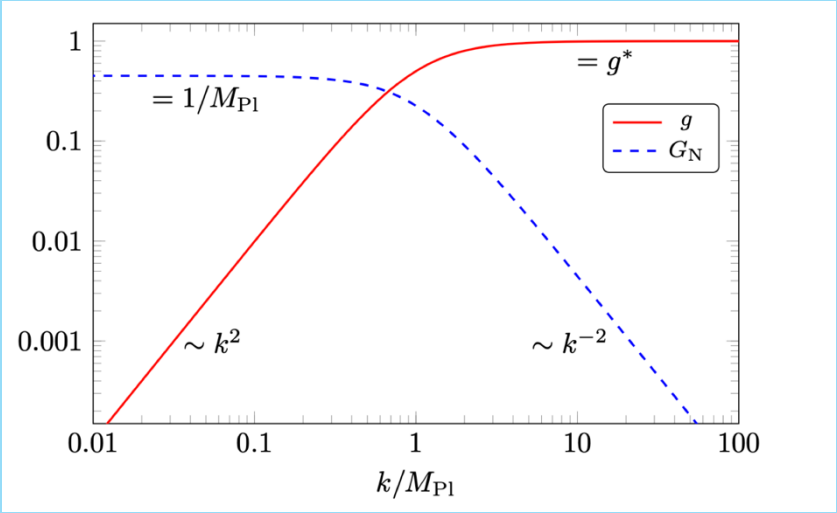
Steven Weinberg

the UV behaviour of gravity may instead be controlled by a non-Gaussian fixed point.

$$\beta_i(g^*) = 0, \qquad g_i(k) \to g_i^* \quad (k \to \infty).$$

$$\beta_g \equiv \partial_t g = \partial_t \left(G_N k^2\right) = 2g + k^2 \partial_t G_N.$$

S. Weinberg, “Ultraviolet divergences in quantum theories of gravitation,” in General Relativity: An Einstein Centenary Survey, edited by S. W. Hawking and W. Israel, Cambridge University Press, Cambridge, pp. 790–831 (1979).



Below Planck scale is constant, Quantum effect is negligible

$$g(k) = \frac{k^2}{M_{\rm Pl}^2 + k^2/g_*}.$$

C. Wetterich, "Exact evolution equation for the effective potential," Physics Letters B, 301 (1): 90–94 (1993).

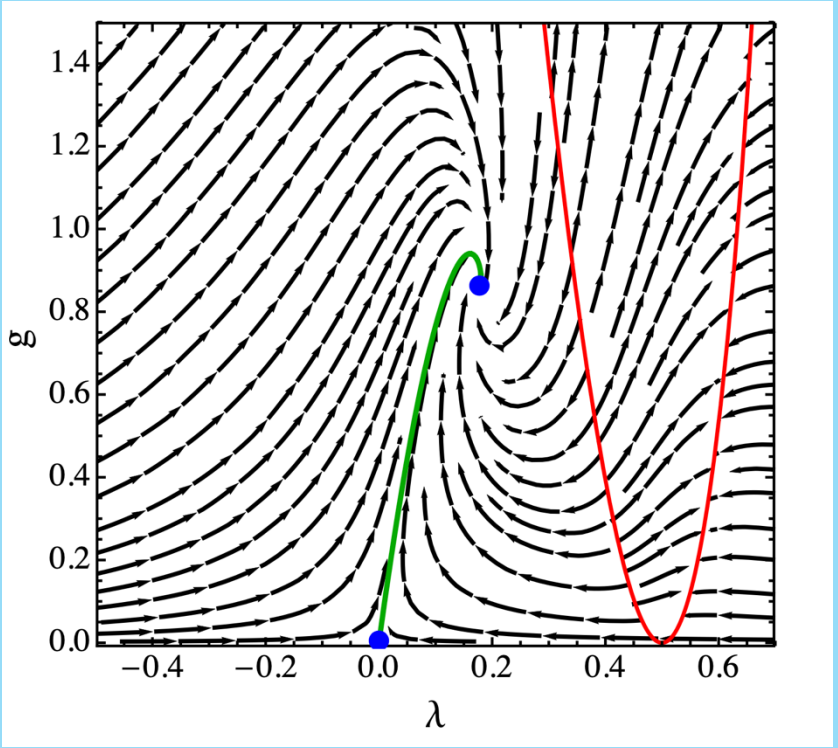
M. Reuter, Phys. Rev. D 57, 971 (1998)

