Field-space Surprises in Multi-field Preheating

Evangelos Sfakianakis

IFAE, Barcelona

June 28 2021

Supported by the “la Caixa” Foundation and EU’s Horizon 2020 programme under the Marie Skłodowska-Curie grant agreement

Current work & TDLI synergies

- I study models of the early universe, inspired by String Theory and BSM physics and compute the observables.
- Synergies with TDLI in model-building, axions, astro-particle physics, Higgs & BSM physics (e.g. neutrinos).

Supervising 1 postdoc and 1 PhD student, under a personal grant of € 300,000.

I. The Big Bang and Cosmic Inflation

$\Rightarrow$ From the Big Bang to inflation: successes and prospects

II. Reheating: A Critical Epoch

$\Rightarrow$ The necessity, importance of reheating. A big unknown of cosmic evolution

III. Realistic Reheating Scenarios

$\Rightarrow$ Multiple fields with non-canonical kinetic terms lead to drastically different behavior

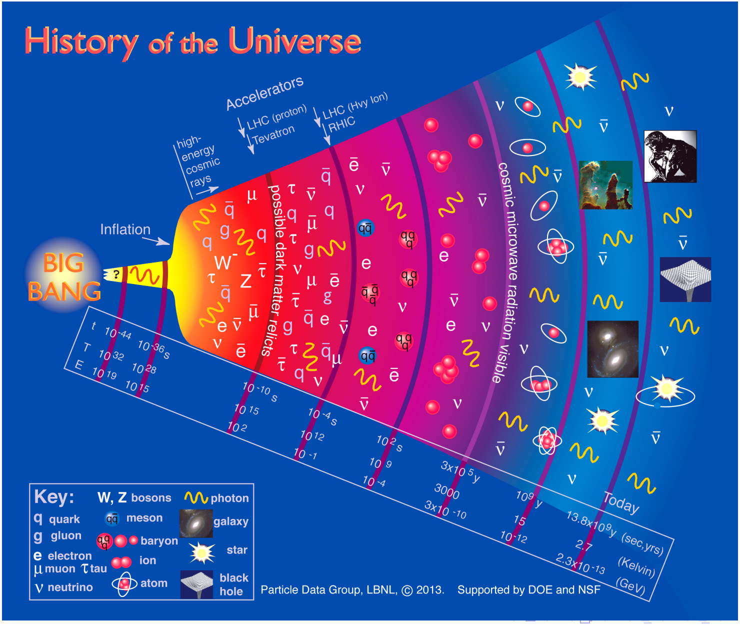
Outline

I. The Big Bang and Cosmic Inflation
$\Rightarrow$ From the Big Bang to inflation: successes and prospects

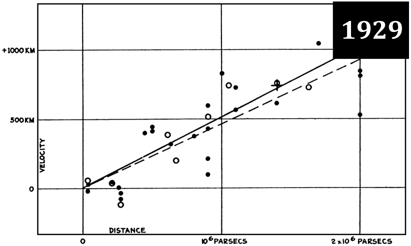
II. Reheating: A Critical Epoch
$\Rightarrow$ The necessity, importance of reheating. A big unknown of cosmic evolution

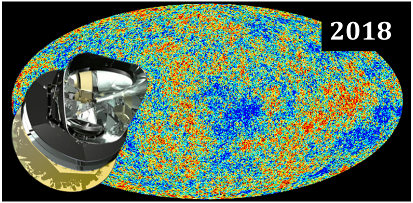
III. Realistic Reheating Scenarios
$\Rightarrow$ Multiple fields with non-canonical kinetic terms lead to drastically different behavior

The Big picture



Cosmology: modern tools for age-old questions





Big Bang: a win at a cost

The universe was in a **hot dense state** in the past, allowing for **Big Bang Nucleosynthesis (BBN)**. It then **expanded and cooled**, forming stars, planets and galaxies.

$\Downarrow$

The Cosmic Microwave Background radiation of

$$T \sim 2.7 \pm \mathcal{O}(10^{-5})\, K$$

was **emitted by an infant universe**.

$\Downarrow$ $\Downarrow$

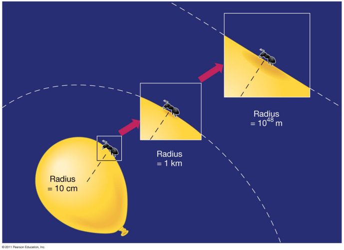
- **Horizon problem:** Separate regions of the universe have the same temperature without being in contact in the past.
- **Flatness problem:** If the universe did not start as extremely flat, it would become more curved as time progressed.

Inflation: two birds with one ... scalar field

PHYSICAL REVIEW D VOLUME 23, NUMBER 2 15 JANUARY 1981

Inflationary universe: A possible solution to the horizon and flatness problems

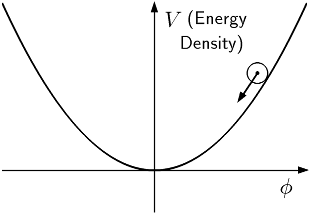
Alan H. Guth*
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305
(Received 11 August 1980)



A scalar field called the "inflaton", slowly rolling down its potential.

$$\ddot{\phi} + 3H\dot{\phi} - \frac{1}{a^2(t)}\nabla^2\phi + \frac{\partial V}{\partial \phi} = 0$$

The motion is **dominated by a drag coefficient** caused by the expansion of the universe $H(t) = \frac{\dot{a}(t)}{a(t)}$.

