1+1 dimensional quantum electrodynamics on the lattice

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QED in two dimensions

- Lagrangian

$$\mathscr{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \sum_{j=1}^{N}\bar{\psi}_j\left[\gamma^\mu(i\partial_\mu - eA_\mu) - m_j\right]\psi_j\ , \qquad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

- $\mu, \nu = 0,1$, first studied by Schwinger for $N = 1,\ m = 0$

Schwinger, PRD 1962

- Invariant under gauge transformation

$$\psi(x) \to e^{i\Lambda(x)}\psi(x), \quad A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu\Lambda(x)$$

- γ^μ are Dirac matrices in 2-dim.

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu},\ \{\gamma_5, \gamma^\mu\} = 0,\ \text{ e.g. } \gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- For one massless fermion ($N = 1, m = 0$), EOM

$$i\gamma^\mu D_\mu\psi = 0, \quad D_\mu = \partial_\mu + ieA_\mu$$

$$\partial_\mu F^{\mu\nu} = j^\nu, \quad j^\nu = e\bar{\psi}\gamma^\nu\psi, \quad \text{with } \partial_\nu j^\nu = 0$$

QED in two dimensions

- In two dimensions, we have ($\varepsilon^{\mu\nu}$ is 2-dim. antisymmetric tensor)

$$\gamma^\mu \gamma_5 = -\,\varepsilon^{\mu\nu}\gamma_\nu$$

- Axial vector and vector current are related to each other

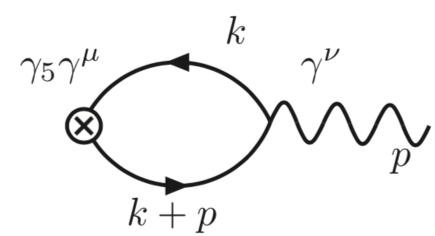
$$J_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi = -\,\varepsilon^{\mu\nu}\bar{\psi}\gamma_\nu\psi = -\,\varepsilon^{\mu\nu}J_\nu$$

- At classical level

$$\partial_\mu J_5^\mu = 0$$

- At quantum level (anomaly)

$$\partial_\mu J_5^\mu = \frac{e^2}{2\pi}\varepsilon^{\mu\nu}F_{\mu\nu} \Rightarrow (\Box + \bar{m}^2)J^\mu = 0,\ \bar{m} = e/\sqrt{\pi}$$



- The theory contains a free massive boson (fermion-antifermion bound state), trivial higher states consisting of n free Schwinger bosons
- In the massive model, these higher states turn into n-boson bound states

Path integral formalism

- Green's function is given by

$$\langle 0|T\{\hat{O}_1(x_1)\hat{O}_2(x_2)\ldots\}|0\rangle = \frac{1}{\int d[A,\psi]e^{iS}}\int d[A,\psi]O_1(x_1)O_2(x_2)\ldots e^{iS}, \quad S=\int d^4x\mathscr{L}$$

- Switch to imaginary time

$$t \to -it_E, \qquad \exp[iS] \to \exp[-S_E]$$

- We then have

$$\langle 0|T\{\hat{O}_1(x_1)\hat{O}_2(x_2)\ldots\}|0\rangle_E = \frac{1}{Z}\int d[A,\psi]O_1(x_1)O_2(x_2)\ldots e^{-S_E}, \quad S_E = i\int d^4x\mathscr{L}_E$$

$$Z=\int d[A,\psi]e^{-S_E}$$

- Minkowskian Green's functions can be reconstructed from their Euclidean counterpart by analytic continuation through Wick rotation
- Euclidean formulation of QFTs can be conveniently realized on a discrete lattice

Lattice action

- We need to discretize the Lagrangian
- Discretized derivative

$$\partial_\mu \psi(x) \to [\psi(x + ae_\mu) - \psi(x - ae_\mu)]/(2a)$$

- To maintain gauge invariance, one needs the parallel transporter

$$U(x - ae_\mu, x + ae_\mu) = e^{ig \int_{x+ae_\mu}^{x-ae_\mu} A_\mu(z) dz_\mu}$$