$$\Gamma_k[g] = 2\kappa^2 Z_k \int d^4x \sqrt{g}\,\left[-R + 2\Lambda_k\right].$$

$$\partial_t \Gamma_k[g] = \frac{1}{2}\mathrm{Tr}\left[\frac{1}{\Gamma^{(2)}_{k,\mathrm{tt}}[g] + R_{k,\mathrm{tt}}}\,\partial_t R_{k,\mathrm{tt}}\right].$$

$$\partial_t \lambda_k = -4\lambda_k + \frac{\lambda_k}{g_k}\partial_t g_k + \frac{5}{4\pi}\frac{g_k\left(1-\frac{1}{6}\eta_g\right)}{1-2\lambda_k}.$$

$$\partial_t g_k = \left(2+\eta_g\right) g_k.$$



$$(g_*, \lambda_*) = (0.86, 0.18)$$

$$\Gamma = \int_x \sqrt{g}\left[-\frac{1}{2}\chi^2 R + \mu^2\chi^2 + \frac{1}{2}K(\chi)\partial^\mu\chi\partial_\mu\chi\right] \qquad K(\chi) = B(\chi/\mu) - 6$$

$$\Gamma = \int d^4x\sqrt{g}\left[V(\chi^2) - \frac{1}{2}F(\chi^2)R + \frac{1}{2}K(\chi^2)\,\partial^\mu\chi\partial_\mu\chi\right]$$

For a theory of quantum gravity one would like to predict these functions.

Functional flow equations are needed.

$$\partial_t\Gamma_k = \frac{1}{2}\,\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + R_k\right)^{-1}\partial_t R_k\right]$$

Scaling solutions in dilaton quantum gravity

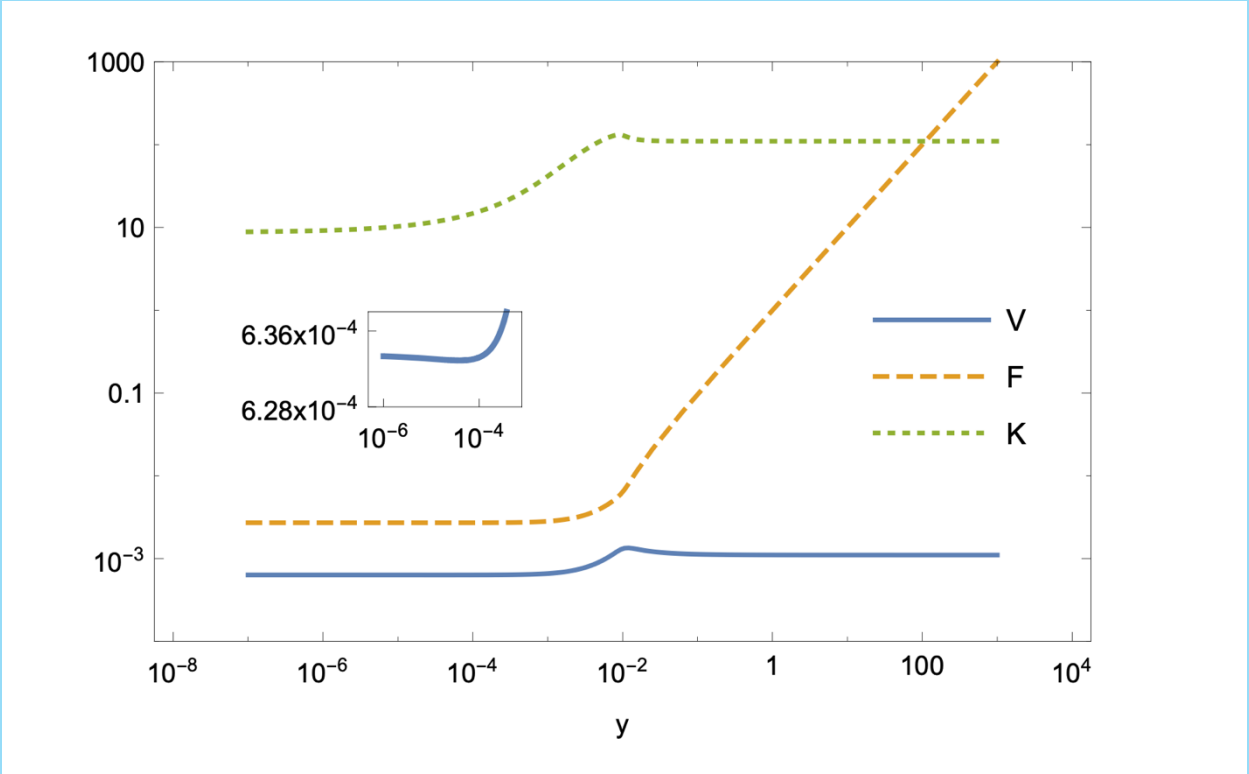
$$\Gamma = \int d^4x\sqrt{g}\left[V(\chi^2) - \frac{1}{2}F(\chi^2)R + \frac{1}{2}K(\chi^2)\,\partial^\mu\chi\partial_\mu\chi\right]$$

