

放射性测量中的统计学 Statistics in Radiation Measurement

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Outline

- Statistical distribution of nuclear decay and radioactivity counts
- Statistical error in radiation measurement
- Time distribution of radioactivity particles and signals

Statistical distribution of nuclear decay and radioactivity counts

Random Variable

- random variable X is
 - a **real function** whose domain is the probability space
 - the set $\{X < x\}$ is an **event** for any real number x
 - the **probability** $P(X = +\infty) = 0$, and $P(X = -\infty) = 0$
- **discrete** random variable
- **continuous** random variable

Distribution Function $D(x)$ and Probability Density Function $p(x)$

- $D(x) = P(X \leq x)$
- For continuous random variable,
 - $D(x) = \int_{-\infty}^x p(\xi) d\xi$
 - $p(x) = D'(x)$
- Obviously,
 - $D(x)$ is **monotone increasing** function.
 - $p(x) \geq 0$
 - $\int_{-\infty}^{+\infty} p(x) dx = 1$

Expectation value, Variance Standard Deviation, Relative Deviation

- For continuous random variable X , its **expectation value** $E(X) = \int_{-\infty}^{+\infty} p(x)x dx$
- and its **variance**

$$\sigma^2 = E\{[X - E(X)]^2\} = E(X^2) - [E(X)]^2$$
- **standard deviation**

$$\sigma = \sqrt{E\{[X - E(X)]^2\}} = \sqrt{E(X^2) - [E(X)]^2}$$
- **relative deviation**

$$\nu = \frac{\sigma}{E(X)}$$

Exercise

- Please give the distribution function, expectation value, and variance of **discrete random variable** X . Suppose X may be x_i with probability $p(x_i)$.



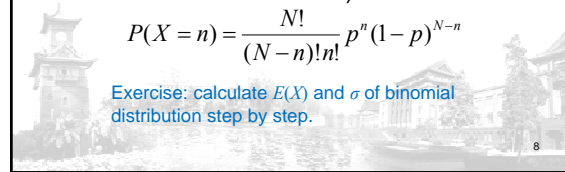
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Binomial Distribution

- Suppose event A occur with probability p in single test, the probability that event A occurs n times if repeating the test N times becomes (Let random variable X be the times that event A occurs):

$$P(X = n) = \frac{N!}{(N-n)!n!} p^n (1-p)^{N-n}$$

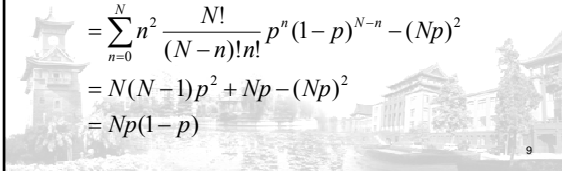
Exercise: calculate $E(X)$ and σ of binomial distribution step by step.



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Binomial Distribution (cont.)

$$\begin{aligned} E(X) &= \sum_{n=0}^N n \cdot P(X = n) \\ &= \sum_{n=0}^N n \cdot \frac{N!}{(N-n)!n!} p^n (1-p)^{N-n} = Np \\ \sigma^2 &= E(X^2) - [E(X)]^2 \\ &= \sum_{n=0}^N n^2 \frac{N!}{(N-n)!n!} p^n (1-p)^{N-n} - (Np)^2 \\ &= N(N-1)p^2 + Np - (Np)^2 \\ &= Np(1-p) \end{aligned}$$



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Example

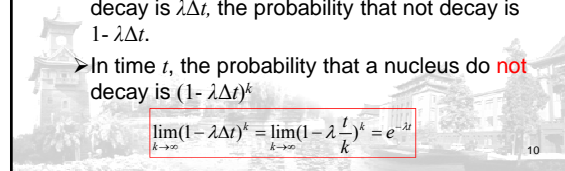
- Suppose N radioactive nuclei the decay constant of which is $\lambda \text{ s}^{-1}$, estimate the probability that n nuclei decay in t seconds.

➤ Let $\Delta t = t/k$, where $k \rightarrow \infty$, $\Delta t \rightarrow 0$.

➤ In a twinkling Δt , the probability that a nucleus decay is $\lambda \Delta t$, the probability that not decay is $1 - \lambda \Delta t$.