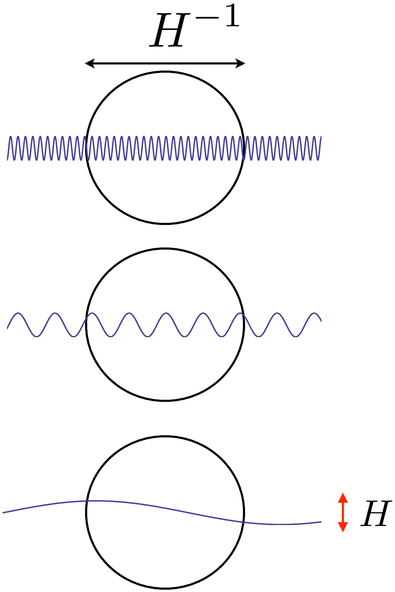


Bonus: fluctuations

fluctuations are always present in **quantum fields** and in the **metric** itself

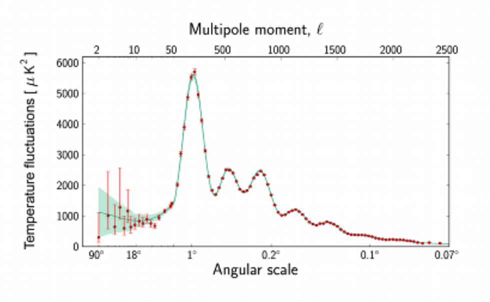
they are stretched **beyond the horizon** by the expansion

they freeze out and become **classical fluctuations**: density perturbations & gravitational waves



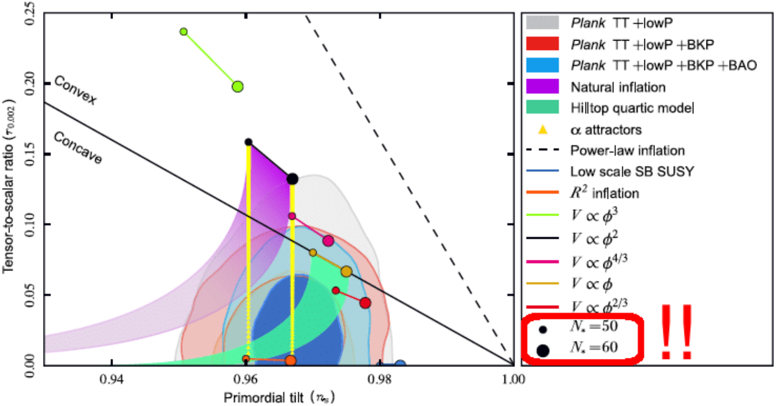
(Simple) Single field inflation:

- **Solves** horizon, flatness, monopole **problems**
- **Explains fluctuations** as stretched quantum mechanical perturbations
- Predicts a **nearly scale invariant** spectrum (of tunable amplitude)
- Predicts **Gaussian** perturbations



- Spectral index not flat by 5σ
- Spectral index running is small
- $|f_{NL}| \lesssim \mathcal{O}(1)$

Hints from the sky



Many models with **different motivation**.

⇓

They all share the **same uncertainty**.

I. The Big Bang and Cosmic Inflation

$\Rightarrow$ From the Big Bang to inflation: successes and prospects

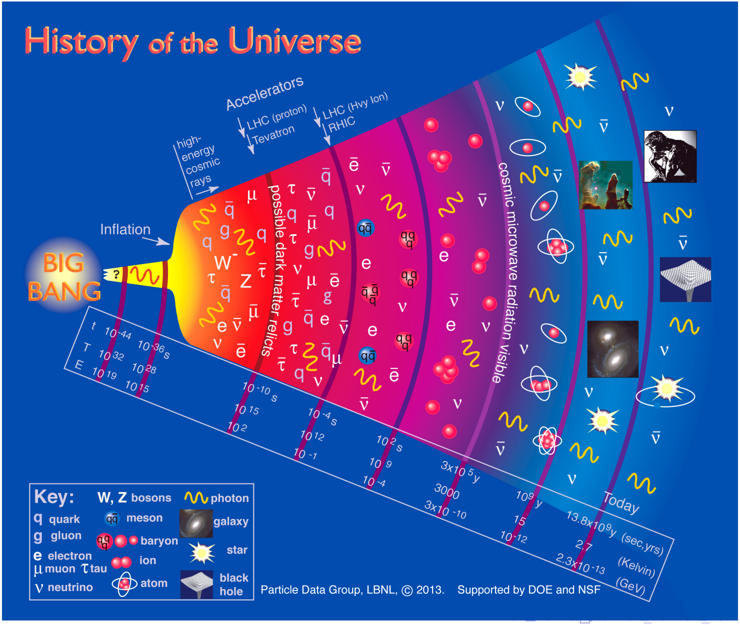
II. Reheating: A Critical Epoch

$\Rightarrow$ The necessity, importance of reheating. A big unknown of cosmic evolution

III. Realistic Reheating Scenarios

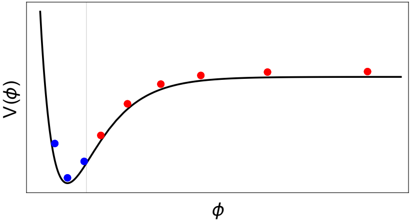
$\Rightarrow$ Multiple fields with non-canonical kinetic terms lead to drastically different behavior

Big picture reminder

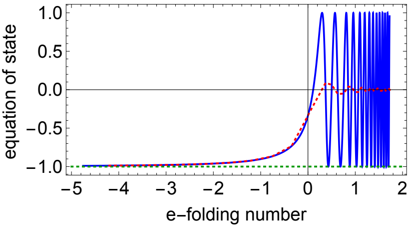


Inflation must end

- The inflaton rolls on a flat potential.
- The inflaton oscillates.



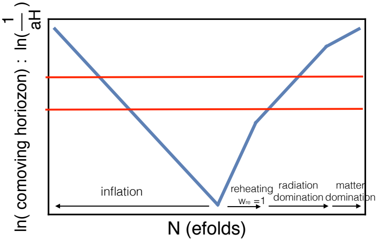
- During inflation: $p \simeq -\rho$
- After inflation: $V(\phi) \approx \frac{1}{2}m^2\phi^2$ and $p \to 0$

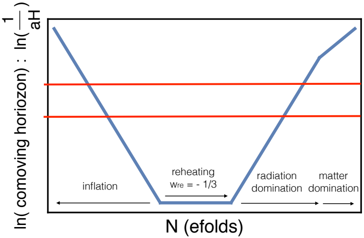


The inflaton **must** transfer its energy to radiative degrees of freedom, setting the stage for BBN.

