The lattice simulation method of SIGW

Xiang-Xi Zeng

- Scalar-induced gravitational waves with non-Gaussianity up to all orders, arXiv: 2508.10812, **X-X Zeng**, Z Ning, R-G Cai, and S-J Wang.
- Scalar-Induced Gravitational Waves Including Isocurvature Perturbations with Lattice Simulations, *JCAP* 03 (2026) 065，arXiv: 2510.02106, **X-X Zeng**.

2026/9/12

2026 New directions EXploring the Unknown beyond the Standard Model

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Contents

- What is SIGW?
- Why lattice simulation?
- Where is the particle physics?
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1 What is SIGW?

What is SIGW?

Starting from the conformal Newtonian gauge

Up to second order, this is all we need

$$\mathrm{d}s^2 = a(\tau)^2\left\{-(1+2\Phi)\mathrm{d}\tau^2 + \left[(1-2\Phi)\delta_{ij} + \frac{1}{2}h_{ij}\right]\mathrm{d}x^i\mathrm{d}x^j\right\}$$

Perturbed Einstein Equations

$$h_{ij}'' + 2\mathcal{H}h_{ij}' - \nabla^2 h_{ij} = -4S_{ij}^{\mathrm{TT}}$$

$$S_{ij} = 4\Phi\partial_i\partial_j\Phi + 2\partial_i\Phi\partial_j\Phi - \frac{4}{3(1+\omega)\mathcal{H}^2}\partial_i(\Phi' + \mathcal{H}\Phi)\partial_j(\Phi' + \mathcal{H}\Phi)$$

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Scalar perturbation sources the GWs

$$\Phi \to h_{ij}$$

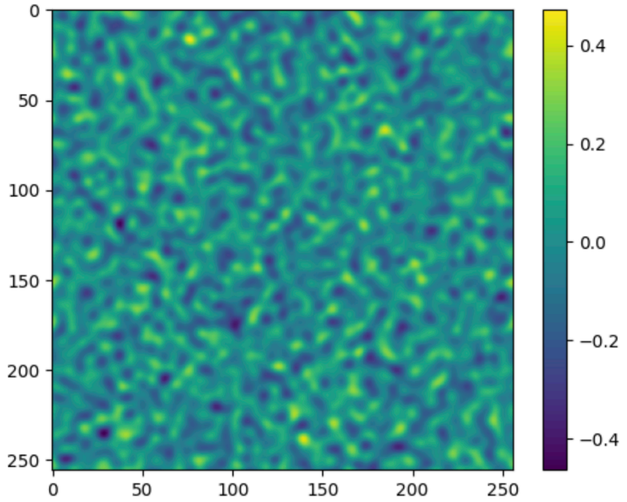
1 What is SIGW?

Why SIGW?