➤ In time t , the probability that a nucleus do **not** decay is $(1 - \lambda \Delta t)^k$

$$\lim_{k \rightarrow \infty} (1 - \lambda \Delta t)^k = \lim_{k \rightarrow \infty} (1 - \lambda \frac{t}{k})^k = e^{-\lambda t}$$



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Example (cont.)

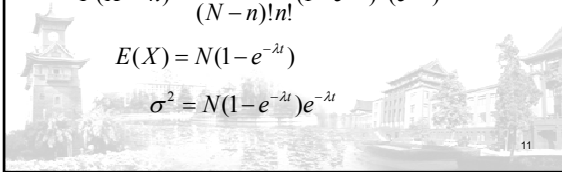
➤ so, in time t , the probability that a nucleus decay is $1 - e^{-\lambda t}$.

➤ Let random variable X be the number of decayed nuclei in time t , then

$$P(X = n) = \frac{N!}{(N-n)!n!} (1 - e^{-\lambda t})^n (e^{-\lambda t})^{N-n}$$

$$E(X) = N(1 - e^{-\lambda t})$$

$$\sigma^2 = N(1 - e^{-\lambda t})e^{-\lambda t}$$



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Poisson Distribution

- For binomial distribution

$$P(X = n) = \frac{N!}{(N-n)!n!} p^n (1-p)^{N-n}$$

- Let $m = Np$, when $N \rightarrow \infty$, $p \ll 1$ 时,

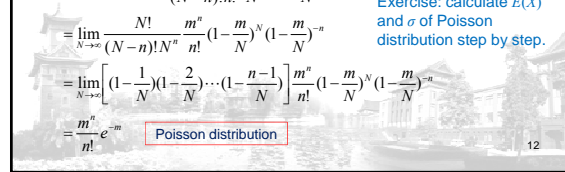
$$\lim_{N \rightarrow \infty} P(X = n) = \frac{N!}{(N-n)!n!} \left(\frac{m}{N}\right)^n \left(1 - \frac{m}{N}\right)^{N-n}$$

$$= \lim_{N \rightarrow \infty} \frac{N!}{(N-n)!N^n} \frac{m^n}{n!} \left(1 - \frac{m}{N}\right)^N \left(1 - \frac{m}{N}\right)^{-n}$$

$$= \lim_{N \rightarrow \infty} \left[\left(1 - \frac{1}{N}\right) \left(1 - \frac{2}{N}\right) \cdots \left(1 - \frac{n-1}{N}\right) \right] \frac{m^n}{n!} \left(1 - \frac{m}{N}\right)^N \left(1 - \frac{m}{N}\right)^{-n}$$

$$= \frac{m^n}{n!} e^{-m} \quad \text{Poisson distribution}$$

Exercise: calculate $E(X)$ and σ of Poisson distribution step by step.



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Poisson Distribution (cont.)

$$\begin{aligned}
 E(X) &= \sum_{n=0}^{\infty} n \cdot P(X=n) = \sum_{n=0}^{\infty} n \cdot \frac{m^n}{n!} e^{-m} \\
 &= m \sum_{n=1}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} = m \\
 \sigma^2 &= E(X^2) - [E(X)]^2 = \sum_{n=0}^{\infty} n^2 \cdot \frac{m^n}{n!} e^{-m} - m^2 \\
 &= m \sum_{n=1}^{\infty} (n-1) \cdot \frac{m^{n-1}}{(n-1)!} e^{-m} + m \sum_{n=1}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} - m^2 \\
 &= m
 \end{aligned}$$

Example

- Suppose N ($N \rightarrow \infty$) radioactive nuclei the decay constant of which is $\lambda \text{ s}^{-1}$, estimate the probability that n nuclei decay in t seconds.

$$\begin{aligned}
 p &= 1 - e^{-\lambda t}, m = Np, P(X=n) = \frac{m^n}{n!} e^{-m} \\
 E(X) &= m = Np = N(1 - e^{-\lambda t}) \\
 \sigma^2 &= m = Np = N(1 - e^{-\lambda t})
 \end{aligned}$$