- The smallest of which defines the link variable on the lattice

$$U(x + ae_\mu, x) \equiv U_{x,\mu}$$

  which satisfies

$$U(x, x + ae_\mu) = U^{-1}(x + ae_\mu, x) = U_{x,\mu}^{-1} = U_{x,\mu}^{*}$$

- All parallel transporters can be constructed from links, for example

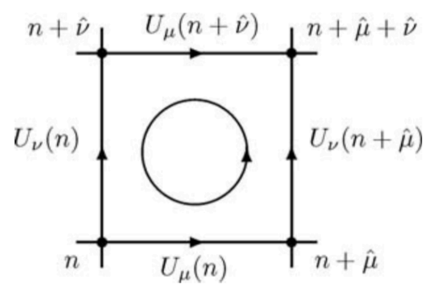
$$U(x - ae_\mu, x + ae_\mu) = U(x - ae_\mu, x)U(x, x + ae_\mu)$$

Lattice action

$$S = i\int d^4x[\mathscr{L}_g + \mathscr{L}_f] = S_g + S_f$$

- Bosonic part can be constructed from the smallest closed loop formed by link variables, called plaquette

$$U_p \equiv U(x, x + ae_\nu)U(x + ae_\nu, x + ae_\mu + ae_\nu) \times U(x + ae_\mu + ae_\nu, x + ae_\mu)U(x + ae_\mu, x)$$



$$S_g[U] = \beta\sum_p\left[1 - \frac{1}{2}(U_p + U_p^\dagger)\right], \quad \beta = \frac{1}{g^2a^2}$$

- Naive lattice fermion

$$D_\mu = [U_{x,\mu}\psi(x + ae_\mu) - U^*_{x-ae_\mu,\mu}\psi(x - ae_\mu)]/(2a)$$

$$Q_n(x,y) = m\delta(x-y) + \frac{1}{2a}\sum_{\mu=1}^{2}\sigma_\mu\left[U_{x,\mu}\delta(x + ae_\mu - y) - U^*_{x-ae_\mu,\mu}\delta(x - ae_\mu - y)\right]$$

- With the free fermion propagator

$$\langle\psi^\dagger(-p)\psi(p)\rangle = \left[\frac{i}{a}\sum_\mu\sigma_\mu sin(p_\mu a) + m\right]^{-1}$$

Fermion doubling problem:
For m=0,
4 poles: $p_1 = (0,0)$, $p_2 = (\pi/a,0)$
$p_3 = (0,\pi/a)$, $p_4 = (\pi/a,\pi/a)$ vs
1 pole: $p = (0,0)$ in continuum

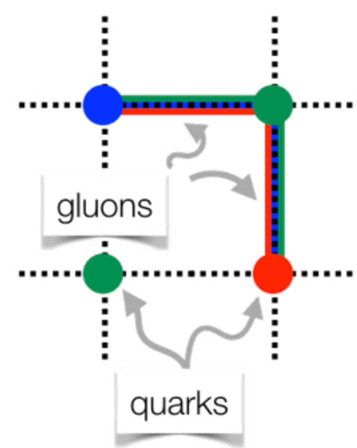
Lattice action

$$S = i\int d^4x[\mathscr{L}_g + \mathscr{L}_f] = S_g + S_f$$

- Add extra terms to kill doublers (example: Wilson fermion)

$$Q_n(x,y) = m\delta(x-y) + \frac{1}{2a}\sum_{\mu=1}^{2}\sigma_\mu\left[U_{x,\mu}\delta(x+ae_\mu-y) - U^*_{x-ae_\mu,\mu}\delta(x-ae_\mu-y)\right]$$
$$-\frac{r}{2}\sum_{\mu=1}^{2}\left[U_{x,\mu}\delta(x+ae_\mu-y) - U^*_{x-ae_\mu,\mu}\delta(x-ae_\mu-y)\right]$$

- Price to get rid of doublers (Nielsen-Ninomiya No-Go theorem), one has to abandon one of
  - Unitarity
  - Locality
  - Chiral symmetry
- To summarize, on a discrete lattice



Lattice action

$$S_g[U] = \frac{1}{g^2}\sum_p \left[1 - \frac{1}{2}(U_p + U_p^\dagger)\right] + \bar{\psi}Q(U)\psi = S_g^l[U] + S_f^l[U,\psi]$$