- Predicting whole functions as scalar potential.
- Dimensionless functions express in terms of dimensionless fields.
  (V/k^4 , F/k^2 , K , $y = \chi^2/k^2$.)
- At fixed point: (infinitely many) dimensionless couplings take fixed values.
- Whole scalar potential is fixed, for arbitrary values of scalar field

T. Henz, J. M. Pawlowski, and C. Wetterich, “Scaling solutions for Dilaton Quantum Gravity,” Physics Letters B 769 (2017) 105–110.

(i) Firstly found a global scaling solution candidate in dilaton gravity

- the solution interpolates between UV and IR regimes



(ii) Einstein-frame effective action becomes k-independent

$$\Gamma = \int d^4x\sqrt{g}\left[V(\chi^2) - \frac{1}{2}F(\chi^2)R + \frac{1}{2}K(\chi^2)\,\partial^\mu\chi\partial_\mu\chi\right]$$

$$g_{\mu\nu} = \omega^2(\chi)\,\tilde{g}_{\mu\nu}. \qquad \hat{V} = \frac{V}{F^2} \qquad \hat{K} = \frac{K}{F} + \frac{6y}{F^2}\left(\frac{\partial F}{\partial y}\right)^2$$

$$\frac{M^2}{k^2\,F(y)}$$

$$\frac{M^2\hat{K}}{2k^2}\,\partial^\mu\chi\,\partial_\mu\chi = \frac{1}{2}\,\partial^\mu\phi\,\partial_\mu\phi, \qquad \hat{K} = \frac{16}{\alpha^2 y}$$

Field Redefinition

$$\Gamma = \int d^4x\sqrt{g}\left(M^4\hat{V}(\phi) - \frac{1}{2}M^2R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi\right).$$

(iii) Cosmological constant vanishes asymptotically

- In the Einstein frame, the potential develops an exponential tail
- The effective cosmological constant goes to zero for $\phi \to \infty$.

$$\frac{1}{y^2} = \exp\left(-\frac{\alpha\phi}{M}\right) \qquad V = M^4\hat{V} = \frac{v_0 M^4}{\xi^2}\exp\left(-\frac{\alpha\phi}{M}\right)$$

If nature is characterized by a scaling solution for which $\hat{V}$ vanishes for $y \to \infty$, the cosmological constant problem is solved, at least for asymptotic time.

However, the treatment of the kinetial remained a central open issue…

1.field-dependent cutoff and gauge symmetry make the flow nontrivial

2. $K(\rho)$ can be negative, while stability still requires only $\frac{K}{\xi} > -6$ (mixing in the kinetic terms for the scalar and the metric.)

3. conformal/Weyl limit should be respected “B=K+6=0”

Scaling solutions for gauge invariant flow equations

$$\Gamma = \int d^4x\sqrt{g}\left(U(\rho) + \frac{K(\rho)}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{F(\rho)}{2}R\right) \qquad \rho = \frac{\phi^2}{2}$$

diffeomorphism symmetry: not all modes are physical

Problem: scalar fluctuation mixes with metric scalar fluctuation.

We want to separate physical scalar-gravity mixing from gauge redundancy.

