Computational algebraic geometry methods for the analytic computation of Feynman integrals

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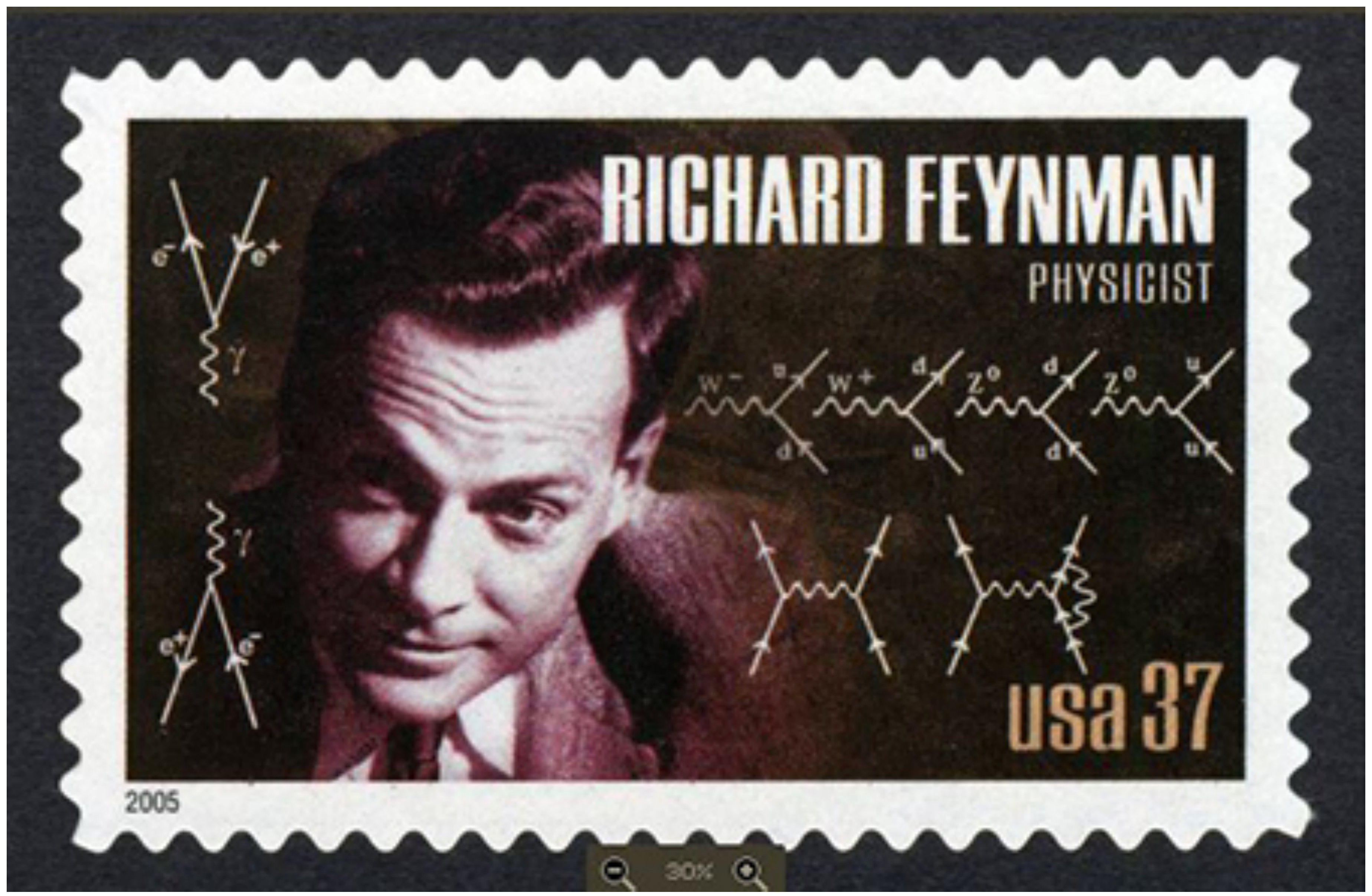
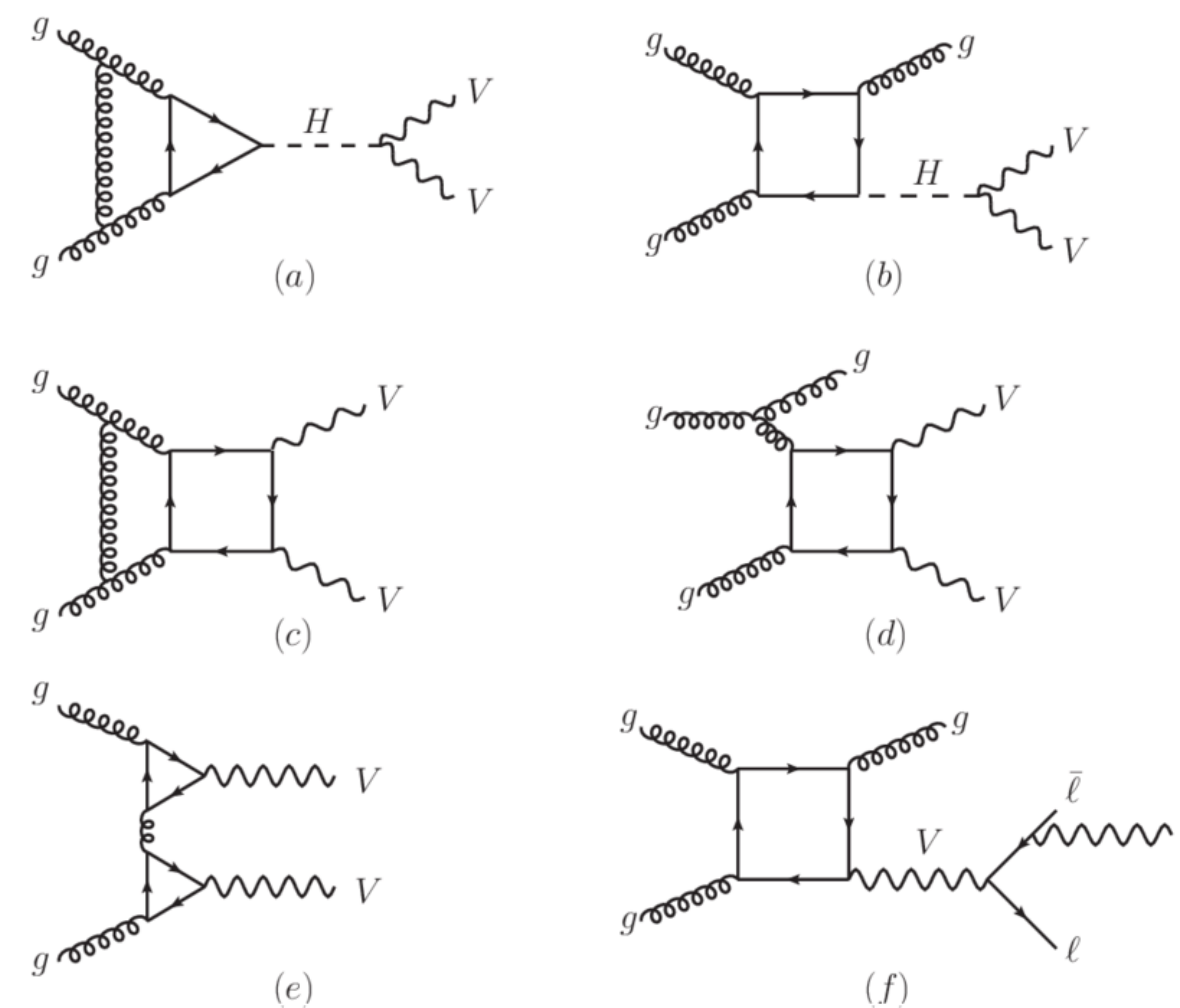
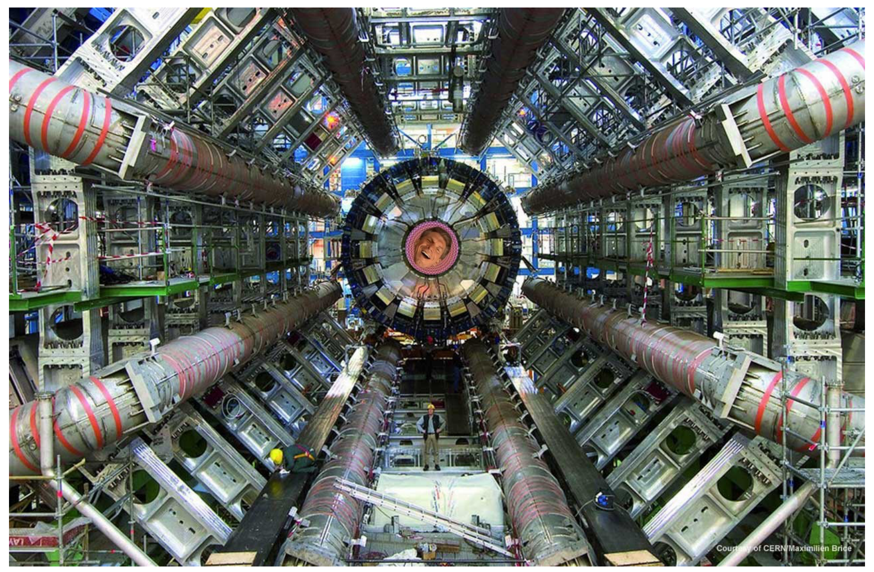
Feynman integrals