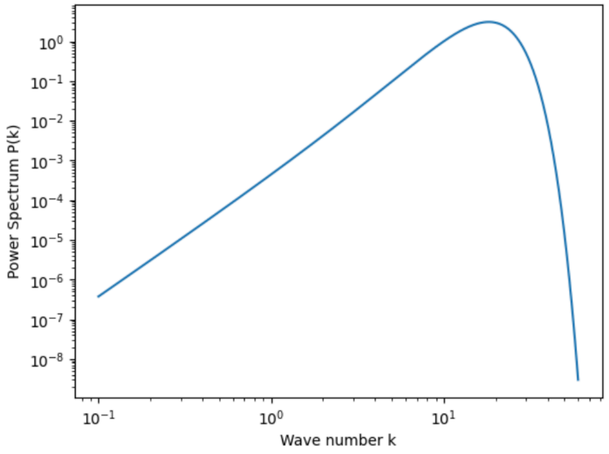
Fingerprint of the universe

$\Phi \sim \zeta \sim \delta\rho/\rho$

Curvature perturbation ζ_g

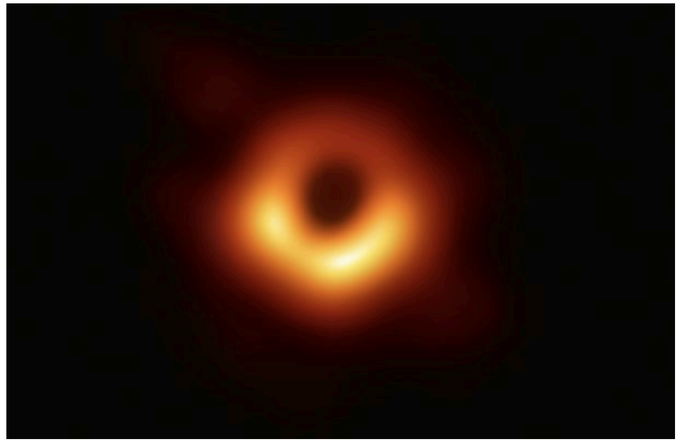


Power spectrum of ζ_g



Scalar-Induced GW (SIGW)

Primordial black holes



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2 Why lattice simulation

Is lattice simulation really necessary?

Isn’t lattice simulation expensive and difficult?

2 Why lattice simulation

Is lattice simulation really necessary?

How to calculate the SIGW?

$$\langle h_{ij}h_{ij}\rangle(k) \sim \int \mathrm{d}v \int \mathrm{d}u I^2(u,v,k,\tau)\langle \zeta\zeta\zeta\zeta\rangle(k)$$

$$h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij} \sim \partial_i\zeta\partial_j\zeta$$

2 Why lattice simulation

Is lattice simulation really necessary?

How to calculate the SIGW?

$$\langle h_{ij}h_{ij}\rangle(k) \sim \int \mathrm{d}v \int \mathrm{d}u I^2(u,v,k,\tau)\langle\zeta\zeta\zeta\zeta\rangle(k)$$

$$\zeta(x) = \zeta_g(x) + F_{\mathrm{NL}}\zeta_g^2 + G_{\mathrm{NL}}\zeta_g^3 + H_{\mathrm{NL}}\zeta_g^4 + \dots$$

$$\langle\zeta(k_1)\zeta(k_2)\zeta(k_3)\zeta(k_4)\rangle \sim \langle\Pi_{i=1}^4(\zeta_g(k_i) + F_{\mathrm{NL}}\int \mathrm{d}^3q_i\zeta_g(\vec{q}_i)\zeta_g(\vec{k}_i - \vec{q}_i))\rangle$$

For F_{NL} order, you need to evaluate 8-multiple integrals.

For G_{NL} order, you need to evaluate 14-multiple integrals.

…

This is the same difficulty in the QFT.

2 Why lattice simulation

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This is the same difficulty in the QFT.

Because we are calculating in momentum space

ζ_g^2 → A number

ζ_g^2 → An integral $\int d^3q\zeta_g(q)\zeta_g(k-q)$

2 Why lattice simulation

Is lattice simulation really necessary? → Does non-Gaussian effects really matter?