C.Wetterich, Gauge-invariant flow equation, Nucl. Phys. B 931 (2018) 262–288

$$h_{\mu\nu} = f_{\mu\nu} + a_{\mu\nu}$$

Physical part

gauge part

Spin-2 graviton($t_{\mu\nu}$) +

Physical scalar $s_{\mu\nu} = \frac{1}{3} P_{\mu\nu}\sigma$

$$a_{\mu\nu} = \bar{D}_\mu v_\nu + \bar{D}_\nu v_\mu - \bar{D}_\mu \bar{D}_\nu u,$$

$$\Gamma_{\rm gf} = \frac{1}{2\alpha}\int_x \sqrt{\bar{g}}\,\bar{g}^{\mu\nu}\Sigma_\mu\Sigma_\nu$$

$$\Sigma_\mu = \bar{D}^\nu h_{\nu\mu} - \frac{\beta+1}{4}\bar{D}_\mu h$$

$$\beta = -1, \qquad \Sigma_\mu = \bar{D}^\nu h_{\mu\nu} = \bar{D}^\nu(f_{\mu\nu} + a_{\mu\nu})$$

$$\alpha \to 0 \qquad \bar{D}^\nu a_{\mu\nu} = 0$$

these modes are fixed by the condition

$$\partial_t \Gamma_k = \pi_2 + \pi_0 + \eta_k$$

$\boldsymbol{\pi_2}$ spin-2 physical sector
$\boldsymbol{\pi_0}$ scalar physical sector
$\boldsymbol{\eta_k}$ measure sector

1. the gauge-invariant flow as a way to organize the physical fluctuations.
2. background-field method with a particular physical gauge fixing
3.Gauge modes are not used as physical propagating modes their effect remains through measure factors.

C.Wetterich, Gauge-invariant flow equation, Nucl. Phys. B 931 (2018) 262–288

We define dimensionless quantities by $u(\tilde{\rho}) = \dfrac{U(\rho)}{k^4}\,, \quad w(\tilde{\rho}) = \dfrac{F(\rho)}{2k^2}\,, \quad \kappa(\tilde{\rho}) = K(\rho)\,, \quad \tilde{\rho} = \dfrac{\rho}{k^2}\,.$

$$\partial_t u = \beta_U = -\partial_\tau u - 4u + \frac{1}{32\pi^2} M_U\,,$$

$$\partial_t w = \beta_F = -\partial_\tau w - 2w - \frac{1}{96\pi^2} M_F\,,$$

$$\partial_t \kappa = \beta_K = -\partial_\tau \kappa + M_K\,.$$

$$-\tilde{\phi}\,\partial_{\tilde{\phi}} = -2\tilde{\rho}\,\partial_{\tilde{\rho}} = \partial_\tau$$

$$\tau = -\log\sqrt{2\tilde{\rho}}$$

dimensionless functions and scaling equations

$$\partial_t u = \partial_t w = \partial_t \kappa = 0$$

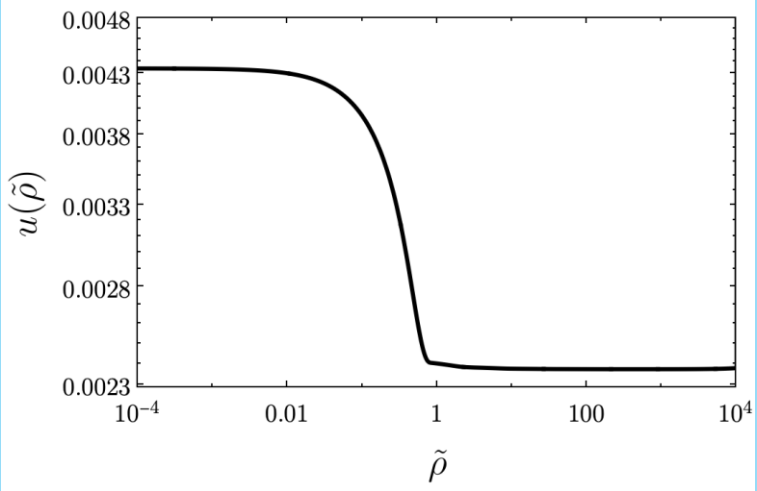
$$\partial_\tau u(\tilde{\rho}) = -4u(\tilde{\rho}) + \frac{1}{32\pi^2} M_U$$