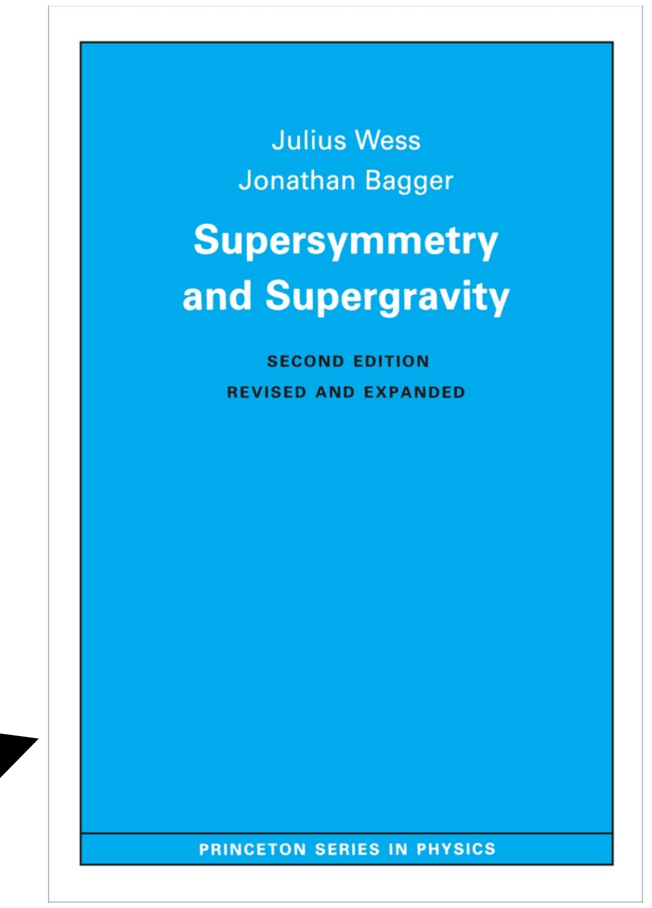


Basis computational tool in quantum field theory

Feynman integrals, nowadays



precision
physics



formal
theory

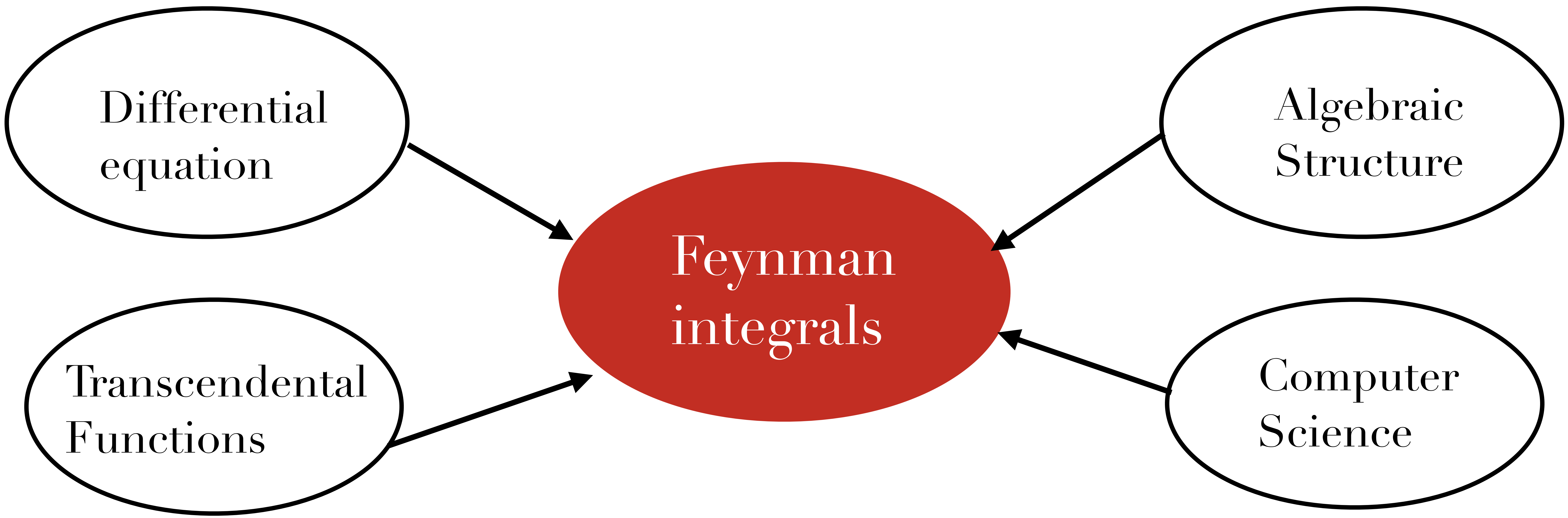
N=8 supergravity
is UV-finite until
five loop

Feynman
integrals

Gravitational wave
template computation

Feynman integrals, nowadays

It is still a basic tool in quantum field theory;
Crucially for precision high-energy physics, formal theories,



The main focus is on multi-loop Feynman Integrals
Significant progress after 2010

New analytic results on Feynman integrals

2-loop Feynman integrals for pp -> Hj
2-loop Feynman integrals for pp -> tt
2-loop Feynman integrals for pp -> 3 jets
2-loop Feynman integrals for pp -> H + 2 jets
3-loop 4-point Feynman integrals with one-massive external line
4-loop form factor Feynman integrals
4-loop analytic electron g-2 Feynman integrals
5-loop 4-point massless Feynman integrals (UV part)
… …

Di Vita, Gehrmann, Laporta, Mastrolia, Pierpaolo, Primo, Schubert, Ulrich
von Manteuffel, Panzer, Schabinger
Bonciani, Del Duca, Frellesvig, Henn, Hidding, Maestri, Moriello, Salvatori, Smirnov
Bern, Carrasco, Chen, Johansson, Roiban, Zeng
Ita, Page, Samuel, Zeng
Chicherin, Gehrmann, Henn, Wasser, **YZ**, Zoia …

Why analytic Feynman integrals?