- Integrate out the fermionic degrees of freedom

$$Z^l = \int d[U,\psi]e^{-S^l} = \int d[U]e^{-S_g^l}\det[Q(U)]$$

- Fermion determinant is computationally more expensive
- A simple way to deal with it: quenched approximation

$$\det[Q(U)] = 1$$

- Fermion pair creation and annihilation processes ignored
- Such fermion loops are expected to have small effects
- Significant simplification in numerical simulations, several orders of magnitude faster

Mass spectrum

- Hadron mass spectrum can be studied from correlation function of an operator with the same quantum number of a given hadron
  - Construct operators with suitable quantum numbers (interpolators)
  - Compute the two-point correlation function
  - Study the large time limit of the correlation function
- Consider

$$C(p,t) = \sum_x e^{ipx} \langle O(x,t) O^\dagger(0,0) \rangle$$

- Insert a complete set of energy eigenstates, it becomes

$$\sum_x \sum_n e^{ipx} \langle 0 | O(x,t) | n \rangle \langle n | O^\dagger(0,0) | 0 \rangle$$

$$= \sum_x \sum_n e^{ipx} \langle 0 | O(x,0) e^{-Ht} | n \rangle \langle n | O^\dagger(0,0) | 0 \rangle$$

$$= \sum_x \sum_n e^{ipx} \langle 0 | O(x,0) | n \rangle \langle n | O^\dagger(0,0) | 0 \rangle e^{-E_n t}$$

Mass spectrum

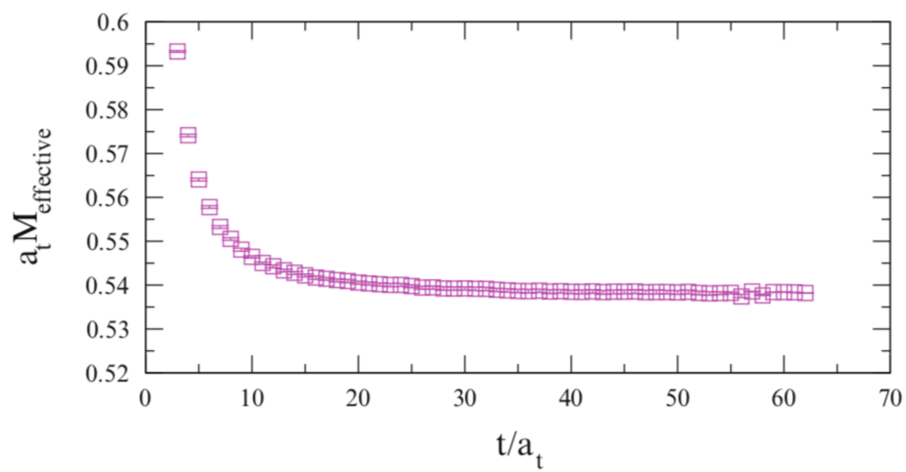
- For finite lattice size, the values of E_n are discrete
- For simplicity, we project to zero momentum $p = 0$

$$C(0,t) = \sum_n c_n e^{-E_n t}, \quad c_n = \sum_x \langle 0 | O(x,0) | n \rangle \langle n | O^\dagger(0,0) | 0 \rangle$$

- In large time limit, the exponential fall-off of the correlation function gives the ground state energy
- One can define the effective mass as

$$am_{\text{eff}} = -\ln \frac{C(0,t)}{C(0,t-1)}$$

- It reaches a plateau at large time separations as the ground state exponential dominates in the correlation function



