放射性测量中的统计学
Statistics in Radiation Measurement

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Outline

- Statistical distribution of nuclear decay and radioactivity counts
- Statistical error in radiation measurement
- Time distribution of radioactivity particles and signals

2

Statistical distribution of nuclear decay and radioactivity counts

3

Random Variable

- random variable X is
  - a real function whose domain is the probability space
  - the set $\{X < x\}$ is an event for any real number x
  - the probability $P(X = +\infty)=0$, and $P(X = -\infty)=0$
- discrete random variable
- continuous random variable

4

Distribution Function $D(x)$ and Probability Density Function $p(x)$

- $D(x) = P(X \le x)$
- For continuous random variable,
  - $D(x) = \int_{-\infty}^{x} p(\xi)\,d\xi$
  - $p(x) = D'(x)$
- Obviously,
  - $D(x)$ is monotone increasing function.
  - $p(x) \ge 0$
  - $\int_{-\infty}^{+\infty} p(x)dx = 1$

5

Expectation value, Variance Standard Deviation, Relative Deviation

- For continuous random variable X, its expectation value $E(X) = \int_{-\infty}^{+\infty} p(x)xdx$
- and its variance

$$\sigma^2 = E\{[X - E(X)]^2\} = E(X^2) - [E(X)]^2$$

- standard deviation

$$\sigma = \sqrt{E\{[X - E(X)]^2\}} = \sqrt{E(X^2) - [E(X)]^2}$$

- relative deviation

$$\nu = \frac{\sigma}{E(X)}$$

6

Exercise

- Please give the distribution function, expectation value, and variance of discrete random variable X. Suppose X may be x_i with probability $p(x_i)$.

Binomial Distribution

- Suppose event A occur with probability p in single test, the probability that event A occurs n times if repeating the test N times becomes (Let random variable X be the times that event A occurs)：

$$P(X=n)=\frac{N!}{(N-n)!n!}p^n(1-p)^{N-n}$$

Exercise: calculate $E(X)$ and σ of binomial distribution step by step.

Binomial Distribution (cont.)

$$E(X)=\sum_{n=0}^{N} n\cdot P(X=n)$$

$$=\sum_{n=0}^{N} n\cdot\frac{N!}{(N-n)!n!}p^n(1-p)^{N-n}=Np$$

$$\sigma^2=E(X^2)-[E(X)]^2$$

$$=\sum_{n=0}^{N} n^2\frac{N!}{(N-n)!n!}p^n(1-p)^{N-n}-(Np)^2$$

$$=N(N-1)p^2+Np-(Np)^2$$

$$=Np(1-p)$$

Example

- Suppose N radioactive nuclei the decay constant of which is λ s^{-1}， estimate the probability that n nuclei decay in t seconds.
  - ➢Let $\Delta t = t/k$, where $k\to\infty$, $\Delta t\to 0$.
  - ➢In a twinkling Δt, the probability that a nucleus decay is $\lambda\Delta t$, the probability that not decay is $1-\lambda\Delta t$.
  - ➢In time t, the probability that a nucleus do not decay is $(1-\lambda\Delta t)^k$

$$\lim_{k\to\infty}(1-\lambda\Delta t)^k=\lim_{k\to\infty}(1-\lambda\frac{t}{k})^k=e^{-\lambda t}$$

Example (cont.)

- ➢so, in time t, the probability that a nucleus decay is $1 - e^{-\lambda t}$.
- ➢ Let random variable X be the number of decayed nuclei in time t, then

$$P(X=n)=\frac{N!}{(N-n)!n!}(1-e^{-\lambda t})^n(e^{-\lambda t})^{N-n}$$

$$E(X)=N(1-e^{-\lambda t})$$

$$\sigma^2=N(1-e^{-\lambda t})e^{-\lambda t}$$

Poisson Distribution

- For binomial distribution

$$P(X=n)=\frac{N!}{(N-n)!n!}p^n(1-p)^{N-n}$$

- Let $m=Np$, when $N\to\infty$, $p<<1$时，

$$\lim_{N\to\infty}P(X=n)=\frac{N!}{(N-n)!n!}(\frac{m}{N})^n(1-\frac{m}{N})^{N-n}$$

$$=\lim_{N\to\infty}\frac{N!}{(N-n)!N^n}\frac{m^n}{n!}(1-\frac{m}{N})^N(1-\frac{m}{N})^{-n}$$

$$=\lim_{N\to\infty}\left[(1-\frac{1}{N})(1-\frac{2}{N})\cdots(1-\frac{n-1}{N})\right]\frac{m^n}{n!}(1-\frac{m}{N})^N(1-\frac{m}{N})^{-n}$$

$$=\frac{m^n}{n!}e^{-m}$$

Poisson distribution

Exercise: calculate $E(X)$ and σ of Poisson distribution step by step.