- Once the analytic expression is obtained, the phase point generation is extremely fast
- Avoid unstable numeric phase points
- Understand the deep structure and hidden symmetry in quantum field theory

- and yes, we can.

see also developments in numerics

Sector decomposition + GPU + quasi-Monte Carlo	https://secdec.readthedocs.io/en/stable/
numeric differential equation	JHEP 05 (2020) 149, Czakon and Niggetiedt

and the reference therein

Mainstream Analytic Methods

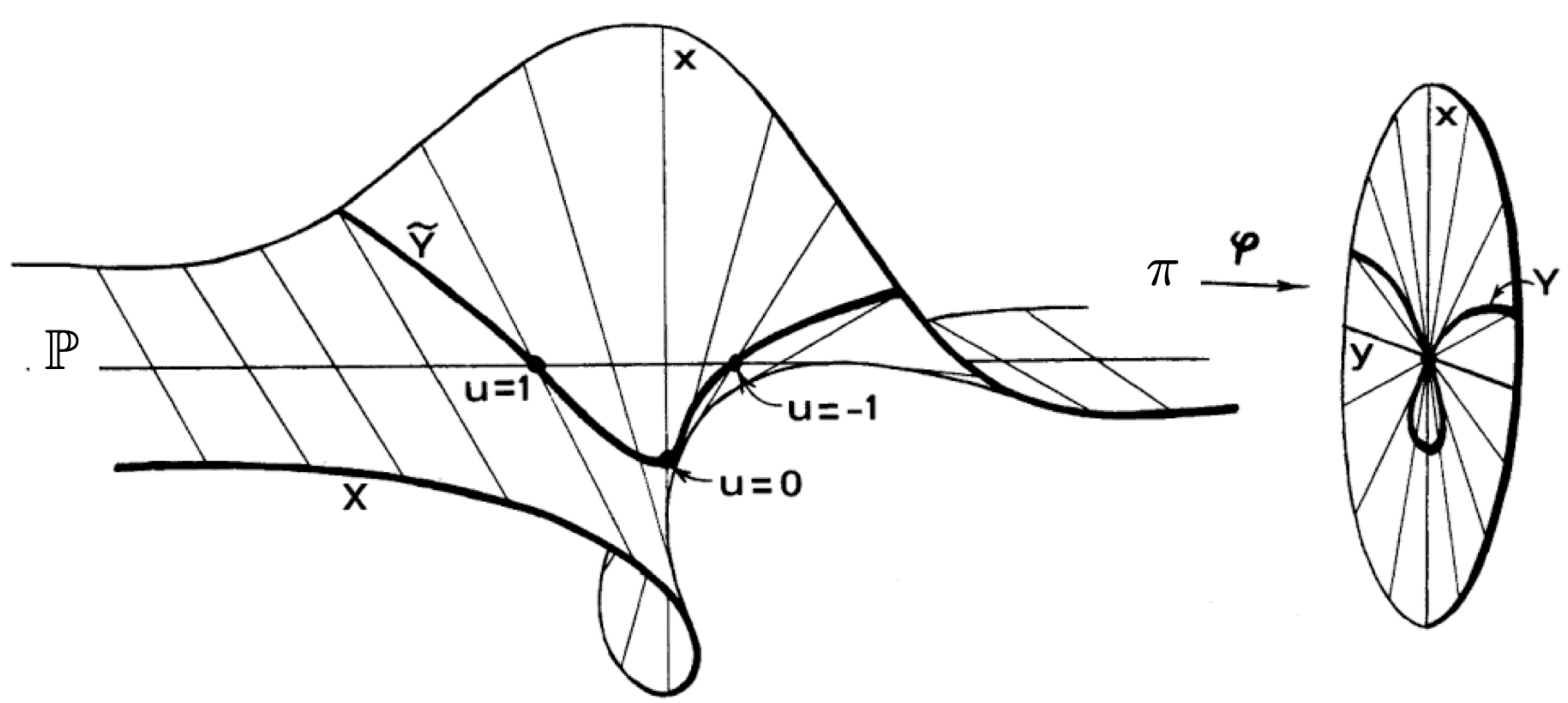
Most used

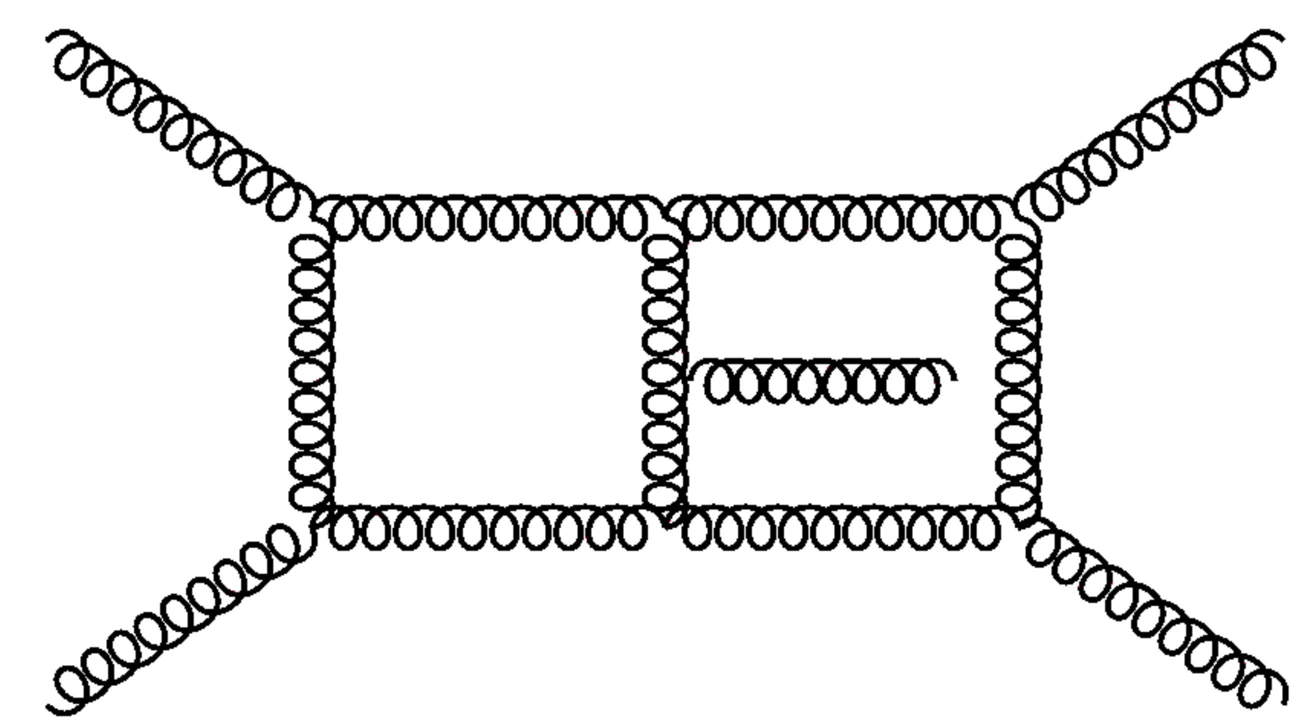
- Canonical differential equation (Henn 2013), in terms of polylogarithm
- Partial fraction + recursive integration (Panzer 2015), package: **HyperInt**, in terms of polylagarithm
- Elliptic Canonical Differetial Equation (Broedel, Duhr, Dulat, Tancredi, 2018), in terms of elliptic polylagarithm