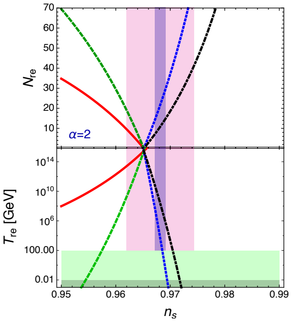
This process is called **reheating**.

Reheating effects





Cook et al. 2015



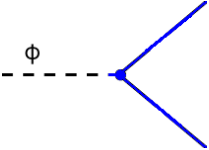
The **reheating history** connects the times of horizon exit & re-entry of perturbations $\Rightarrow$ **shifts CMB observables**

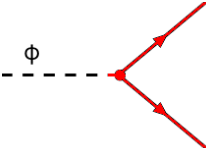
"The value of $\mathcal{N}_*$ is not well constrained and depends on unknown details of reheating"

CMB-S4 Science Book, 2016

Introduce couplings $\mu\phi\chi^2$ or $h\phi\bar{\psi}\psi$ and assume $m_\phi \gg m_\chi, m_\psi$

$$\Gamma_{\phi\to\chi\chi} = \frac{\mu^2}{8\pi m_\phi} \qquad \Gamma_{\phi\to\bar{\psi}\psi} = \frac{h^2 m_\phi}{8\pi}$$





We can describe the decays as an extra friction term

$$\ddot{\phi} + 3H\dot{\phi} + \Gamma\dot{\phi} + m^2\phi = 0$$

Reheating occurs for $\Gamma > H$.

Parametric resonance: preheating

Bose enhancement changes the game. Take $\mathcal{L} \subset -\frac{1}{2} g \phi^2 \chi^2$

$$\ddot{\chi}_k + 3H\dot{\chi}_k + \left(\frac{k^2}{a^2} + 2g\phi^2 \right) \chi_k = 0$$

Neglect the expansion $(H = 0)$ and take $\phi(t) = \Phi_0 sin(mt)$

$$\boxed{\ddot{\chi}_k + \left[k^2 + g\Phi_0^2 \sin^2(mt)\right] \chi_k = 0}$$

An equation of the form $\dot{x} = A(t)x$, where $A(t)$ is **periodic**, $A(t+T) = A(t)$, has solutions of the form

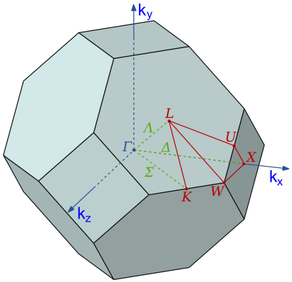
$$x(t) = c_1 P(t) e^{\mu t} + c_2 P(t) e^{-\mu t}$$

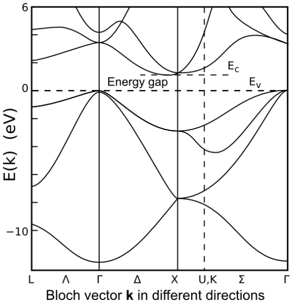
where μ is called the **Floquet exponent**.

In crystals the potential is **periodic in space** $V(\vec{x}) = V(\vec{x} + \vec{x}_0)$

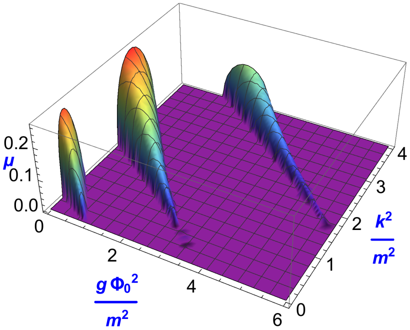
$$\Downarrow$$

The Schroedinger equation has solutions $\psi(x) \propto e^{\mu x}$
leading to **bands** and **band-gaps**

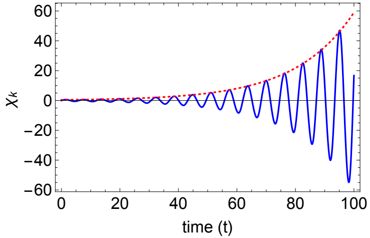




Floquet charts

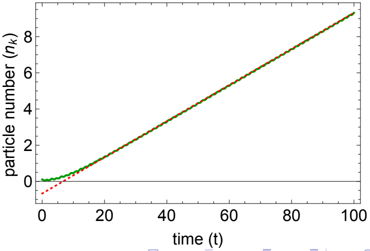


Exponentially growing mode-function



We can read off the regions where the Floquet chart leads to amplification $\chi_k(t) \sim e^{\mu_k t}$.

and particle number



Kofman, Linde, Starobinsky [9704452]

Non-adiabaticity

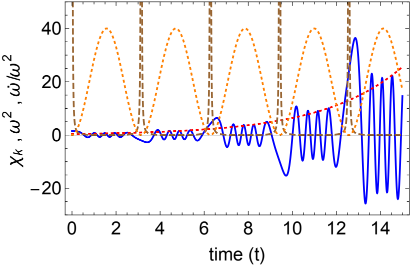
$$\ddot{\chi} + \omega^2(t)\chi = 0$$

For $\omega^2 \gg 1/T$ and $\frac{\dot{\omega}}{\omega^2} \ll 1$

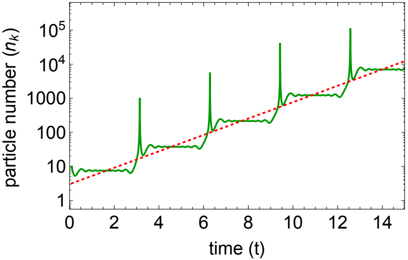
$$\chi \simeq \frac{1}{\sqrt{\omega}} \exp\left[\pm i \int \omega dt\right]$$

When the **adiabaticity condition** is violated, we get a sudden **burst of particle production**.