Poisson Distribution (cont.)

$$E(X)=\sum_{n=0}^{\infty} n\cdot P(X=n)=\sum_{n=0}^{\infty} n\cdot \frac{m^n}{n!}e^{-m}$$
$$=m\sum_{n=1}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}=m$$
$$\sigma^2=E(X^2)-[E(X)]^2=\sum_{n=0}^{\infty} n^2\cdot\frac{m^n}{n!}e^{-m}-m^2$$
$$=m\sum_{n=1}^{\infty}(n-1)\cdot\frac{m^{n-1}}{(n-1)!}e^{-m}+m\sum_{n=1}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}-m^2$$
$$=m$$

13

Example

- Suppose N ($N\rightarrow\infty$) radioactive nuclei the decay constant of which is λ s^{-1}， estimate the probability that n nuclei decay in t seconds.

$$p=1-e^{-\lambda t}, m=Np, P(X=n)=\frac{m^n}{n!}e^{-m}$$
$$E(X)=m=Np=N(1-e^{-\lambda t})$$
$$\sigma^2=m=Np=N(1-e^{-\lambda t})$$

14

Binomial Distribution VS Poisson Distribution

	Binomial Distribution	**Poisson Distribution**
$P(X=n)$	$\frac{N!}{(N-n)!n!}(1-e^{-\lambda t})^n(e^{-\lambda t})^{N-n}$	$\frac{m^n}{n!}e^{-m}, m=N(1-e^{-\lambda t})$
$E(X)$	$N(1-e^{-\lambda t})$	$N(1-e^{-\lambda t})$
σ^2	$N(1-e^{-\lambda t})e^{-\lambda t}$	$N(1-e^{-\lambda t})$

15

Exercise

- There is a source containing N radioactive nuclei, the decay constant of which is λ s^{-1},
  - estimate its activity A,
  - and calculate the expectation number of decayed nuclei in time t from activity A.
  - compare the result with that from Poisson distribution, and explain the difference.

16

Exercise (solution)

- $A=\lambda N$
- according to the definition of activity, the expectation number of decayed nuclei is $At=\lambda Nt$
- from Poisson distribution, $E(X)=N(1-e^{-\lambda t})$
- notice that when λt is very small, $1-e^{-\lambda t}\approx\lambda t$
- so, $E(X)\approx\lambda Nt$, which is the same as the result from activity.

17

Gaussian Distribution

- Random variable X follow Gaussian distribution if its probability density function becomes

$$p(x)=\frac{1}{\sqrt{2\pi}\cdot\sigma}e^{-\frac{(x-m)^2}{2\sigma^2}}$$

- As we all know, its expectation is m and variance σ.

18

From Poisson to Gaussian

- Suppose a random variable X following a Poisson distribution with parameter m, X would follow a Gaussian distribution when $m>>1$.
- Usually we can use Gaussian distribution approximately instead of Poisson distribution when $m\geq 20$

19

Transforming and Combining Random Variable

- Linear transformations of random variable
  - $E(cX) = cE(X)$
  - $\sigma^2(cX) = c^2\sigma^2(X)$
- Sum and difference of independent random variable
  - $E(X \pm Y) = E(X) \pm E(Y)$
  - $\sigma^2(X \pm Y) = \sigma^2(X) + \sigma^2(Y)$
- Product of independent random variable
  - $E(XY) = E(X)E(Y)$

20

Transforming and Combining Random Variable

- the sum of several independent Poisson random variable also follows Poisson distribution.
- the sum of several independent Gaussian random variable also follows Gaussian distribution

21

Cascade Random Variable

- Let X be a random variable for test condition A, and Y for test condition B. X, Y are mutually independent.
- take a test under condition A, and obtain a possible value x of X
- take tests under condition B x times, and obtain x possible values $y_1,y_2,\ldots y_x$.
- so $z= y_1+y_2+\ldots+y_x$ is a possible value of random variable Z. Z is cascade random variable of X and Y.

22

- For cascade random variable Z, the following equation could be proved:

$$E(Z) = E(X)E(Y)$$

$$\sigma^2(Z) = [E(Y)]^2\sigma^2(X) + E(X)\sigma^2(Y)$$

$$v^2(Z) = \frac{\sigma^2(Z)}{[E(Z)]^2} = v^2(X) + \frac{1}{E(X)}v^2(Y)$$

when $E(X)$ is comparatively large, the contribution of $v^2(Y)$ for $v^2(Z)$ could be ignored.