Sometimes magic

- Mellin-Barnes
- Dimension Recursion Relations (Lee, Smirnov 2012),
  usually numeric, but sometimes provide analytic result for complicated integrals
- Integral Bootstrap (Chicherin, Henn, Mitev 2017)
  can easily get the "nice" integrals, eg. conformal, in an integral family

This talk is about using computational algebraic geometry methods for
the analytic Feynman integral computations





Outline

- Analytic Feynman integral computation, an overview
- Computational Algebraic Geometry method for Feynman integrals

  1. Module lift method for UT integral searching
  2. Syzygy method for complicated IBP computations

Based on

Bendle, Boehm, Heymann, Ma, Rahn, Wittman, Ristau, Wu, **YZ** 2021

"Two-loop five-point integration-by-parts relations in a usable form", 2104.06866

Boehm, Wittmann, Xu, Wu and **YZ**

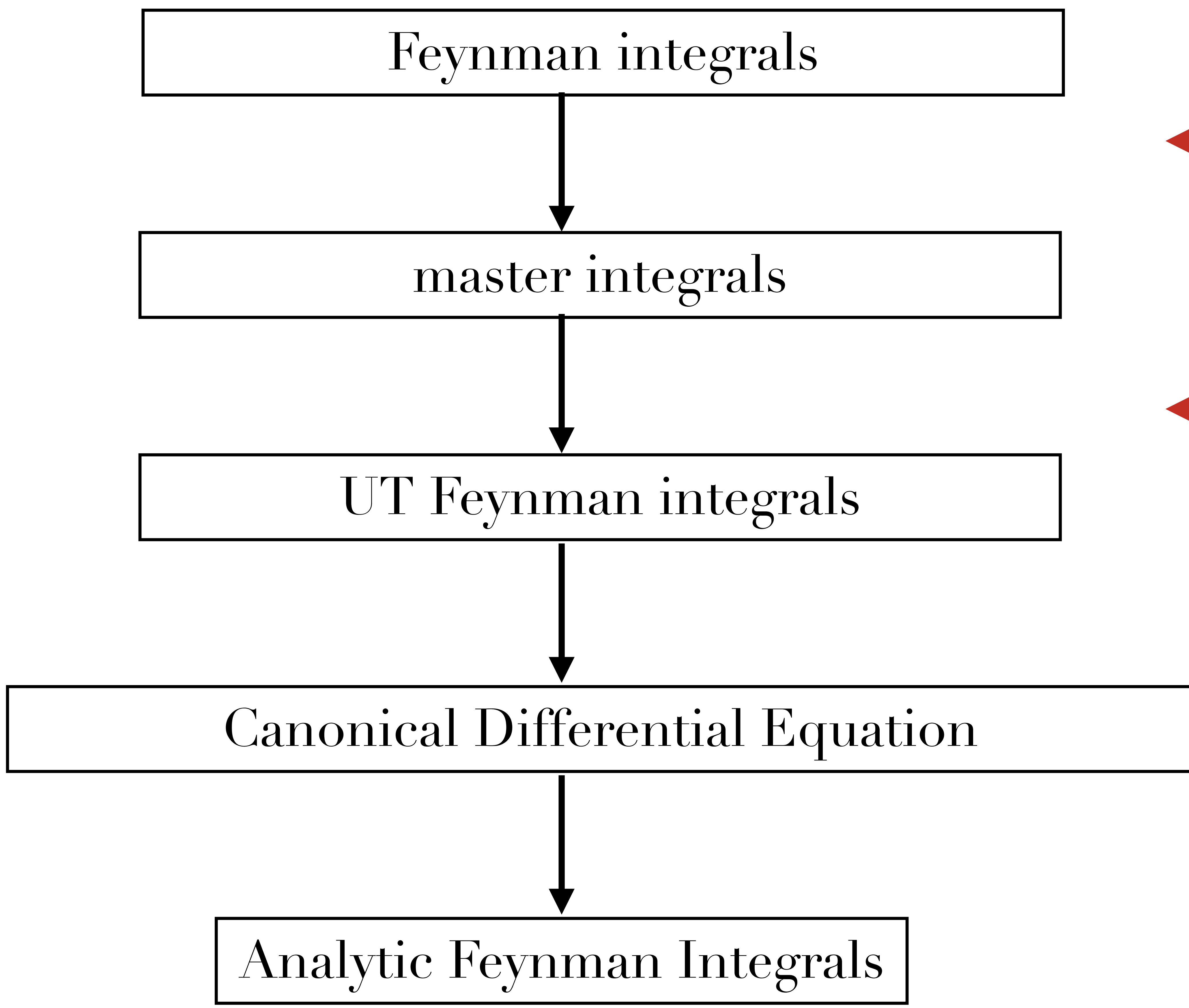
"IBP reduction coefficients made simple " JHEP 12 (2020) 054

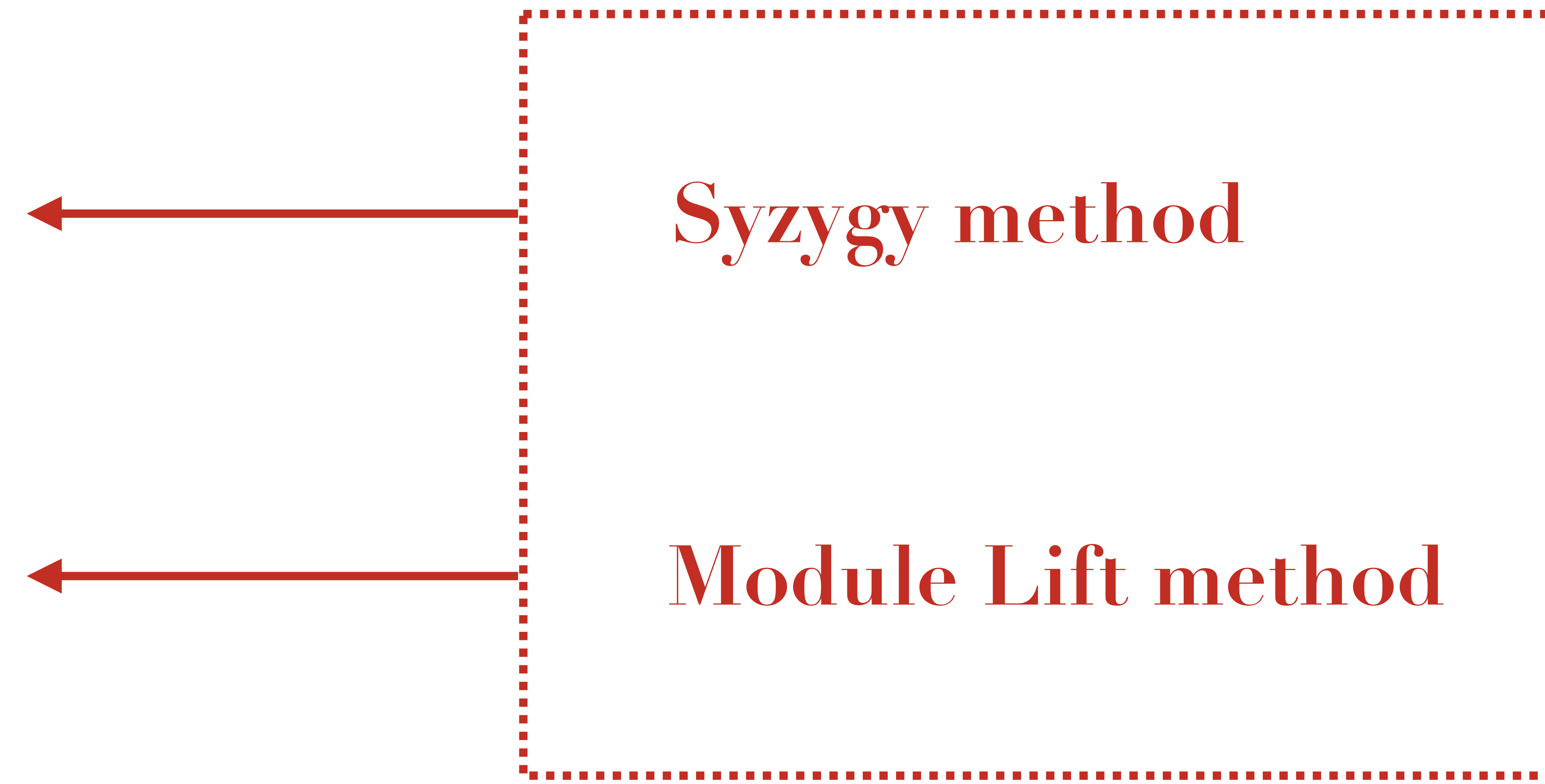
Chicherin, Gehrmann, Henn, Wasser, **YZ**, Zoia