Adiabaticity violation:

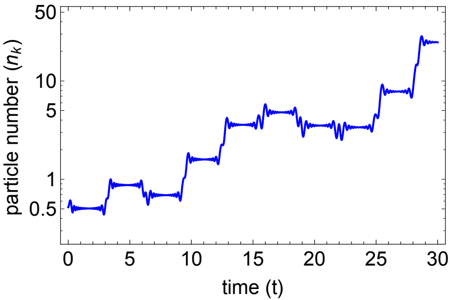


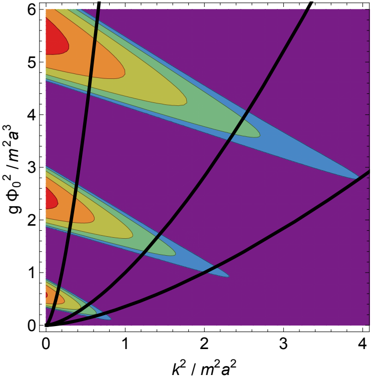
Particle production at $\dot{\omega}/\omega^2 \gg 1$



$$\ddot{\chi}_k + 3H\dot{\chi}_k + \left(\frac{k^2}{a^2} + 2g\phi^2\right)\chi_k = 0$$

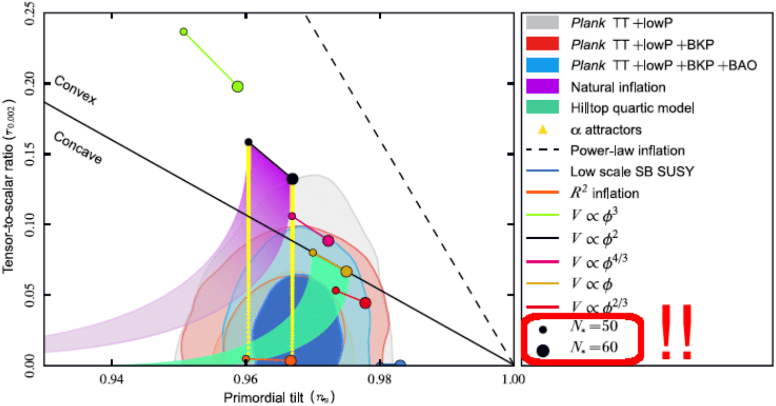
Stochastic resonance:





Quantitative differences and qualitative similarities
$\Rightarrow$ **Floquet theory is still useful**

Anticipating upcoming data



The time of horizon-exit is being constrained, begging for a **better understanding of reheating**.

Outline

I. The Big Bang and Cosmic Inflation

$\Rightarrow$ From the Big Bang to inflation: successes and prospects

II. Reheating: A Critical Epoch

$\Rightarrow$ The necessity, importance of reheating. A big unknown of cosmic evolution

III. Realistic Reheating Scenarios

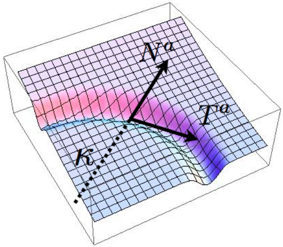
$\Rightarrow$ Multiple fields with non-canonical kinetic terms lead to drastically different behavior

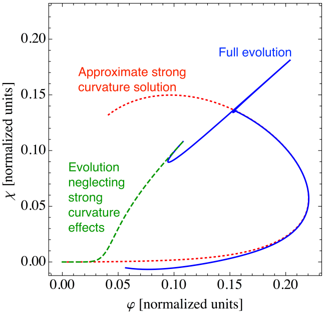
General Model-building

At high energies, we expect

- multiple fields and
- more complicated couplings, e.g. $\mathcal{L} \subset f(\phi)(\partial\chi)^2 + \tilde{f}(\chi)(\partial\phi)^2$

leading to interesting inflationary dynamics.





During inflation, field-space features received significant attention (van Tent et al 2003, Achucarro et al 2010, ...).

Recent **novel trajectories** supported by field-space curvature reveal interesting connections to the Swampland program (a **whole other talk** I'd love to give!)

"Family tree" of this work

inflation (80's)

$\Downarrow$

preheating (late 90's)

$\Downarrow$

field-space effects (2000's - ...)

$\Downarrow$

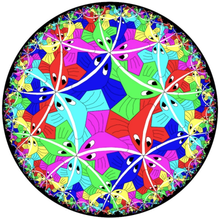
Higgs inflation (2008) $+$ **α-attractors (2010's)**

$\Downarrow$

Field-space effects in multi-field preheating,
focusing on Higgs(-like) inflation & α-attractors

Hyperbolic manifolds & α-attractors

Hyperbolic space on an "Escher disk"



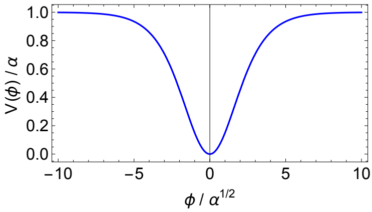
$$\mathcal{L} = \frac{\alpha}{2}\frac{(\partial r)^2 + r^2(\partial\theta)^2}{(1-r^2)^2} + V(r,\theta)$$
$$= \frac{1}{2}(\partial\Phi)^2 + \tilde{V}(\Phi) + [\ldots\theta\ldots]$$

where

$$V(r) = \frac{1}{2}m^2r^2 + \ldots \Rightarrow \tilde{V}(\Phi) \sim \tanh^2(\Phi/\sqrt{\alpha}) + \ldots$$