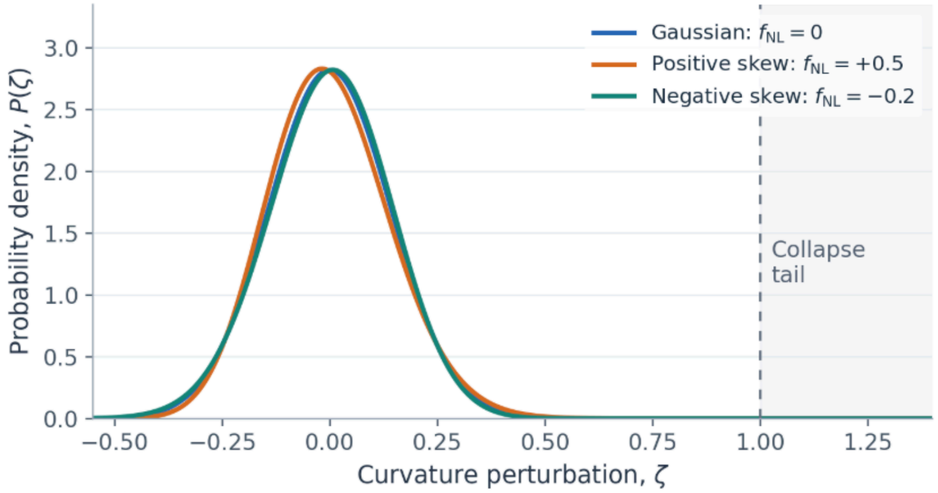
↓

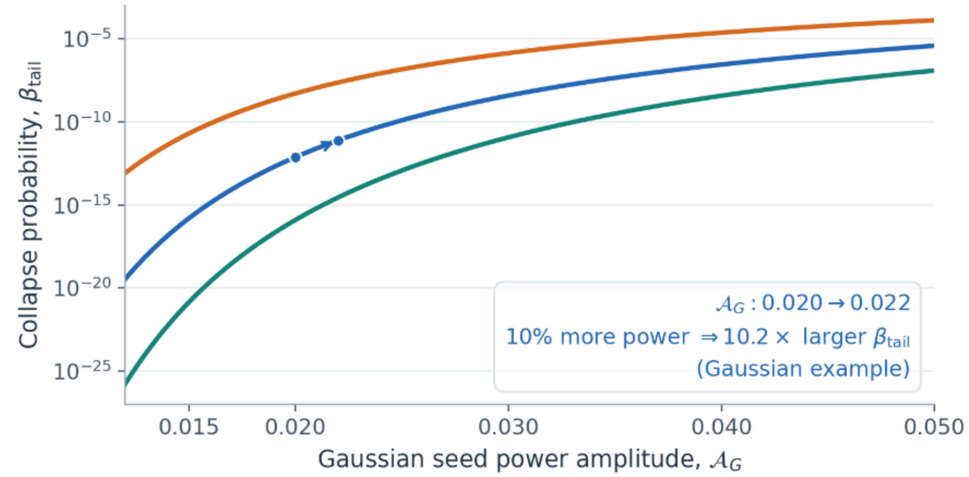
Related to abundant PBH formation

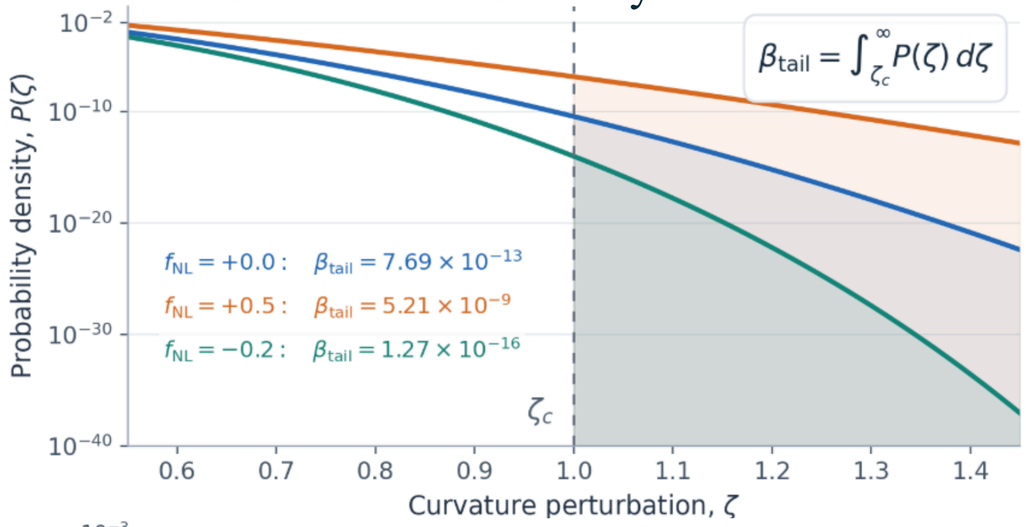
Some inflation models, like USR, curvaton