Binomial Distribution VS Poisson Distribution

	Binomial Distribution	Poisson Distribution
$P(X=n)$	$\frac{N!}{(N-n)!n!} (1-e^{-\lambda t})^n (e^{-\lambda t})^{N-n}$	$\frac{m^n}{n!} e^{-m}, m = N(1 - e^{-\lambda t})$
$E(X)$	$N(1 - e^{-\lambda t})$	$N(1 - e^{-\lambda t})$
σ^2	$N(1 - e^{-\lambda t}) e^{-\lambda t}$	$N(1 - e^{-\lambda t})$

Exercise

- There is a source containing N radioactive nuclei, the decay constant of which is $\lambda \text{ s}^{-1}$,
 - estimate its activity A ,
 - and calculate the expectation number of decayed nuclei in time t from activity A .
 - compare the result with that from Poisson distribution, and explain the difference.

Exercise (solution)

- $A = \lambda N$
- according to the definition of activity, the expectation number of decayed nuclei is $At = \lambda Nt$
- from Poisson distribution, $E(X) = N(1 - e^{-\lambda t})$
- notice that when λt is very small, $1 - e^{-\lambda t} \approx \lambda t$
- so, $E(X) \approx \lambda Nt$, which is the same as the result from activity.

Gaussian Distribution

- Random variable X follow Gaussian distribution if its probability density function becomes

$$p(x) = \frac{1}{\sqrt{2\pi} \cdot \sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

- As we all know, its expectation is m and variance σ .

From Poisson to Gaussian

- Suppose a random variable X following a Poisson distribution with parameter m , X would follow a Gaussian distribution when $m \gg 1$.
- Usually we can use Gaussian distribution approximately instead of Poisson distribution when $m \geq 20$



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Transforming and Combining Random Variable

- Linear transformations of random variable
 - $E(cX) = cE(X)$
 - $\sigma^2(cX) = c^2\sigma^2(X)$
- Sum and difference of **independent** random variable
 - $E(X \pm Y) = E(X) \pm E(Y)$
 - $\sigma^2(X \pm Y) = \sigma^2(X) + \sigma^2(Y)$
- Product of **independent** random variable
 - $E(XY) = E(X)E(Y)$



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Transforming and Combining Random Variable

- the sum of several **independent** Poisson random variable also follows Poisson distribution.
- the sum of several **independent** Gaussian random variable also follows Gaussian distribution



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Cascade Random Variable

- Let X be a random variable for test condition A, and Y for test condition B. X , Y are mutually independent.
- take a test under condition A, and obtain a possible value x of X
- take tests under condition B x times, and obtain x possible values y_1, y_2, \dots, y_x .
- so $z = y_1 + y_2 + \dots + y_x$ is a possible value of random variable Z . Z is **cascade random variable** of X and Y .



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- For cascade random variable Z , the following equation could be proved:

$$E(Z) = E(X)E(Y)$$

$$\sigma^2(Z) = [E(Y)]^2 \sigma^2(X) + E(X) \sigma^2(Y)$$

$$\nu^2(Z) = \frac{\sigma^2(Z)}{[E(Z)]^2} = \nu^2(X) + \frac{1}{E(X)} \nu^2(Y)$$

when $E(X)$ is comparatively large, the contribution of $\nu^2(Y)$ for $\nu^2(Z)$ could be ignored.



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Characteristic of Cascade Random Variable

- Cascade random variable Z of X and Y is also Bernoulli random variable, if X and Y are both Bernoulli random variable.
 - $p_z = p_x p_y$
- Cascade random variable Z of X and Y also follows Poisson distribution, if X follows Poisson distribution and Y follows Bernoulli distribution.
 - $m_z = m_x p_y$



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Exercise

- A source contains two kinds of radioactive nuclei: ^{137}Cs with activity 1000 Bq and ^{60}Co with activity 10000 Bq. Now a detector with efficiency of 50% is used to record the γ rays emitted by the source 1 minutes.
 - Estimate the expectation record result and its statistic error.
 - Give the distribution of the record result of several repeating experiments.