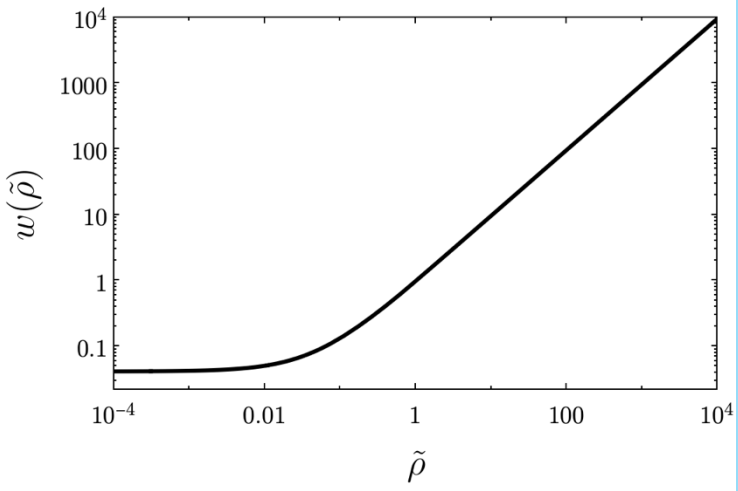
$$\partial_\tau w(\tilde{\rho}) = -2w(\tilde{\rho}) - \frac{1}{96\pi^2} M_F$$

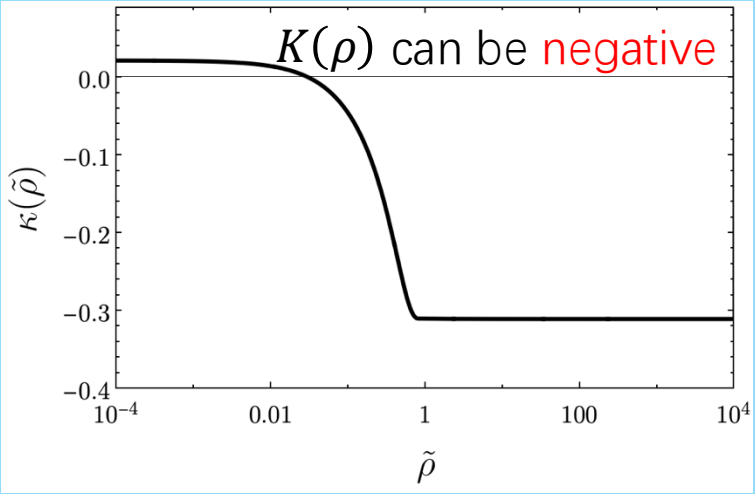
$$\partial_\tau \kappa(\tilde{\rho}) = M_K\,.$$

Global scaling solution = interpolation between two asymptotic regimes

The ultraviolet fixed point is not characterized only by the couplings at the origin.
It requires the existence of a global scaling solution that can be extended to the whole field range.







- $u(\tilde{\rho})$: crossover between two plateaus
- $w(\tilde{\rho})$: linear growth for large $\tilde{\rho}$
- $\kappa(\tilde{\rho})$: nontrivial but regular, negative converging behavior.

Asymptotic behavior in two regime

IR fixed point and Large field expansion

$$\tilde{\rho} = \frac{\rho}{k^2} \to \infty \qquad k \to 0$$

$$w(\tilde{\rho}) \sim \xi_\infty \tilde{\rho} \qquad F(\rho) \sim 2\xi_\infty \rho \sim \xi_\infty \phi^2 \qquad G_{\text{eff}} \sim \frac{1}{F} \to 0$$

Infrared fixed point: free graviton + free scalar

$$u(\tilde{\rho}) \to u_\infty = \frac{3}{128\pi^2}, \qquad w(\tilde{\rho}) \to \xi_\infty \tilde{\rho}, \qquad \kappa(\tilde{\rho}) \to \kappa_\infty$$

$$U(\rho) \to u_\infty k^4 \to 0 \qquad (k \to 0)$$

scale-invariant form of dilaton quantum gravity

The ξ_∞ and κ_∞ are free parameters in this limit (IR)

$$u(\tilde{\rho}) \to u_\infty = \frac{3}{128\pi^2}, \qquad w(\tilde{\rho}) \to \xi_\infty \tilde{\rho}, \qquad \kappa(\tilde{\rho}) \to \kappa_\infty$$

scalar anomalous dimension vanishes

$$\partial_\tau u(\tilde{\rho}) = -4u(\tilde{\rho}) + \frac{1}{32\pi^2} M_U$$

$$\partial_\tau w(\tilde{\rho}) = -2w(\tilde{\rho}) - \frac{1}{96\pi^2} M_F$$

$$\partial_\tau \kappa(\tilde{\rho}) = M_K \,.$$

$$-\tilde{\phi}\,\partial_{\tilde{\phi}} = -2\tilde{\rho}\,\partial_{\tilde{\rho}} = \partial_\tau$$

infrared endpoint is characterized by the ratio $\epsilon_\infty = \dfrac{\kappa_\infty}{\xi_\infty}$

how the scaling solution leaves this fixed point when $\tilde{\rho}$ is large but finite?