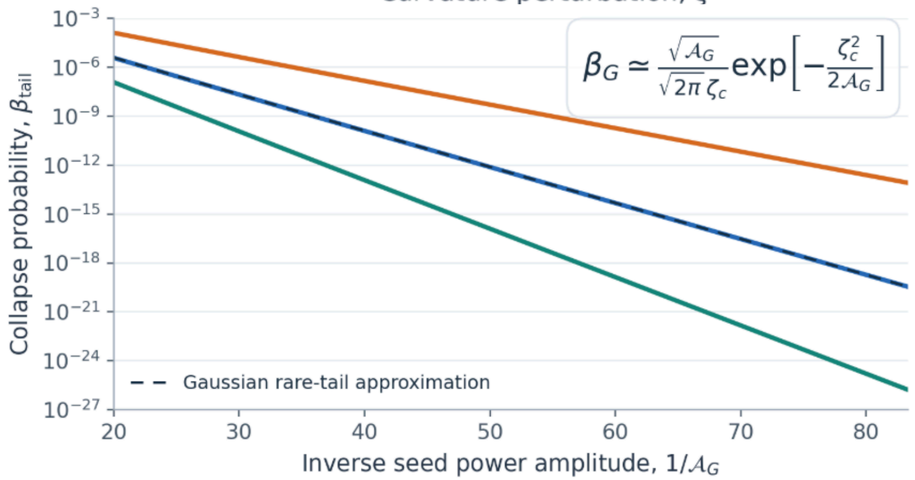
2 Why lattice simulation

Is lattice simulation really necessary? → Does non-Gaussian effects really matter?







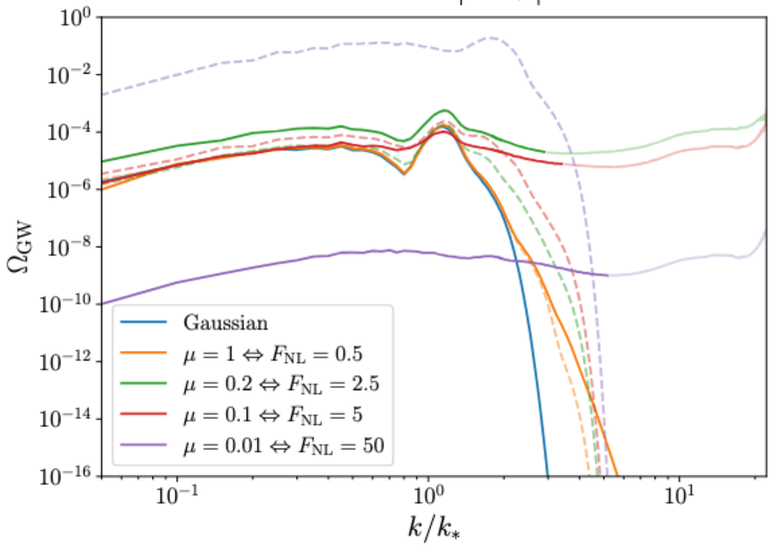


2 Why lattice simulation

Is lattice simulation really necessary? → Does non-Gaussian effects really matter?