"*All master integrals for three-jet production at NNLO*", PhysRevLett. 123 (2019), no. 4 041603
"*Analytic result for a two-loop five-particle amplitude*", PhysRevLett. 122 (2019), no. 12 121602

Analytic Feynman integral computation
an overview

Canonical Differential Equation for Analytic Feynman integrals





It is usually easier to compute Feynman integrals with differential equations than by a direct integration.

From Feynman integrals to master integrals

For a scattering process, there are a huge number of integrals

After a tensor reduction

$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{1}{D_1^{\alpha_1} \dots D_k^{\alpha_k}}, \quad \alpha_i \in \mathbb{Z}$$

negative index means the numerator

IBP reduction

at the two/three loop orders,

IBP reduction can reduce **millions** of Feynman integrals to **hundreds** of master integrals.

$$\boxed{\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{\partial}{\partial l_i^\mu} \frac{v_i^\mu}{D_1^{\alpha_1} \dots D_k^{\alpha_k}} = 0}$$

Chetyrkin, Tkachov 1981

IBP reduction

$$\boxed{\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{\partial}{\partial l_i^\mu} \frac{v_i^\mu}{D_1^{\alpha_1} \ldots D_k^{\alpha_k}} = 0}$$

Usually by choosing different vectors, we also have a huge number of IBP relations
To get the complete reduction: Gaussian Elimination (Laporta algorithm 2000)

Two issues:

The Gaussian Elimination is computationally heavy, sometimes
the most time consuming step for a scattering amplitude computation

The IBP reduction coefficients may be too large to use.

Problem 1: Can we shorten the IBP reduction coefficients?

Differential Equation for Feynman Integrals

$d = 4 - 2\epsilon$ $\quad$ $I(\bar{x}, \epsilon)$ master integrals as a column vector

(Kotikov 1991)

$$\frac{\partial}{\partial x_i} I(\bar{x}, \epsilon) = A_i(\bar{x}, \epsilon) I(\bar{x}, \epsilon)$$

Can be used both numerically and analytically
eg. Gehrmann, Remiddi 1999,
Papadopoulos 2014,
Liu, Ma, Wang, 2018 …

is similar to Christoffel symbols in general relativity
transform inhomogeneously with a change of master integrals

Different choices of the master integrals change the DE dramatically.
The simplest choice is
the integrals with uniform transcendental (UT) weights, which gives

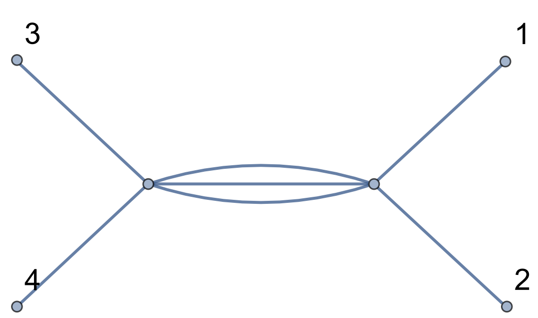
Canonical Differential Equation

What is Uniformly transcendental (UT) integrals?

$$\mathcal{T}(\log)=1, \mathcal{T}(\pi)=1, \mathcal{T}(\zeta_n)=n, \mathcal{T}(\mathrm{Li}_n)=n, \ldots, \mathcal{T}(f_1 f_2)=\mathcal{T}(f_1)+\mathcal{T}(f_2)$$

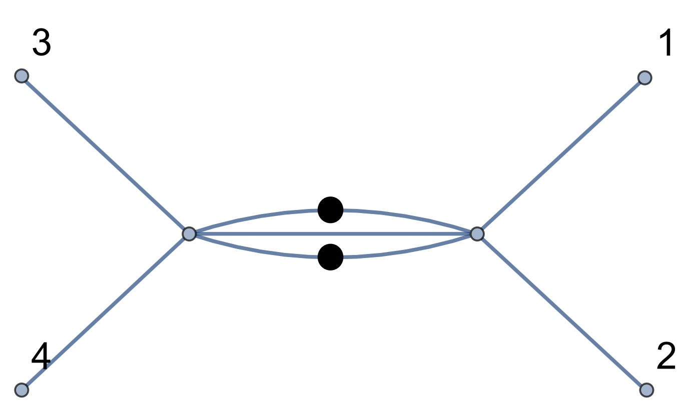
also require the 1st derivative has the transcendental weight n-1

$$I = (\text{overall normalization}) \times \sum_{k=0}^{\infty} \epsilon^k f_k, \quad \mathcal{T}(f_k)=k$$



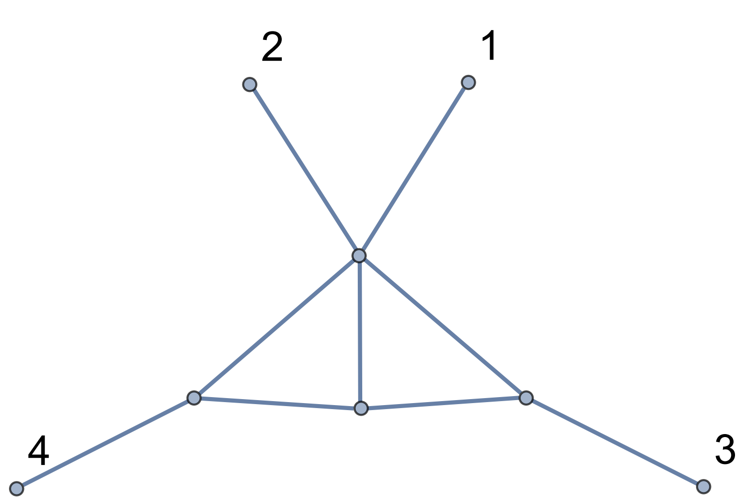
$$(s_{12})^{1-2\epsilon}\left(\frac{1}{4\epsilon}+\frac{13}{8}+\frac{1}{48}\left(345-2\pi^2\right)\epsilon+\frac{1}{96}\left(-256\zeta(3)+2595-26\pi^2\right)\epsilon^2+O\left(\epsilon^3\right)\right)$$

not UT



$$(s_{12})^{-1-2\epsilon}\left(-\frac{1}{\epsilon^2}+\frac{\pi^2}{6}+\frac{32\zeta(3)\epsilon}{3}+\frac{19\pi^4\epsilon^2}{120}+O\left(\epsilon^3\right)\right)$$