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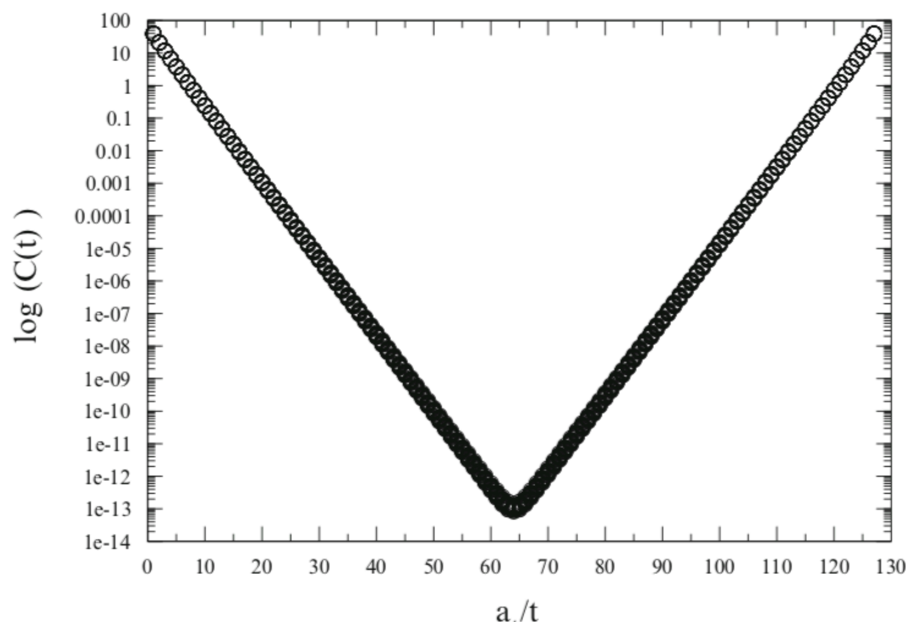
- In large time limit, the exponential fall-off of the correlation function gives the ground state energy
- If one chooses periodic boundary condition, then

$$C(0,t) = \sum_n c_n \left[e^{-E_n t} + e^{-E_n (T-t)} \right]$$



Mass spectrum

- Two-point function calculation (for $O_\Gamma(x) = \bar{\psi}(x)\Gamma\psi(x)$)

$$\langle \mathcal{O}_\Gamma(x)\mathcal{O}_\Gamma(0)\rangle = \bar{\psi}(x)\Gamma\bar{\psi}(0)\psi(x)\Gamma\psi(0) = tr[\Gamma S(x,0)\Gamma S(0,x)]$$

- $S(x, y)$ is the quark propagator, and computed as the inverse of the fermion matrix

$$QS(x,y) = \delta(x,y)$$

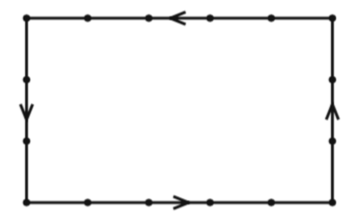
- Taking the meson interpolator with $\Gamma = \gamma_5$ as an example

$$\begin{aligned}\langle O_{\rm PS}(0)O_{\rm PS}(t)\rangle &= \sum_{\mathbf{x}} \left[\bar{\psi}(\mathbf{x},t)\gamma_5\Psi(\mathbf{x},t)\right]\left[\bar{\psi}(0,0)\gamma_5\Psi(0,0)\right] \\ &= \sum_{\mathbf{x}} \mathrm{Tr}\left[S_F(0,0;\mathbf{x},t)\underbrace{\gamma_5 S_F(\mathbf{x},t;0,0)\gamma_5}_{=S_F^\dagger(0,0;\mathbf{x},t)}\right]\end{aligned}$$

Wilson loop

- A Wilson loop is a loop formed by link variables, the simplest example is a plaquette
- For a Wilson loop which has length m in spatial direction and n in temporal direction

$$W(m,n) = \langle U(x, x+mae_\nu)U(x+mae_\nu, x+nae_\mu+mae_\nu) \times U(x+nae_\mu+mae_\nu, x+nae_\mu)U(x+nae_\mu, x)\rangle$$



- In the continuum limit, it becomes

$$W(m,n) \to W(r,t) = \langle e^{ie\oint_C dx_\mu A_\mu}\rangle$$

- C is a rectangular contour with $r = ma$ and $t = na$
- It represents the probability amplitude of creating an infinitely heavy (static) fermion-antifermion pair at $t_0 = 0$ (with separation r) and annihilating at time t

Wilson loop

- For large t, one expects

$$W(r, t \to \infty) \to e^{-V(r)t} = e^{-V(r)na}$$

- $V(r)$ is the static potential, it has a linear behavior with the distance as the Coulomb potential in D spatial dimensions

$$V(r) \to r^{2-D}$$

- It can be extracted by computing

$$aV(r) = \ln \frac{W(r, t-1)}{W(r, t)} \Bigg|$$