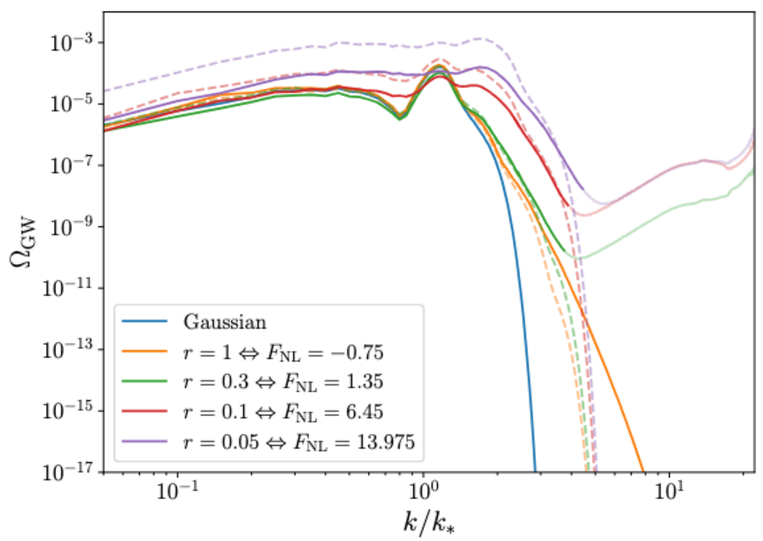
USR

$\zeta = -\mu \log\left|1 - \frac{\zeta_g}{\mu}\right|$



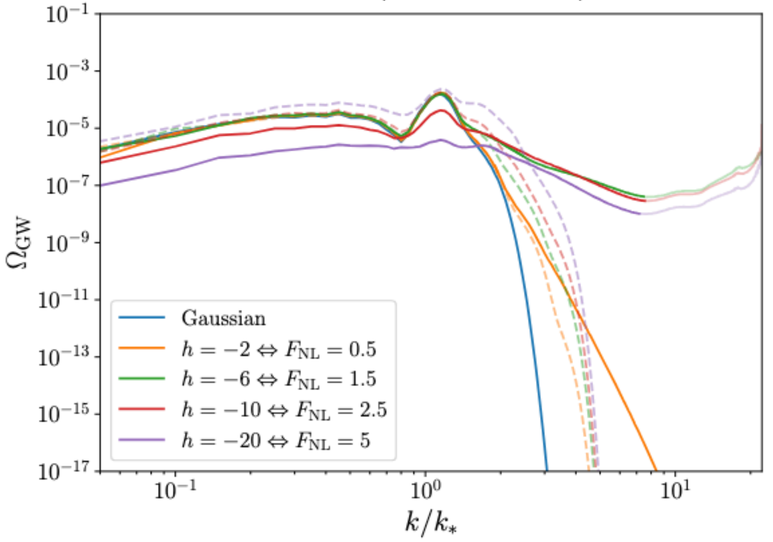
Curvaton model

$\zeta = \log[X(r, \zeta_g)]$



USR with a step

$\zeta = -\frac{2}{|h|}\left(\sqrt{|1 - |h|\zeta_g|} - 1\right)$



Full-order

Finite order ($F_{\rm NL}$)

1.Amplitude
2.UV behavior
3.Peak location

2 Why lattice simulation

Isn't lattice simulation expensive and difficult? → Depends on the lattice number N $N > 512$ Expensive

$N \leq O(300)$ Laptop

`SimuSIGW` (Provide CUDA acceleration) `pip install SimuSIGW`

(Easily setup)

↓ You can choose

- `run_evolution` : CPU FFT pseudospectral backend. FPS
- `run_evolution_fd` : CPU Numba finite-difference backend, with `fd_order=2` or `fd_order=4`. FD
- `run_evolution_torch` : optional PyTorch dense Fourier-matrix backend for the original GPU-style calculation. FPS
- `run_evolution_torch_fft` : optional PyTorch FFT pseudospectral backend. FPS

↓

You can find the tutorial on the Github website

FPS method $N = 128,256$

FD method $N = 256,512$

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The SIGW is good, but the main topic of this workshop is particle physics.

3 The particle physics

Can we use SIGW to detect the particle physics?

Or

How particle physics influence the SIGW?

→ There are two ways:

(1) the equation state and the sound speed of the cosmic fluid

(2) the scalar perturbation (Boltzmann equation)

$$h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij} = -4S^{\mathrm{TT}}_{ij}$$