UT



$$(s_{12})^{-1-2\epsilon}\left(-\frac{1}{4\epsilon^4}+\frac{\pi^2}{24\epsilon^2}+\frac{8\zeta(3)}{3\epsilon}+\frac{19\pi^4}{480}+O\left(\epsilon^1\right)\right)$$

UT

UT basis is also good for numeric computations

UT Integral $\Rightarrow$ Canonical DE

$$\tilde{I} = (\text{overall normalization}) \times \sum_{k=0}^{\infty} \epsilon^k f_k, \quad \mathcal{T}(f_k) = k$$

$$\boxed{\frac{\partial}{\partial x_i}\tilde{I}(\bar{x},\epsilon) = \epsilon \tilde{A}_i(\bar{x})\tilde{I}(\bar{x},\epsilon)}$$

Canonical DE

Feynman integrals are the iterated integration of rational functions $\Rightarrow$ polylogarithm functions

$$\tilde{I}(x) = \underbrace{P\exp\left(\epsilon\int_{\mathcal{C}} dA\right)}_{\text{path-ordered}}\tilde{I}(x_0)$$

Analogy of perturbation theory of QFT (Dyson series)

Chen's (陈国才) iterated integrals, homotopically invariant

UT Integral $\Rightarrow$ Canonical DE

Henn 2013

Symbol letters

$$\frac{\partial}{\partial x_i}\tilde{I}(\bar{x},\epsilon) = \epsilon\tilde{A}_i(\bar{x})\tilde{I}(\bar{x},\epsilon)$$

$$= \epsilon\left(\sum_{l=1} \frac{\partial \log(W_l)}{\partial x_i} m_l\right)\tilde{I}(\bar{x},\epsilon)$$

Proportional to $\boldsymbol{\epsilon}$

Constant matrix

The first line ensures that the equation can be solved perturbatively in $\boldsymbol{\epsilon}$

The second line ensures that the solution is the polylogarithm function in symbol letters

Analogue of the interaction picture in quantum mechanics

$$\frac{\partial}{\partial x_i}I(\bar{x},\epsilon) = A_i(\bar{x},\epsilon)I(\bar{x},\epsilon)$$

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = (H_0 + \epsilon H_1)|\psi\rangle$$

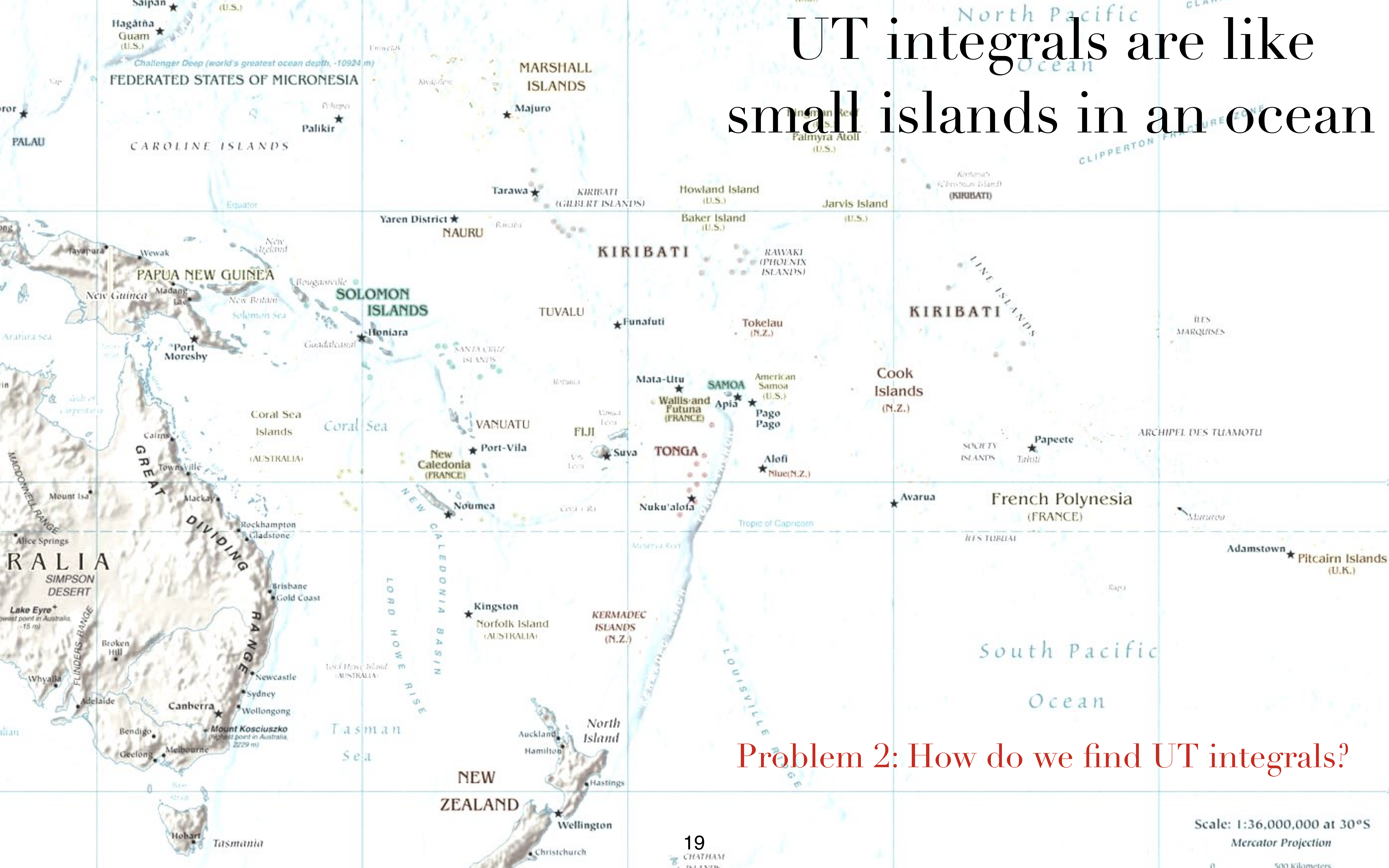
$$\tilde{I}(\bar{x},\epsilon) = T(\bar{x},\epsilon)I(\bar{x},\epsilon)$$

$$|\psi\rangle_I = e^{iH_0 t}|\psi\rangle$$

$$\frac{\partial}{\partial x_i}\tilde{I}(\bar{x},\epsilon) = \epsilon\tilde{A}_i(\bar{x})\tilde{I}(\bar{x},\epsilon)$$