Large field expansion we need: $y = \frac{1}{\tilde{\rho}}$

expand the scaling solution around the IR fixed point as:

$$u(y) = u_\infty + \frac{s_1}{\xi_\infty} y + \cdots$$
$$w(y) = \xi_\infty y^{-1} + t_0 + \frac{t_1}{\xi_\infty} y + \cdots$$
$$\kappa(y) = \varepsilon_\infty \xi_\infty + k_1 y + \cdots$$

Scaling equations →

$$t_0 = \frac{A_w}{2B}, \qquad k_1 = \frac{C_\kappa}{2B}, \qquad s_1 = \frac{C_u}{6B}, \qquad t_1 = \frac{C_w}{4B^2}.$$

Once B is chosen, the local IR scaling solution is fixed.

The conformal point B=0 is identified, but not fully controlled.

The large-field expansion shows that the local IR scaling solution is not unique, but forms a one-parameter family B

$$\hat{K}k^2 = \frac{1}{2w}\left(\kappa + \frac{6\tilde{\rho}}{w}\left(\frac{\partial w}{\partial \tilde{\rho}}\right)^2\right) > 0 \qquad \text{Constraint} \qquad \hat{K} = \frac{B}{\phi^2}.$$

$\kappa_0 > 0$ *Small field* $B = \frac{\kappa_\infty}{\xi} + 6 > 0$ *Large field*

Stability : $B > 0 \iff \varepsilon_\infty > -6$

Hence, even a negative large-field kinetial is possible within the stable regime.

But this is only a local asymptotic solution near $\tilde{\rho} \to \infty$.

Small field expansion and UV fixed point

$$\tilde{\rho} = \frac{\rho}{k^2} \to 0 \qquad k \to \infty$$

$$u(\tilde{\rho}) = u_0 \qquad w(\tilde{\rho}) = w_0 \qquad \kappa(\tilde{\rho}) = \kappa_0$$

$$\partial_\tau u(\tilde{\rho}) = -4u_0 - \frac{5u_0}{24\pi^2(u_0 - w_0)} - \frac{3u_0}{160\pi^2(u_0 - 4w_0)} + \frac{79}{480\pi^2},$$

$$\partial_\tau w(\tilde{\rho}) = -2w_0 - \frac{25u_0}{64\pi^2(u_0 - w_0)} + \frac{5u_0^2}{1344\pi^2(u_0 - 4w_0)^2} - \frac{9u_0}{896\pi^2(u_0 - 4w_0)} + \frac{1207}{2688\pi^2},$$

$$\partial_\tau \kappa(\tilde{\rho}) = \frac{(5u_0 - 23w_0)}{240\pi^2(u_0 - 4w_0)^2}\kappa_0.$$

$$u_{0*} = 0.0055$$
$$w_{0*} = 0.0277$$

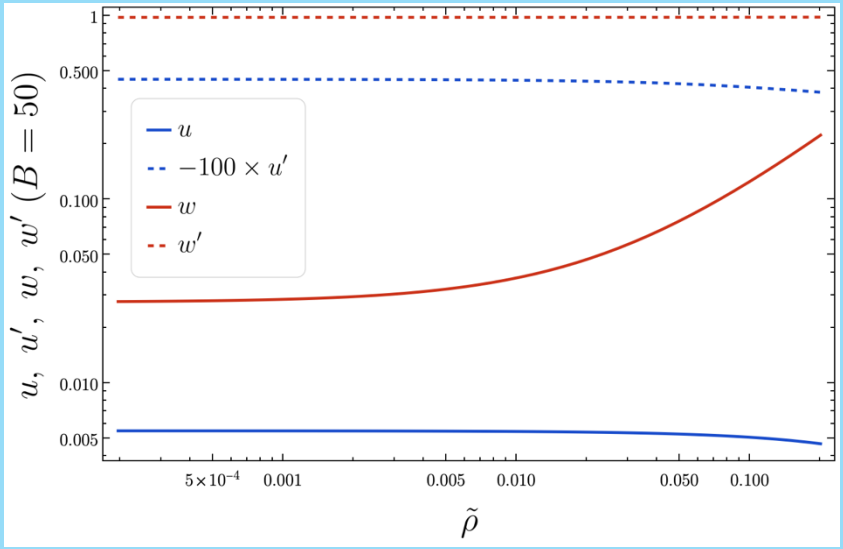
But the κ-equation has no stable constant solution with $\kappa_0 > 0$.