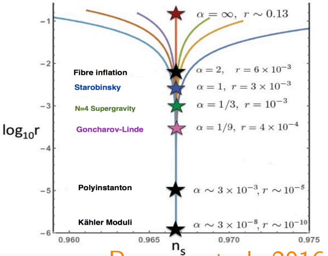
α-attractors lead to "universal" predictions

$$n_s \simeq 1 - \frac{2}{N}, \quad r \simeq \frac{12\alpha}{N^2}$$



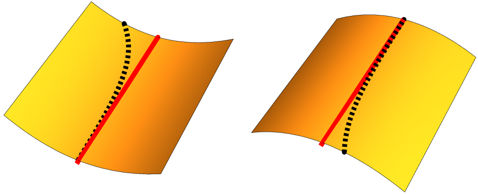
α-attractors and geodesics

- **String theory** compactifications: Fibre inflation
- **Supergravity**, e.g. E- and T-model

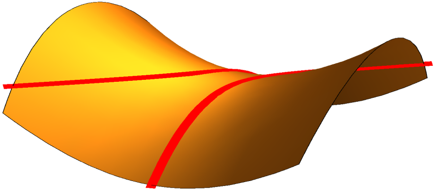


Burgess et al. 2016

"Usual" behavior for $m^2 > 0$ & $m^2 < 0$

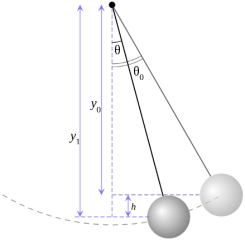


Geodesics diverge, similarly to a $m^2 < 0$



Effective Mass-squared: Ingredients

For motion **along a single-field attractor** ϕ, quantization is simple for the second field χ



$$\ddot{\chi}_k + 3H\dot{\chi}_k \left(\frac{k^2}{a^2} + m^2_{\text{eff},\chi} \right) \chi_k = 0$$

$$m^2_{\text{eff},\chi} \simeq m^2_{1,\chi} + m^2_{2,\chi}$$

$$m^2_{1,\chi} \equiv \mathcal{G}^{\chi K} \left(\mathcal{D}_\chi \mathcal{D}_K V \right) \longleftrightarrow \text{potential gradient – “traditional” mass}$$

$$m^2_{2,\chi} \equiv \frac{1}{2} \mathcal{R} \dot{\phi}^2 \longleftrightarrow \text{non-trivial field-space manifold}$$

α-attractors: the two-field T-model

Complex fields in supergravity lead to the **2-field Lagrangian**

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu\chi\partial^\mu\chi + e^{2b(\chi)}\partial_\mu\phi\partial^\mu\phi\right) - V(\phi,\chi)$$

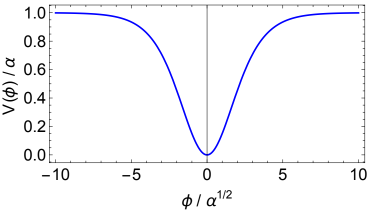
For single-field motion $\chi = 0$

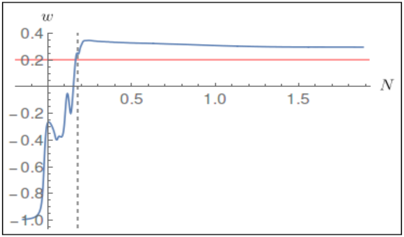
$$V(\phi,\chi=0) = \mu^2\alpha\left|\tanh(\phi/\sqrt{6\alpha})\right|^2$$

The **field-space Ricci scalar** is

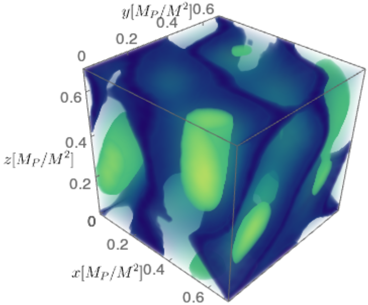
$$\mathcal{R} = -\frac{4}{3\alpha}$$

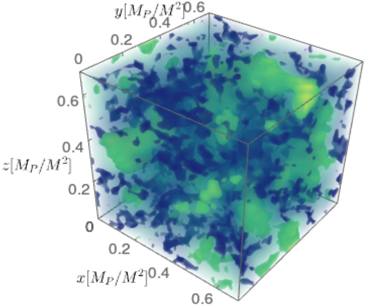
Smaller $\alpha \Rightarrow$ highly curved manifold



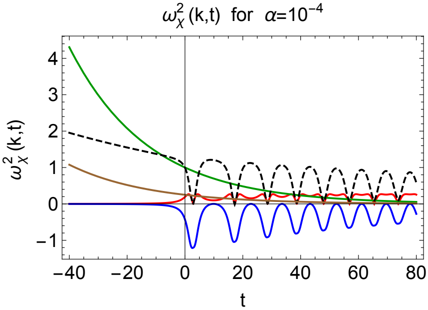


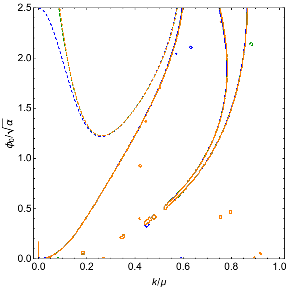
Lattice simulations
(Krajewski et al, 2018)
showed
very efficient preheating
for $\alpha \ll 1$





Effective frequency





$$m_{2,\chi}^2 = \frac{1}{2} R \left(\frac{d\phi}{dt}\right)^2 \propto -\frac{1}{\alpha} \times \left(\frac{\sqrt{\alpha}}{\mathcal{O}(1)}\right)^2 = -\mathcal{O}(1)$$

During each background oscillation
the χ field undergoes **tachyonic amplification**.

$\Rightarrow$ **Preheating is faster for larger curvature**.

Higgs(-like) inflation

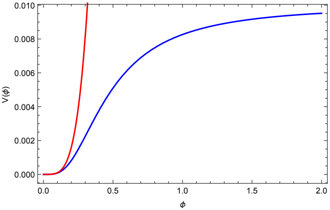
The Standard Model Higgs boson as the inflaton

Fedor Bezrukov [a,b], Mikhail Shaposhnikov [a]

[a] *Institut de Théorie des Phénomènes Physiques, École Polytechnique Fédérale de Lausanne, CH-1015 Lausanne, Switzerland*
[b] *Institute for Nuclear Research of Russian Academy of Sciences, Prospect 60-letiya Oktyabrya 7a, Moscow 117312, Russia*

$$\mathcal{L} \subset \frac{1}{2} M_{\rm Pl}^2 R \to \frac{1}{2} M_{\rm Pl}^2 R + \xi H^\dagger H R \sim \frac{1}{2} M_{\rm Pl}^2 R + \xi h^2 R$$

The conformal transformation from the Jordan to the Einstein frame leads to a flat potential.