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Statistical error in radiation measurement

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statistical error of radiation measurement data

- Usually, particle count follows Poisson distribution, so $\sigma^2 = m$
- When m is comparatively large,

$$\sigma^2 = m \approx \bar{n} \approx n_i$$
- $\bar{n} = \sum_{i=1}^k n_i / k$, where n_i is the result of the i -th measuring
- sample standard deviation:

$$\sigma_s = \sqrt{\frac{1}{k-1} \sum_{i=1}^k (n_i - \bar{n})^2}$$

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- Statistical error is the main error in radiation measurement. Considering only statistical error, the result of a single measuring could be expressed as :

$$n_i \pm \sigma = n_i \pm \sqrt{n_i}$$

- It means the result of any single measuring in same condition would be in $(n_i - \sqrt{n_i}, n_i + \sqrt{n_i})$ with probability of 68.3%.
- Its relative deviation is

$$\nu = \frac{\sigma}{m} = \frac{\sqrt{m}}{m} = \frac{1}{\sqrt{m}} \approx \frac{1}{\sqrt{n_i}}$$

The larger the count is, the smaller the relative error is!

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Error Propagation

- Independent random variable X_1, X_2, \dots, X_n
- with standard deviation $\sigma_1, \sigma_2, \dots, \sigma_n$,
- $Y = f(X_1, X_2, \dots, X_n)$
- then

$$\sigma^2(Y) = \left(\frac{\partial Y}{\partial X_1}\right)^2 \sigma_1^2 + \left(\frac{\partial Y}{\partial X_2}\right)^2 \sigma_2^2 + \dots + \left(\frac{\partial Y}{\partial X_n}\right)^2 \sigma_n^2$$

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Example

	Standard Deviation	Relative Deviation
$Y = aX_1 \pm bX_2$	$\sqrt{(a\sigma_1)^2 + (b\sigma_2)^2}$	$\frac{\sqrt{(a\sigma_1)^2 + (b\sigma_2)^2}}{ax_1 \pm bx_2}$
$Y = X_1 * X_2$	$x_1 x_2 \sqrt{\left(\frac{\sigma_1}{x_1}\right)^2 + \left(\frac{\sigma_2}{x_2}\right)^2}$	$\sqrt{\left(\frac{\sigma_1}{x_1}\right)^2 + \left(\frac{\sigma_2}{x_2}\right)^2}$
$Y = X_1 / X_2$	$\frac{x_1}{x_2} \sqrt{\left(\frac{\sigma_1}{x_1}\right)^2 + \left(\frac{\sigma_2}{x_2}\right)^2}$	$\sqrt{\left(\frac{\sigma_1}{x_1}\right)^2 + \left(\frac{\sigma_2}{x_2}\right)^2}$

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Statistical Error of Count Rate

- N counts in time t , so count rate $n=N/t$.

$$\sigma(n) = \sqrt{\frac{\sigma^2(N)}{t^2}} = \sqrt{\frac{N}{t^2}} = \sqrt{\frac{n}{t}} \quad \nu(n) = \frac{\sigma(n)}{n} = \frac{\sqrt{\frac{n}{t}}}{n} = \sqrt{\frac{1}{nt}} = \sqrt{\frac{1}{N}}$$

count rate result : $n \pm \frac{n}{\sqrt{N}}$

Relative deviation of count rate
1) relates to only total counts N ;
2) equals that of total counts N .

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Statistical Error of Average Counts for multiple measures

- k count results N_1, N_2, \dots, N_k from k times measure, each in time t .

- Mean counts: $\bar{N} = \frac{1}{k} \sum_{i=1}^k N_i$

- Variance: $\sigma^2(\bar{N}) = \frac{1}{k^2} \sum_{i=1}^k \sigma^2(N_i) = \frac{1}{k^2} \sum_{i=1}^k N_i = \frac{\bar{N}}{k}$

- relative deviation: $\nu(\bar{N}) = \frac{\sigma(\bar{N})}{\bar{N}} = \frac{1}{\sqrt{k\bar{N}}} = \frac{1}{\sqrt{\sum_{i=1}^k N_i}}$

- average counts: $\bar{N} \pm \sqrt{\bar{N}/k}$

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Exercise

- For the same condition on last slides, calculate the average count rate and its standard deviation as well as relative deviation.

A) measuring once in time kt
B) measuring k times and each in time t
would have the same relative deviation as long as the total counts are the same for two kinds of measurement.