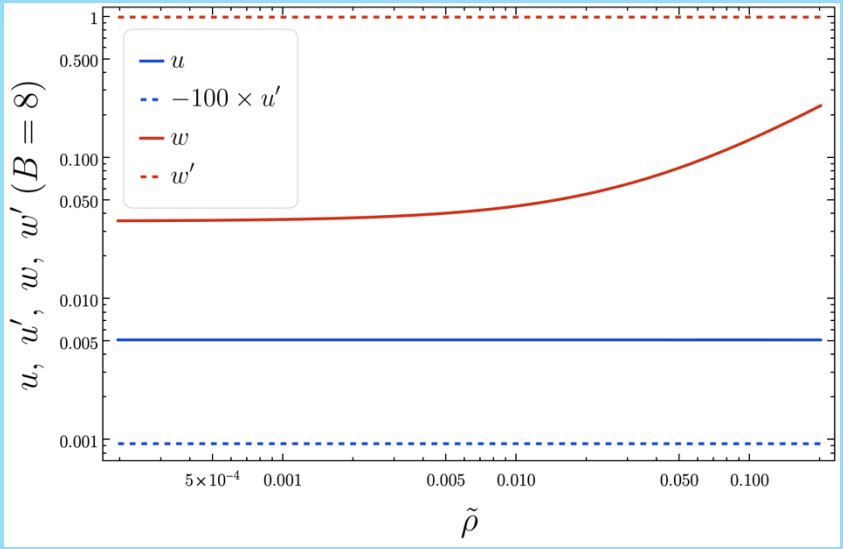
Hence, **the UV scaling solution cannot be constant in $\tilde{\rho}$.**

$$u(\tilde{\rho}) = u_0 + c_1\tilde{\rho} + \cdots ,$$
$$w(\tilde{\rho}) = w_0 + w_1\tilde{\rho} + \cdots ,$$
$$\kappa(\tilde{\rho}) = \kappa_0 + \kappa_1\tilde{\rho} + \cdots .$$

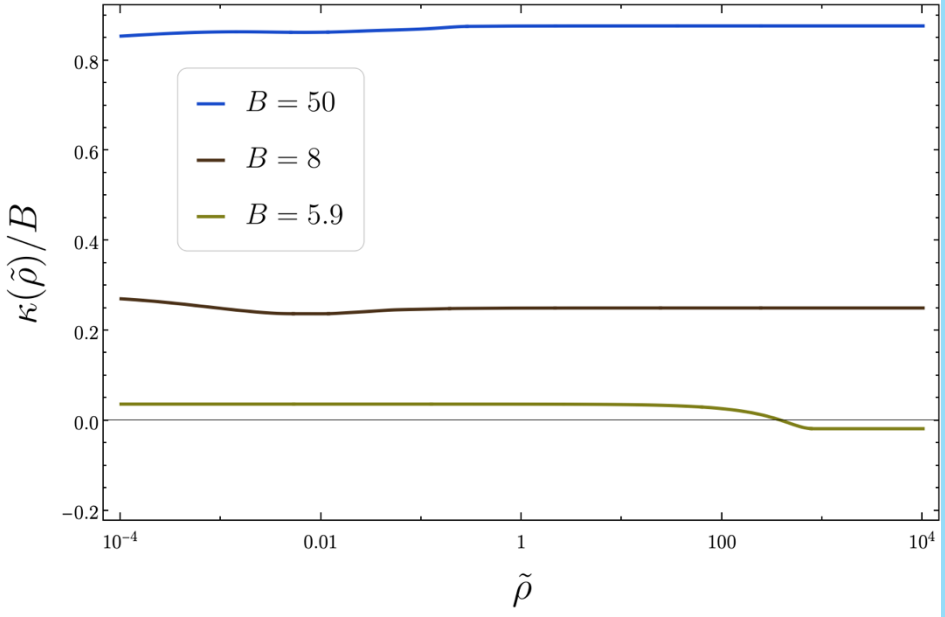
The UV fixed-point equations are not closed in u_0, w_0, κ_0 alone, because derivatives also enter the flow kernels.

both u'(0) and w'(0) are nonzero and depend on the infrared parameter B





After this, we can ask which values of B lead to globally acceptable solutions.