Multiple fields **necessarily** lead to

$$\mathcal{L} \subset \mathcal{G}_{IJ} \partial_\mu \phi^I \partial^\mu \phi^J \to \mathcal{R}_{\rm field-space}$$

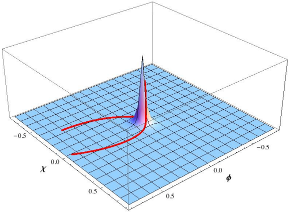
$$n_s \simeq 1 - \frac{2}{N}, \qquad r \simeq \frac{12}{N^2}$$

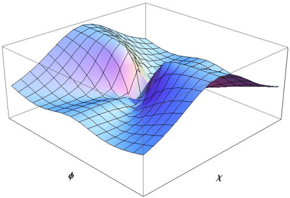
Higgs-like inflation

$$S_{\rm Jordan} = \int d^4x \sqrt{-\tilde{g}} \left[f(\phi^I)\tilde{R} - \frac{1}{2}\delta_{IJ}\tilde{g}^{\mu\nu}\partial_\mu\phi^I\partial_\nu\phi^J - \tilde{V}(\phi^I) \right]$$

$$\Downarrow \boxed{g_{\mu\nu}(x) \propto f(\phi^I(x))\, \tilde{g}_{\mu\nu}(x)} \Downarrow f(\phi^I) \subset \xi\phi^2$$

$$S_{\rm Einstein} = \int d^4x \sqrt{-g} \left[\frac{M_{\rm Pl}^2}{2} R - \frac{1}{2}\mathcal{G}_{IJ} g^{\mu\nu}\partial_\mu\phi^I\partial_\nu\phi^J - V(\phi^I) \right]$$





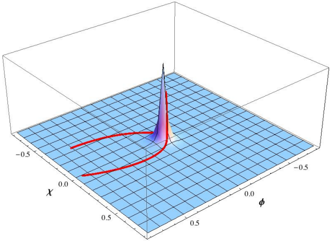
- $\frac{1}{2}\mathcal{G}_{IJ}\partial_\mu\phi^I\partial^\mu\phi^J$ leads to a **locally curved manifold**.
- **The potential has flat directions**, where inflation proceeds.

Effective Mass-squared reminder

$$\ddot{\chi}_k + 3H\dot{\chi}_k + \left(\frac{k^2}{a^2} + m^2_{\mathrm{eff},\chi}\right)\chi_k = 0 \qquad m^2_{\mathrm{eff},\chi} \simeq m^2_{1,\chi} + m^2_{2,\chi}$$

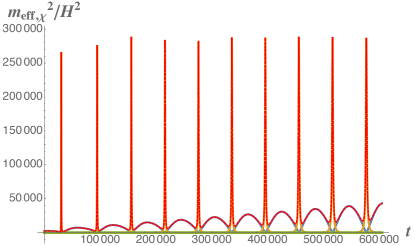
$$m^2_{1,\chi} \equiv \mathcal{G}^{\chi K}\left(\mathcal{D}_\chi \mathcal{D}_K V\right) \longleftrightarrow \text{potential gradient – “traditional” mass}$$

$$m^2_{2,\chi} \equiv \frac{1}{2}\mathcal{R}\dot{\phi}^2 \longleftrightarrow \text{non-trivial field-space manifold}$$



The field-space Ricci $\mathcal{R}$ “spikes” at the origin.

An "unusual" way for **adiabaticity violation** for $m^2_{2,\chi} \propto \mathcal{R}\dot{\phi}^2$



We define

$$\mathcal{A} \equiv \frac{\Omega'}{\Omega^2}$$

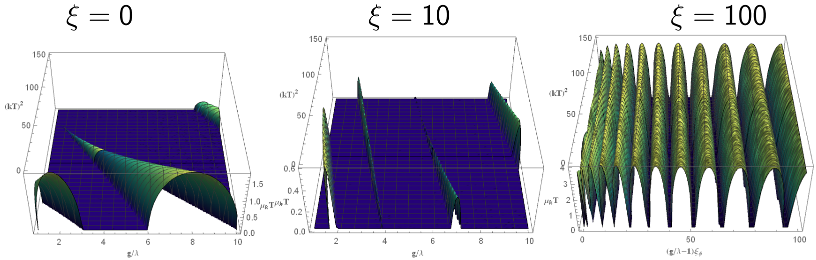
where

$$\Omega^2 = k^2 + a^2 m^2_{\text{eff},\chi}$$

Adiabaticity is violated for $\Omega' \gg \Omega^2$, rather than $\Omega \approx 0$.

A broad range of wavenumbers is excited $k \lesssim \xi H_{\text{end}}$

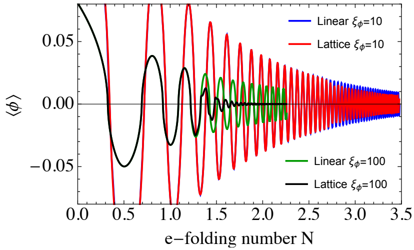
Linear analysis (VERY briefly)

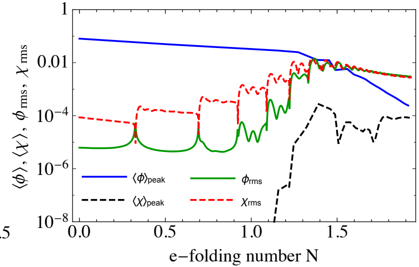


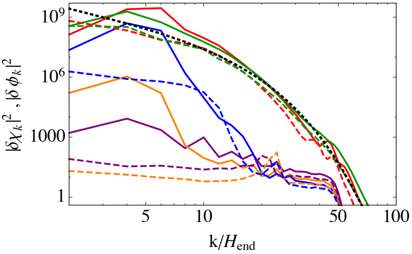
Dense instability bands hint at **efficient particle production**