23

Characteristic of Cascade Random Variable

- Cascade random variable Z of X and Y is also Bernoulli random variable, if X and Y are both Bernoulli random variable.
  - $p_z=p_xp_y$
- Cascade random variable Z of X and Y also follows Poisson distribution, if X follows Poisson distribution and Y follows Bernoulli distribution.
  - $m_z=m_xp_y$

24

Exercise

- A source contain two kinds of radioactive nuclei: ^{137}Cs with activity 1000 Bq and ^{60}Co with activity 10000 Bq. Now a detector with efficiency of 50% is used to record the γ rays emitted by the source 1 minutes.
  - Estimate the expectation record result and its statistic error.
  - give the distribution of the record result of several repeating experiments.

25

Statistical error in radiation measurement

26

statistical error of radiation measurement data

- Usually, particle count follows Poisson distribution, so $\sigma^2 = m$
- When m is comparatively large,

$$\sigma^2 = m \approx \bar{n} \approx n_i$$

- $\bar{n} = \sum_{i=1}^{k} n_i / k$, where n_i is the result of the i-th measuring
- sample standard deviation:

$$\sigma_s = \sqrt{\frac{1}{k-1}\sum_{i=1}^{k}(n_i - \bar{n})^2}$$

27

- Statistical error is the main error in radiation measurement. Considering only statistical error, the result of a single measuring could be expressed as :

$$n_i \pm \sigma = n_i \pm \sqrt{n_i}$$

- It means the result of any single measuring in same condition would be in $(n_i - \sqrt{n_i}, n_i + \sqrt{n_i})$ with probability of 68.3%.
- Its relative deviation is

$$\nu = \frac{\sigma}{m} = \frac{\sqrt{m}}{m} = \frac{1}{\sqrt{m}} \approx \frac{1}{\sqrt{n_i}}$$

The larger the count is, the smaller the relative error is!

28

Error Propagation

- Independent random variable $X_1, X_2, ..., X_n$
- with standard deviation $\sigma_1, \sigma_2, ..., \sigma_n$,
- $Y=f(X_1, X_2, ..., X_n)$
- then

$$\sigma^2(Y) = \left(\frac{\partial Y}{\partial X_1}\right)^2 \sigma_1^2 + \left(\frac{\partial Y}{\partial X_2}\right)^2 \sigma_2^2 + \cdots + \left(\frac{\partial Y}{\partial X_n}\right)^2 \sigma_n^2$$

29

Example

	Standard Deviation	Relative Deviation
$Y=aX_1 \pm bX_2$	$\sqrt{(a\sigma_1)^2+(b\sigma_2)^2}$	$\frac{\sqrt{(a\sigma_1)^2+(b\sigma_2)^2}}{ax_1 \pm bx_2}$
$Y=X_1 * X_2$	$x_1x_2\sqrt{(\frac{\sigma_1}{x_1})^2+(\frac{\sigma_2}{x_2})^2}$	$\sqrt{(\frac{\sigma_1}{x_1})^2+(\frac{\sigma_2}{x_2})^2}$
$Y=X_1/X_2$	$\frac{x_1}{x_2}\sqrt{(\frac{\sigma_1}{x_1})^2+(\frac{\sigma_2}{x_2})^2}$	$\sqrt{(\frac{\sigma_1}{x_1})^2+(\frac{\sigma_2}{x_2})^2}$

30

Statistical Error of Count Rate

- N counts in time t, so count rate $n=N/t$.

$$\sigma(n)=\sqrt{\frac{\sigma^2(N)}{t^2}}=\sqrt{\frac{N}{t^2}}=\sqrt{\frac{n}{t}} \qquad \nu(n)=\frac{\sigma(n)}{n}=\frac{\sqrt{\frac{n}{t}}}{n}=\sqrt{\frac{1}{nt}}=\sqrt{\frac{1}{N}}$$

$$\text{count rate result}: n\pm\frac{n}{\sqrt{N}}$$

Relative deviation of count rate
1) relates to only total counts N;
2) equals that of total counts N.

31

Statistical Error of Average Counts for multiple measures

- k count results N_1, N_2, ..., N_k from k times measure, each in time t.
- Mean counts: $\bar{N}=\frac{1}{k}\sum_{i=1}^{k}N_i$
- Variance: $\sigma^2(\bar{N})=\frac{1}{k^2}\sum_{i=1}^{k}\sigma^2(N_i)=\frac{1}{k^2}\sum_{i=1}^{k}N_i=\frac{\bar{N}}{k}$
- relative deviation: $\nu(\bar{N})=\frac{\sigma(\bar{N})}{\bar{N}}=\frac{1}{\sqrt{k\bar{N}}}=\frac{1}{\sqrt{\sum_i N_i}}$
- average counts: $\bar{N}\pm\sqrt{\bar{N}/k}$

32

Exercise

- For the same condition on last slides, calculate the average count rate and its standard deviation as well as relative deviation.