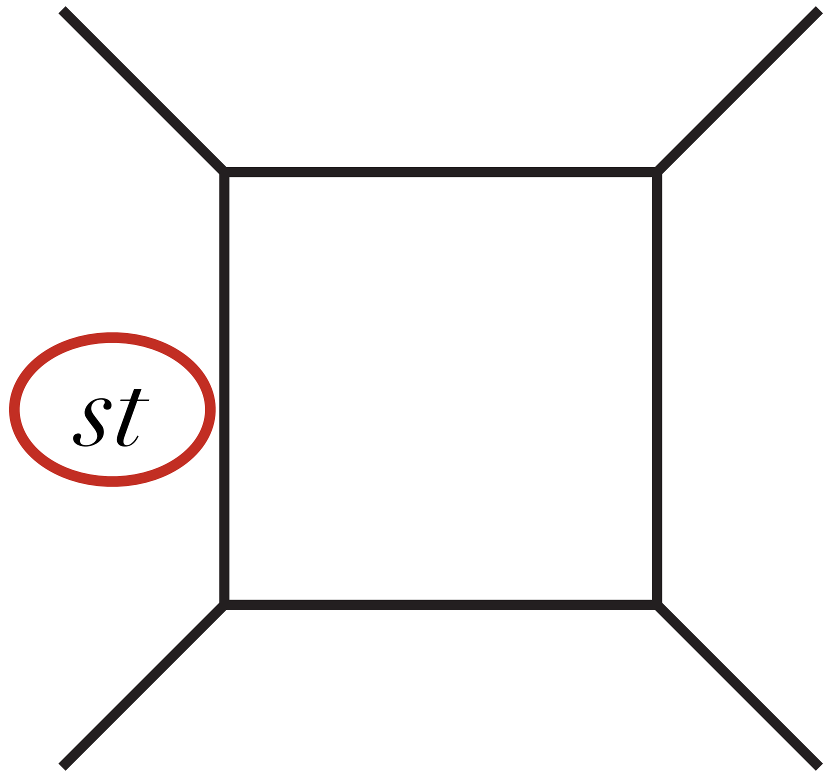
$$i\hbar\frac{\partial}{\partial t}|\psi\rangle_I = \epsilon H_I(t)|\psi\rangle_I$$

UT integrals are like small islands in an ocean

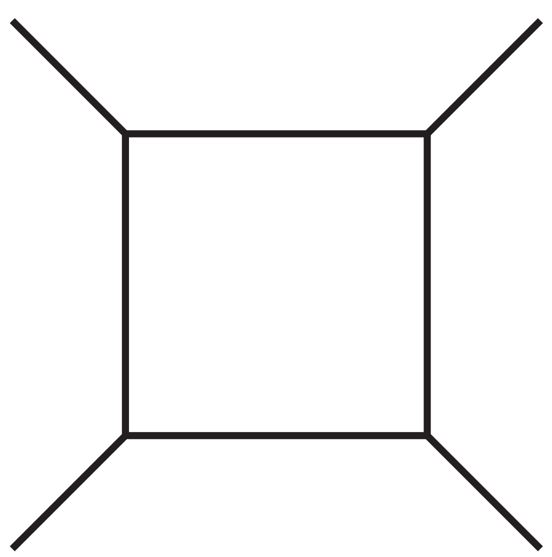


Problem 2: How do we find UT integrals?

To find UT integrals



A massless scale box integral with the overall factor “st” is UT



$$= \frac{4}{s\,t\,\epsilon^2} - \frac{2\,\mathrm{Log}[\frac{t}{s}]}{(s\,t)\,\epsilon} - \frac{4\,\pi^2}{3\,(s\,t)} + \left(\frac{\mathrm{Log}[\frac{t}{s}]\,\left(13\,\pi^2 + 6\,\mathrm{Log}[\frac{t}{s}]\,\left(i\,\pi + \mathrm{Log}[\frac{t}{s}]\right)\right)}{6\,s\,t} - \frac{2\,\pi\,\left(\pi + i\,\mathrm{Log}[\frac{t}{s}]\right)\,\mathrm{Log}[1+\frac{t}{s}]}{s\,t} + \frac{2\,i\,\pi\,\mathrm{Log}[1+\frac{t}{s}]^2}{s\,t} + \frac{2\,\mathrm{Log}[\frac{t}{s}]\,\mathrm{PolyLog}[2,-\frac{s}{t}]}{s\,t} + \frac{2\,\mathrm{PolyLog}[3,1+\frac{s}{t}]}{s\,t} + \frac{4\,\mathrm{PolyLog}[3,-\frac{t}{s}]}{s\,t} + \frac{2\,\mathrm{PolyLog}[3,1+\frac{t}{s}]}{s\,t} - \frac{40\,\mathrm{Zeta}[3]}{3\,s\,t}\right)\epsilon + O[\epsilon]^2$$

But we’d better to “guess" UT integrals before we get the analytic result

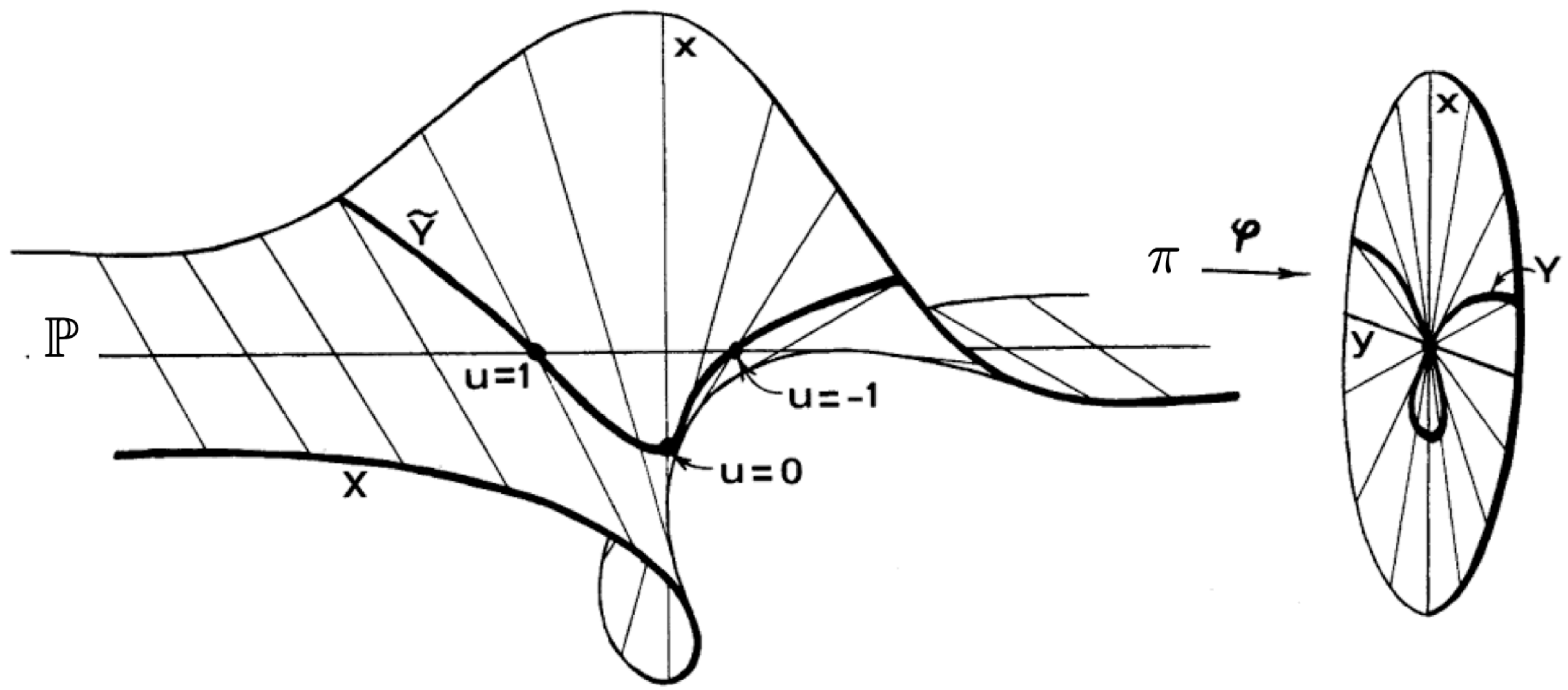
Applications of Computational algebraic geometry

One main difficulty for analytic scattering amplitude computation is to deal with polynomials, rational functions:

Simplify the differential equation
polynomial coefficients for IBP reductions
…

The key to polynomial/rational function problems is computational algebraic geometry

Algebraic Geometry



Algebraic geometry originates from the study of curves/surfaces defined by multivariate polynomial equations.