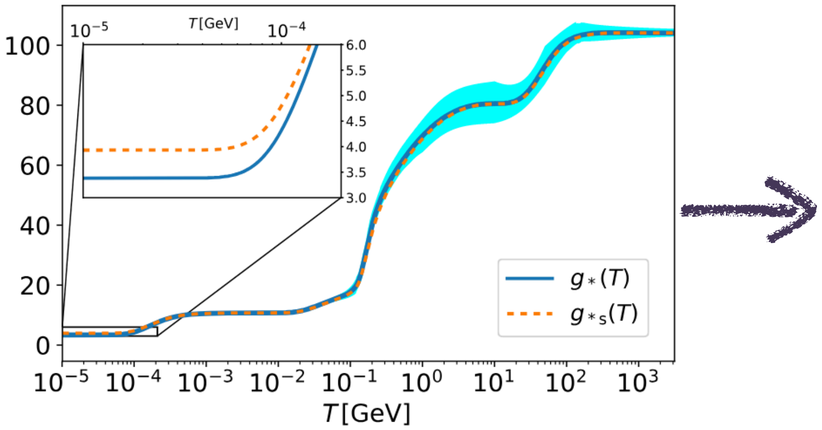
$$S_{ij} = 4\Phi\partial_i\partial_j\Phi + 2\partial_i\Phi\partial_j\Phi - \frac{4}{3(1+\omega)\mathcal{H}^2}\partial_i(\Phi' + \mathcal{H}\Phi)\partial_j(\Phi' + \mathcal{H}\Phi)$$

$$\Phi'' + 3\mathcal{H}(1+c_s^2)\Phi' + (3\mathcal{H}^2(c_s^2 - \omega) - c_s^2\nabla^2)\Phi = 0$$