Need for **lattice simulations**

Lattice results

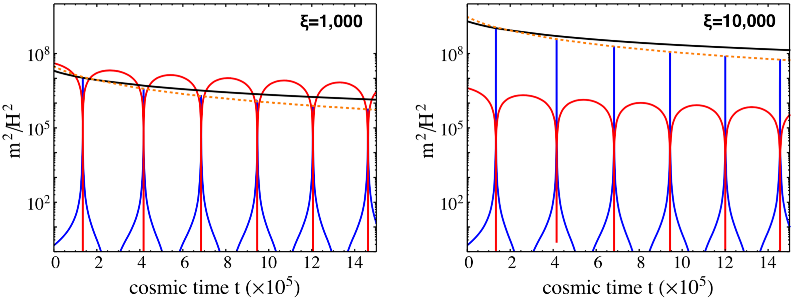






Non-minimal couplings quickly lead to a thermal radiation bath while preserving CMB predictions

Higgs inflation is a multi-field non-minimally coupled model with known SM couplings ⇒ the inflaton decays into W, Z bosons.

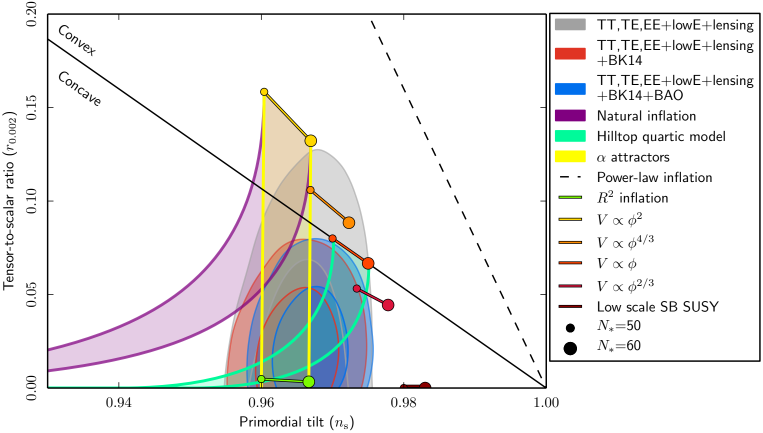


$$m_{\text{spike}} \sim \xi H_{\text{end}}$$

$$m_B \sim \frac{10^5}{\sqrt{\xi}} H_{\text{end}}$$

For $\xi \gtrsim 10^3$ preheating completes within **ONE** oscillation

Thank you . . .



Understanding **preheating** in major plateau models
reduces theoretical **error-bars** of the $n_s - r$ plot
& allows for **comparison of Higgs inflation models**

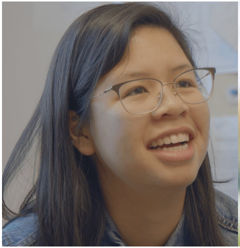
Some References

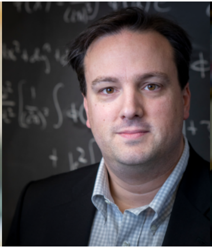
- DeCross, Kaiser, Prabhu, Prescod-Weinstein & **EIS**, arXiv: 1510.08553 [astro-ph.CO], 1610.08868 [astro-ph.CO], 1610.08916 [astro-ph.CO]
- Krajewski, Turzynski & Wieczorek, arXiv: 1801.01786 [astro-ph.CO]
- Ema, Jinno, Mukaida & Nakayama, arXiv: 1609.05209 [hep-ph]
- Iarygina, **EIS** , Wang & Achucarro, arXiv: 1810.02804 [astro-ph.CO], 2005.00528 [astro-ph.CO]
- **EIS** & van de Vis, arXiv: 1810.01304 [hep-ph]
- Nguyen, van de Vis, **EIS**, Giblin & Kaiser, arXiv:1905.12562 [hep-ph], 2005.00433 [astro-ph.CO]
- Babichev, Gorbunov, Ramazanov & Reverberi, arXiv: 2006.02225 [hep-ph]
- Ema, Jinno, Nakayama & van de Vis, arXiv: 2102.12501 [hep-ph]

Acknowledgments

Some of my fantastic preheating collaborators!

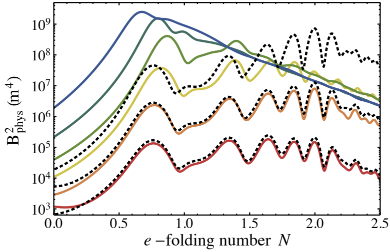


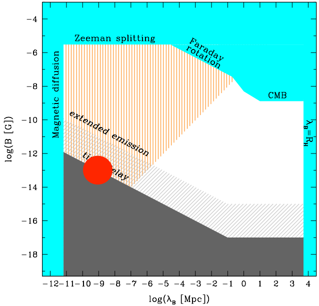




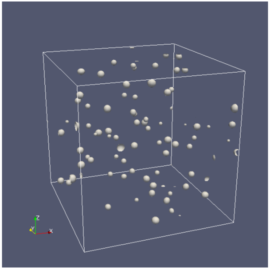

More preheating examples

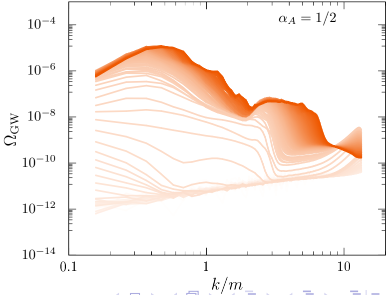
Helical magnetic fields





Oscillons & Gravitational Waves





The Big Bang theory says nothing about

- **what banged,**
- **why it banged,**
- **or what happened before it banged.**

Standard Single Field Slow Roll

Action:

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\rm pl}^2}{2}R + \frac{1}{2}(\partial\phi)^2 - V(\phi)\right]$$

Equation of motion:

$$\ddot{\phi} + 3H\dot{\phi} - e^{-2Ht}\nabla^2\phi = -\frac{\partial V}{\partial\phi}$$

Slow Roll conditions:

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \sim \left(\frac{V'}{V}\right)^2 \ll 1 \quad , \quad |\eta| \equiv \left|\epsilon - \frac{\ddot{\phi}}{H\dot{\phi}}\right| \sim \left|\frac{V''}{V}\right| \ll 1$$

Observables:

$$n_s = 1 - 6\epsilon + 2\eta \qquad r = 16\epsilon$$

Time delay formalism a la Guth & Pi

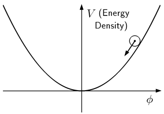
Take the case of a single scalar field. If the field has quantum fluctuations $\delta\phi(\vec{x},t)$ on top of a classical trajectory $\phi_0(t)$, then one can write

$$\phi(\vec{x},t) = \phi_{\rm cl}(t) + \delta\phi(\vec{x},t) = \phi_{\rm cl}(t) - \delta\tau(\vec{x})\dot{\phi}_{\rm cl}(t)$$
$$\Rightarrow \boxed{\phi(\vec{x},t) = \phi_{\rm cl}(t - \delta\tau(\vec{x}))}$$

Intuitively inflation ends on different times at different places.

The time delay field $\delta\tau(\vec{x})$ is given by

$$\boxed{\delta\tau(\vec{x}) = \frac{\delta\phi(\vec{x},t)}{\dot{\phi}_{\rm cl}(t)}}$$



and is related to the density perturbations or temperature fluctuations

$$\frac{\delta T(\vec{x})}{T} = \frac{\delta\rho(\vec{x})}{\rho} \propto \delta\tau(\vec{x})$$

General Model set-up

Action

$$\mathcal{S} = \int d^3x dt \sqrt{-g} \left[\frac{M_{\rm pl}^2}{2} R - \frac{1}{2} \mathcal{G}_{IJ} g^{\mu\nu} \partial_\mu \phi^I \partial_\nu \phi^J - V(\phi^I) \right]$$

Background fields:

$$\boxed{\mathcal{D}_t \dot{\phi}^I + 3H\dot{\phi}^I + \mathcal{G}^{IK} V_{,K} = 0}$$

where $\mathcal{D}_t \dot{\phi}^I \equiv \ddot{\phi}^I + \Gamma^I_{JK} \dot{\phi}^J \dot{\phi}^K$

Fluctuations:

$$\boxed{\ddot{Q}^I_k + 3H\dot{Q}^I_k + \left[\frac{k^2}{a^2} \delta^I_J + \mathcal{M}^I_J \right] Q^I_k = 0}$$

$$\hat{X}^{\phi}(x^{\mu}) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[\left(v_k e_1^{\phi} \hat{b}_{\mathbf{k}} + w_k e_2^{\phi} \hat{c}_{\mathbf{k}} \right) e^{i\mathbf{k}\cdot\mathbf{x}} + \right.$$
$$\left. + \left(v_k^* e_1^{\phi} \hat{b}_{\mathbf{k}}^{\dagger} + w_k^* e_2^{\phi} \hat{c}_{\mathbf{k}}^{\dagger} \right) e^{-i\mathbf{k}\cdot\mathbf{x}} \right],$$
$$\hat{X}^{\chi}(x^{\mu}) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[\left(y_k e_1^{\chi} \hat{b}_{\mathbf{k}} + z_k e_2^{\chi} \hat{c}_{\mathbf{k}} \right) e^{i\mathbf{k}\cdot\mathbf{x}} + \right.$$
$$\left. + \left(y_k^* e_1^{\chi} \hat{b}_{\mathbf{k}}^{\dagger} + z_k^* e_2^{\chi} \hat{c}_{\mathbf{k}}^{\dagger} \right) e^{-i\mathbf{k}\cdot\mathbf{x}} \right]$$

$$\left(v_k'' + \Omega_{(\phi)}^2 v_k \right) e_1^{\phi} = -a^2 \mathcal{M}^{\phi}{}_{\chi} y_k e_1^{\chi},$$
$$\left(w_k'' + \Omega_{(\phi)}^2 w_k \right) e_2^{\phi} = -a^2 \mathcal{M}^{\phi}{}_{\chi} z_k e_2^{\chi},$$
$$\left(y_k'' + \Omega_{(\chi)}^2 y_k \right) e_1^{\chi} = -a^2 \mathcal{M}^{\chi}{}_{\phi} v_k e_1^{\phi},$$
$$\left(z_k'' + \Omega_{(\chi)}^2 z_k \right) e_2^{\chi} = -a^2 \mathcal{M}^{\chi}{}_{\phi} w_k e_2^{\phi},$$

Quantizing the fluctuations

For motion **along a single-field attractor** ϕ,
quantization is simple for the second field χ

$$\hat{Q}^{\phi}(x^{\mu}) = \sqrt{\mathcal{G}^{\phi\phi}}\, a(t) \int \frac{d^3k}{(2\pi)^{3/2}} \left[v_k \hat{b}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + c.c. \right]$$

$$\hat{Q}^{\chi}(x^{\mu}) = \sqrt{\mathcal{G}^{\chi\chi}}\, a(t) \int \frac{d^3k}{(2\pi)^{3/2}} \left[z_k \hat{c}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + c.c. \right]$$

Re-write as a "harmonic" oscillator

$$\delta\tilde{\phi}_k'' + \Omega^2_{(\phi)}(k,\tau)\, \delta\tilde{\phi}_k \simeq 0$$

$$\delta\tilde{\chi}_k'' + \Omega^2_{(\chi)}(k,\tau)\, \delta\tilde{\chi}_k \simeq 0$$