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Error of Net Count Rate (Background Subtracted)

- measuring background: N_b in t_b
- measuring sample: N_s in t_s
- net count rate: $n_0 = N_s/t_s - N_b/t_b$
- standard deviation:

$$\sigma(n_0) = \sqrt{\frac{N_s}{t_s^2} + \frac{N_b}{t_b^2}} = \sqrt{\frac{n_s}{t_s} + \frac{n_b}{t_b}}$$

$$\text{result: } n_0 \pm \sigma(n_0) = (n_s - n_b) \left[1 \pm \frac{1}{n_s - n_b} \sqrt{\frac{n_s}{t_s} + \frac{n_b}{t_b}} \right]$$

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Combination of Measurement Result with Unequal Precision

- Considering k independent measuring, the i -th counts is N_i in time t_i .

$$\bar{n} = \frac{\sum_i W_i n_i}{\sum_i W_i} \quad W_i = \frac{\lambda^2}{\sigma_n^2} \quad n_i = \frac{N_i}{t_i} \quad \sigma_n^2 = \frac{n_i}{t_i} \quad \lambda: \text{any constant}$$

- We can using $\lambda^2 = \bar{n} \approx n_i$, then $W_i = t_i$.

$$\bar{n} = \frac{\sum_i t_i n_i}{\sum_i t_i} = \frac{\sum_i N_i}{\sum_i t_i}$$

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Combination of Measurement Result with Unequal Precision (cont.)

$$\sigma(\bar{n}) = \sqrt{\frac{\sum_i \sigma^2(N_i)}{\left(\sum_i t_i\right)^2}} = \sqrt{\frac{\sum_i N_i}{\left(\sum_i t_i\right)^2}} = \sqrt{\frac{\bar{n}}{\sum_i t_i}}$$

$$\nu(\bar{n}) = \frac{\sigma(\bar{n})}{\bar{n}} = \sqrt{\frac{1}{\bar{n} \sum_i t_i}} = \sqrt{\frac{1}{\sum_i N_i}}$$

$$\text{result: } \bar{n} \pm \sqrt{\frac{\bar{n}}{\sum_i t_i}}$$

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Optimization of Measuring time and conditions

- measuring background: N_b in t_b
- measuring sample: N_s in t_s
- net count rate: $n_0 = N_s/t_s - N_b/t_b$
- standard deviation:

$$\sigma(n_0) = \sqrt{\frac{N_s}{t_s^2} + \frac{N_b}{t_b^2}} = \sqrt{\frac{n_s}{t_s} + \frac{n_b}{t_b}}$$

Find the best t_s and t_b which lead to minimum $\sigma(n_0)$ when $T = t_s + t_b$ is fixed.

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Solution

- Let

$$\frac{d}{dt_s} \left(\frac{n_s}{t_s} + \frac{n_b}{T - t_s} \right) = 0 \Rightarrow \frac{t_b}{t_s} = \sqrt{\frac{n_b}{n_s}}$$

$$t_b = \frac{\sqrt{n_b}}{\sqrt{n_s} + \sqrt{n_b}} T \quad t_s = \frac{\sqrt{n_s}}{\sqrt{n_s} + \sqrt{n_b}} T$$

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Time Distribution of Radiation Particles and Signals

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Time Interval of Successive Nuclear Event

- A source of activity m emits a γ photon per nucleus decay. Give the distribution which the time interval T of successive γ emission follows.
- in time t , the numbers of γ photons follows Poisson distribution with parameter mt :

$$P_N(N = n) = \frac{(mt)^n}{n!} e^{-mt}$$

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- $T < t$ means at least one γ photon was emitted in time t .

$$P_T(T < t) = \sum_{k=1}^{\infty} P_N(N = k) = 1 - P_N(N = 0) = 1 - e^{-mt}$$

$$p_T(t) = \frac{d}{dt} P_T(T < t) = m e^{-mt} \quad (t > 0)$$

The time interval of successive nuclear events follows exponential distribution.

$$E(t) = \frac{1}{m}, \sigma^2(t) = \frac{1}{m^2}$$

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Question

- A detector has dead time τ . if the detector recorded n counts, estimate the counting loss.

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Summary



- Nuclear events number follows **Poisson** distribution, And its time interval follows **exponential** distribution.
- For a Poisson distribution with parameter m , its expectation and variance are both m .
- The larger the total counts are recorded, the less error the result contains.

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Homework



- P25: 1,2,3,4,5,6,10

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