The numerical analysis shows a critical value $B_c \simeq 5.7$

$B > B_c$ the solution remains stable for all $\tilde{\rho}$.

$B < B_c$ the stability condition is violated and numerical instabilities develop.

Different B lead to different UV data, so a continuous family remains possible…

So far all discussions based on canonical scaling field $\tilde{\rho} = \frac{\rho}{k^2}$,
Works well for IR.

But in UV, the scalar field can acquire an anomalous dimension η.
Then the scaling field has to be renormalized.

$$\phi_R = K^{1/2}\phi \qquad \tilde{\rho}_R = K\tilde{\rho}$$

Scalar field anomalous dimension

1. For a $\tilde{\rho}$-independent extended Reuter fixed point

$$u = u_0, \qquad w = w_0, \qquad \kappa = \kappa_0(k).$$

$$k\partial_k \kappa_0 = -\eta \kappa_0 \qquad \phi_R = K^{1/2}\phi \qquad \tilde{\rho}_R = K\tilde{\rho}$$

$$(\partial_t - \underbrace{(2+\eta)\tilde{\rho}\partial_{\tilde{\rho}}}) u = -4u + \frac{1}{32\pi^2} M_U$$

$\partial_\tau^{(\eta)}$ Flow equation remains the same

similar for the other flow equations

$$\partial_\tau \kappa(\tilde{\rho}) = \frac{5u_0 - 23w_0}{240\pi^2(u_0 - 4w_0)^2}\kappa_0 \qquad \partial_\tau \to k\partial_k + \partial_\tau \qquad \eta = 0.0116$$

gravity induced anomalous dimension

K -running is absorbed into field renormalization.

2. For dilaton quantum gravity

for strongly diverging $\kappa(\tilde{\rho})$ the series expansion for $(\tilde{\rho} \to 0)$ may be rearranged.

$$u(\tilde{\rho}) = u_0 \qquad w(\tilde{\rho}) = w_0 + \xi\tilde{\rho} \qquad \kappa(\tilde{\rho}) = \kappa_0 + \frac{\kappa_{-1}}{\tilde{\rho}}$$

From scaling equations: $\qquad u_0^* = 0.00587, \qquad w_0^* = 0.0278.$

K depends on the field $\quad \Rightarrow \quad$ linear field renormalization is not enough.

$$(\partial_t + \partial_\tau)\kappa(\tilde{\rho}) = \frac{1}{(4\pi)^2}\left[\left(\frac{9\kappa_0^2\tilde{\rho}}{\kappa_{-1}^2} - \frac{25\kappa_0}{24\kappa_{-1}} + \frac{2}{3\tilde{\rho}}\right) + \left(-\frac{5\kappa_0^2\tilde{\rho}}{2\kappa_{-1}^2} + \frac{3\kappa_0}{2\kappa_{-1}} - \frac{2}{3\tilde{\rho}}\right) + \frac{5u_0 - 23w_0}{15(u_0 - 4w_0)^2}\,\kappa_0 + \frac{65u_0 - 296w_0}{180(u_0 - 4w_0)^2}\,\frac{\kappa_{-1}}{\tilde{\rho}}\right].$$

$$\tilde{\eta} = -\left(2 + \frac{1}{(4\pi)^2} \frac{296 w_0 - 65 u_0}{180 (u_0 - 4 w_0)^2}\right) \approx -2.026.$$

In the IR, the anomalous dimension goes to zero, while in the UV it can be nonzero. So, in general, η should depend on the field.

Nonlinear scaling field required, not pursued here.

Summary

A global scaling solution is found in the two-derivative scalar–tensor truncation of dilaton quantum gravity.

Within the present truncation, a continuous B-family of global solutions cannot be excluded.

The gauge-invariant flow gives access to the scalar kinetial $\boldsymbol{\kappa}(\widetilde{\boldsymbol{\rho}})$, including stable regions with negative K.

Summary

The UV endpoint cannot, in general, be described only by constant couplings u_0 , w_0 , κ_0. Field derivatives such as $u'(0)$ and $w'(0)$ are part of the fixed-function structure.

Possible anomalous scaling of the scalar field may appear near the UV endpoint. For dilaton quantum gravity, this would require a field-dependent anomalous dimension or a nonlinear scaling field.

Field-dependent anomalous dimension / nonlinear scaling field remains open

The End

Thank you for Listening!