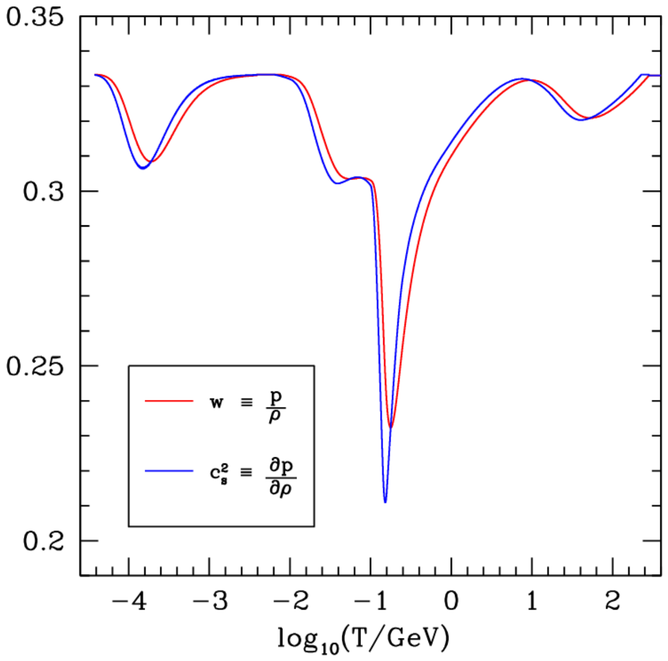
3 The particle physics

(1) the equation state and the sound speed of the cosmic fluid

For example, the standard model

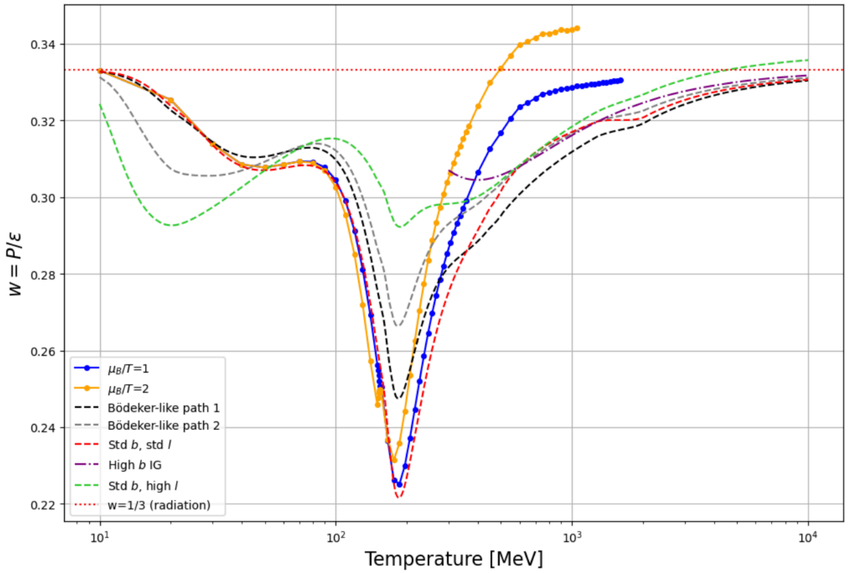


K. Abe, Y. Tada, I. Ueda, JCAP, 2010.06193



I. Musco, K. Jedamzik, S. Young, PRD, 2303.07980

Beyond the standard model



M. Gonin, O. Ivanytskyi, D. Blaschke, G.Hasinger, 2604.12581

3 The particle physics

(1) the equation state and the sound speed of the cosmic fluid

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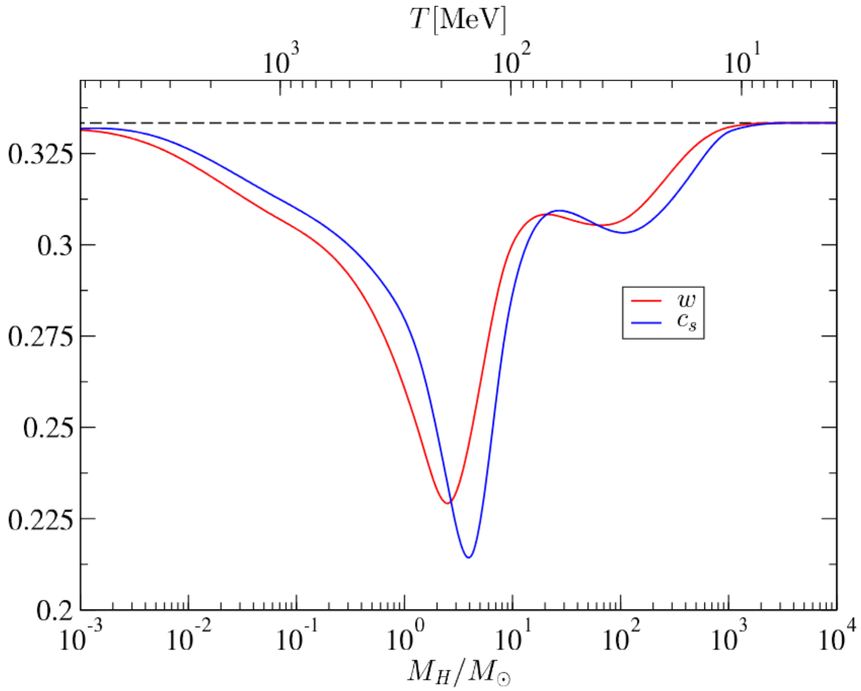

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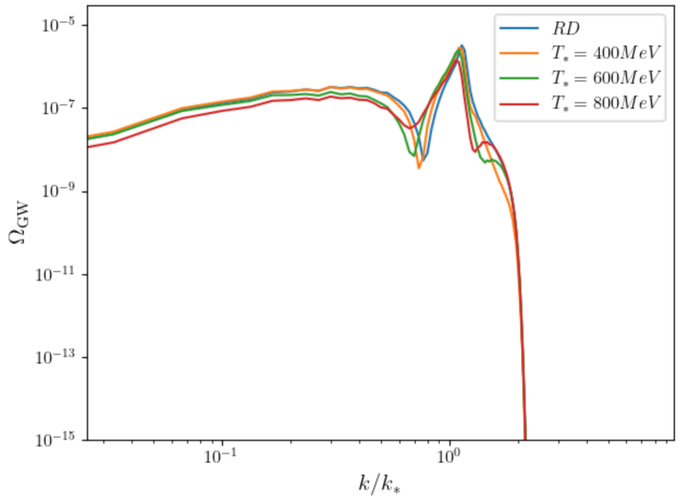
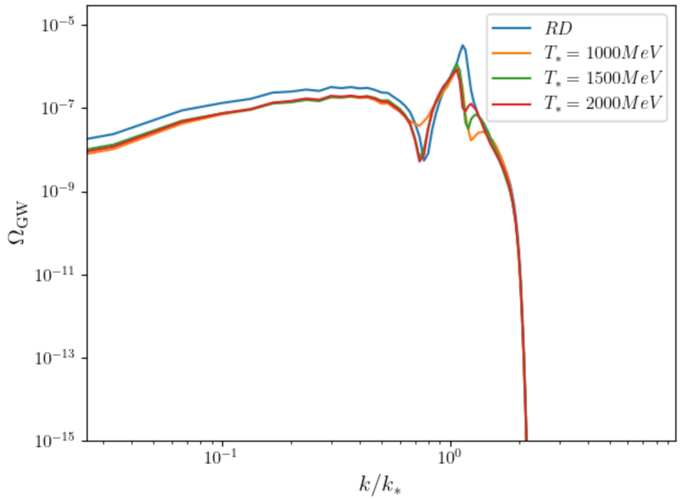