A)measuring once in time kt
B)measuring k times and each in time t
would have the same relative deviation as long as the total counts are the same for two kinds of measurement.

33

Error of Net Count Rate (Background Subtracted)

- measuring background: N_b in t_b
- measuring sample: N_s in t_s
- net count rate: $n_0 = N_s/t_s - N_b/t_b$
- standard deviation:

$$\sigma(n_0)=\sqrt{\frac{N_s}{t_s^2}+\frac{N_b}{t_b^2}}=\sqrt{\frac{n_s}{t_s}+\frac{n_b}{t_b}}$$

$$\text{result}: n_0\pm\sigma(n_0)=(n_s-n_b)\left[1\pm\frac{1}{n_s-n_b}\sqrt{\frac{n_s}{t_s}+\frac{n_b}{t_b}}\right]$$

34

Combination of Measurement Result with Unequal Precision

- Considering k independent measuring, the i-th counts is N_i in time t_i.

$$\bar{n}=\frac{\sum_i W_i n_i}{\sum_i W_i} \quad W_i=\frac{\lambda^2}{\sigma_n^2} \quad n_i=\frac{N_i}{t_i} \quad \sigma_n^2=\frac{n_i}{t_i} \quad \lambda:\text{any contant}$$

- We can using $\lambda^2=\bar{n}\approx n_i$,then $W_i=t_i$.

$$\bar{n}=\frac{\sum_i t_i n_i}{\sum_i t_i}=\frac{\sum_i N_i}{\sum_i t_i}$$

35

Combination of Measurement Result with Unequal Precision (cont.)

$$\sigma(\bar{n})=\sqrt{\frac{\sum_i \sigma^2(N_i)}{\left(\sum_i t_i\right)^2}}=\sqrt{\frac{\sum_i N_i}{\left(\sum_i t_i\right)^2}}=\sqrt{\frac{\bar{n}}{\sum_i t_i}}$$

$$\nu(\bar{n})=\frac{\sigma(\bar{n})}{\bar{n}}=\sqrt{\frac{1}{\bar{n}\sum_i t_i}}=\sqrt{\frac{1}{\sum_i N_i}}$$

$$\text{result}: \bar{n}\pm\sqrt{\frac{\bar{n}}{\sum_i t_i}}$$

36

Optimization of Measuring time and conditions

- measuring background: N_b in t_b
- measuring sample: N_s in t_s
- net count rate: $n_0 = N_s/t_s - N_b/t_b$
- standard deviation:

$$\sigma(n_0) = \sqrt{\frac{N_s}{t_s^2} + \frac{N_b}{t_b^2}} = \sqrt{\frac{n_s}{t_s} + \frac{n_b}{t_b}}$$

Find the best t_s and t_b which lead to minimum $\sigma(n_0)$ when $T = t_s + t_b$ is fixed.

37

Solution

- Let

$$\frac{d}{dt_s}\left(\frac{n_s}{t_s} + \frac{n_b}{T - t_s}\right) = 0 \Rightarrow \frac{t_b}{t_s} = \sqrt{\frac{n_b}{n_s}}$$

$$t_b = \frac{\sqrt{n_b}}{\sqrt{n_s} + \sqrt{n_b}} T \qquad t_s = \frac{\sqrt{n_s}}{\sqrt{n_s} + \sqrt{n_b}} T$$

38

Time Distribution of Radiation Particles and Signals

39

Time Interval of Successive Nuclear Event

- A source of activity m emits a γ photon per nucleus decay. Give the distribution which the time interval T of successive γ emission follows.
- in time t, the numbers of γ photons follows Poisson distribution with parameter mt:

$$P_N(N = n) = \frac{(mt)^n}{n!} e^{-mt}$$

40

- $T<t$ means at least one γ photon was emitted in time t.

$$P_T(T < t) = \sum_{k=1}^{\infty} P_N(N = k) = 1 - P_N(N = 0) = 1 - e^{-mt}$$

$$p_T(t) = \frac{d}{dt} P_T(T < t) = me^{-mt} \, (t > 0)$$

The time interval of successive nuclear events follows exponential distribution.

$$E(t) = \frac{1}{m}, \sigma^2(t) = \frac{1}{m^2}$$

41

Question

- A detector has dead time τ. if the detector recorded n counts, estimate the counting loss.

42

Summary

- Nuclear events number follows Poisson distribution, And its time interval follows exponential distribution.
- For a Poisson distribution with parameter m, its expectation and variance are both m.
- The larger the total counts are recorded, the less error the result contains.

43

Homework

- P25: 1,2,3,4,5,6,10

44