P. Du, S. Kumar, J. Zhong, X.-X. Zeng, to appear

3 The particle physics

(2) the scalar perturbation

Boltzmann Equation: $\frac{df}{d\lambda} = C[f(\lambda)] + I[f(\lambda)]$

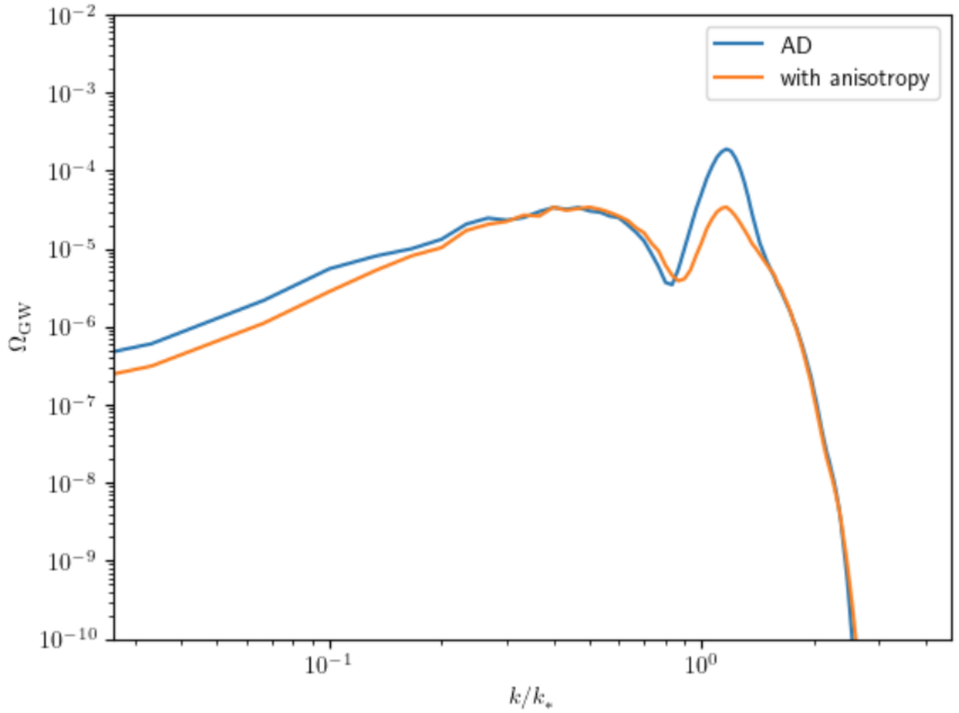
↓

Anisotropic stress Π_{ij}

↓

$$\Psi - \Phi = 8\pi G a^2 \bar{p} \Pi \,.$$

free streaming neutrinos



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4 Summary

Take home message:

(1) we can treat all order non-Gaussianaity now; therefore, a more consistent constraint on PBH abundance is viable.

(2) The tool SimuSIGW

Still be updating (the next version will include the isocurvature and the time dependent quantity)

(3) SIGW can detect partcle interactions

(i) the equation state and the sound speed of the cosmic fluid

(ii) the scalar perturbation

Thanks for your attention!