Modern algebraic geometry generalised these geometric objects to abstract objects (like scheme), and has been applied to number theory, complex analysis , topology and physics.

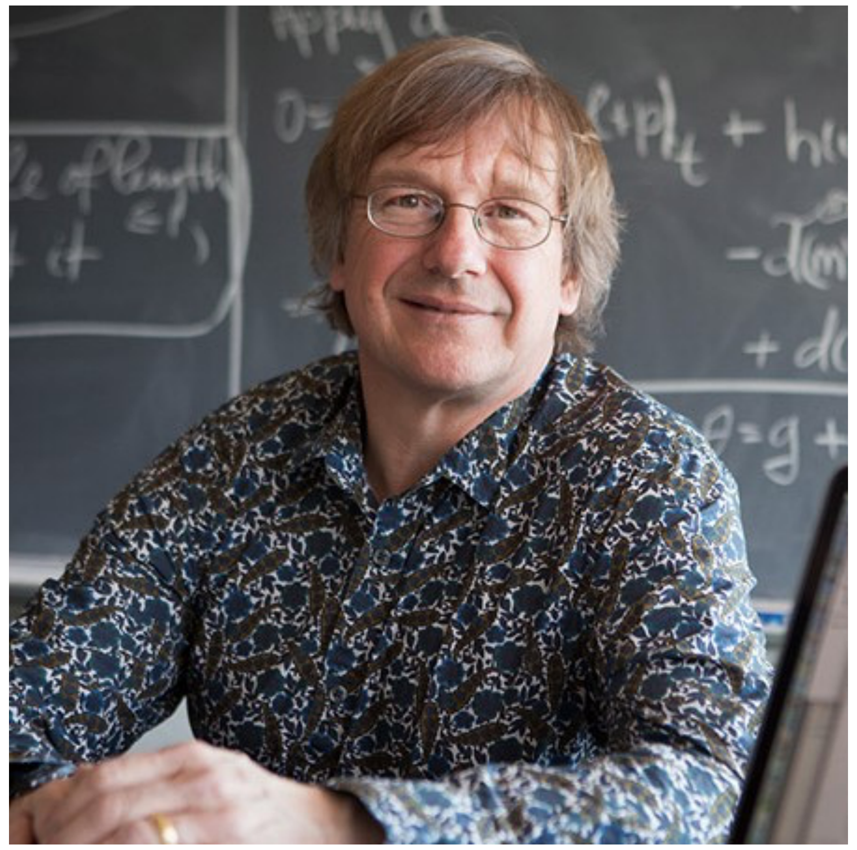
Computational Algebraic Geometry (CAG)

Algebraic Geometry

Computer Algorithms

originated in 1970s
thrive from 2000s

Computational Algebraic Geometry



Personally I learnt CAG from Professor Michael Stillman.

Bruno Buchberger, Frank-Olaf Schreyer, Jean-Charles Faugère
David Eisenbud, Michael Stillman, Daniel Grayson, Wolfram Decker …

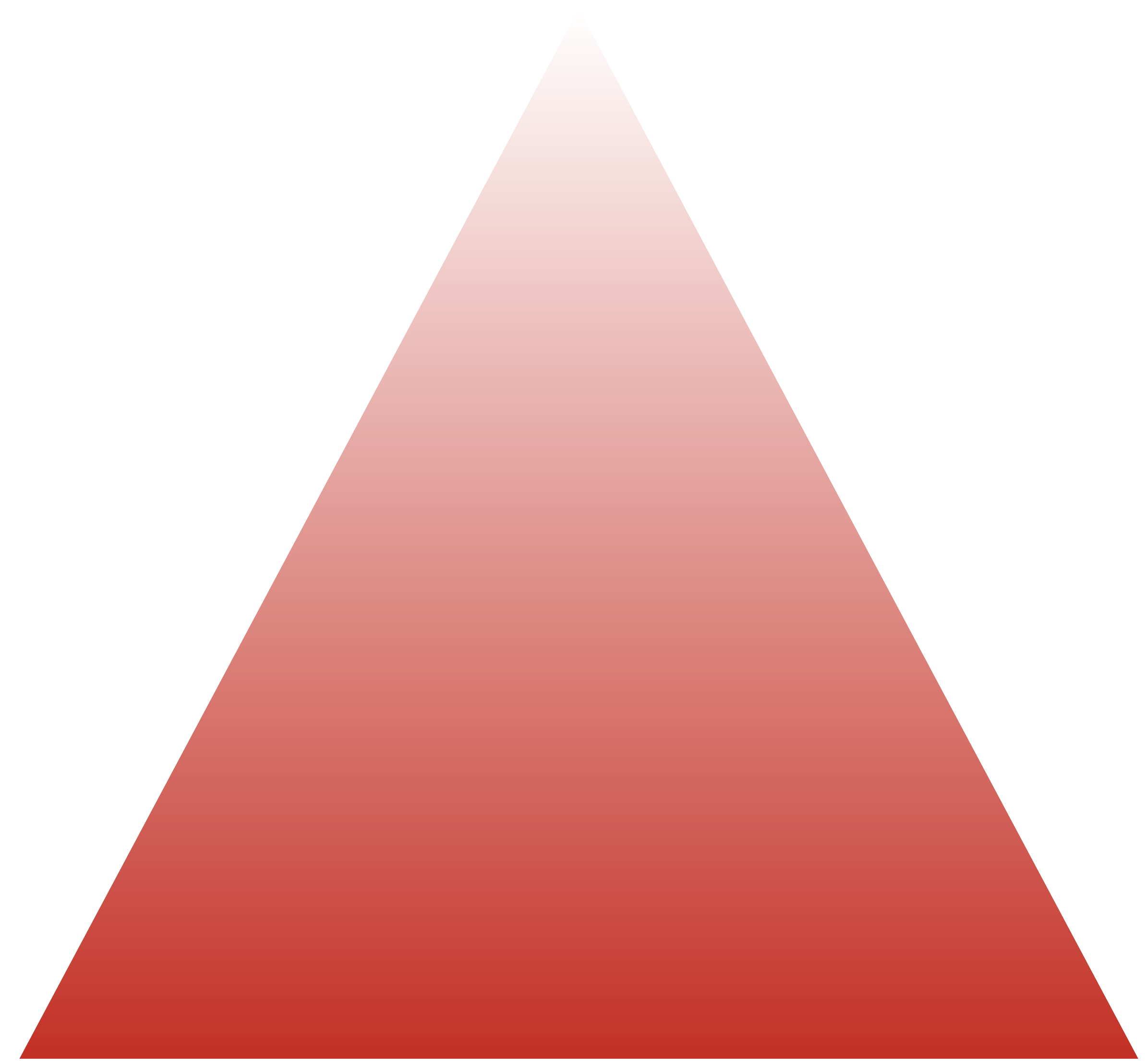
CAG in one slide

“Gaussian elimination”
for several multivariate polynomials

Buchberger algorithm

Groebner basis

Trinity of CAG



Syzygy

Solve homogeneous linear equations
with polynomial solutions

Schreyer algorithm

Lift

Solve inhomogeneous linear equations
with polynomial solutions

Application of Computational Algebraic Geometry for UT integral searching

UT integral searching

We show how
to use computational
algebraic geometry
to help these computations

based on the properties of Feynman integrals

4D leading singularity (residue) Henn 2013, 2014

Baikov leading singularity (residue) Chicherin, Gehrmann, Henn, Wasser, **YZ**, Zoia, 2019

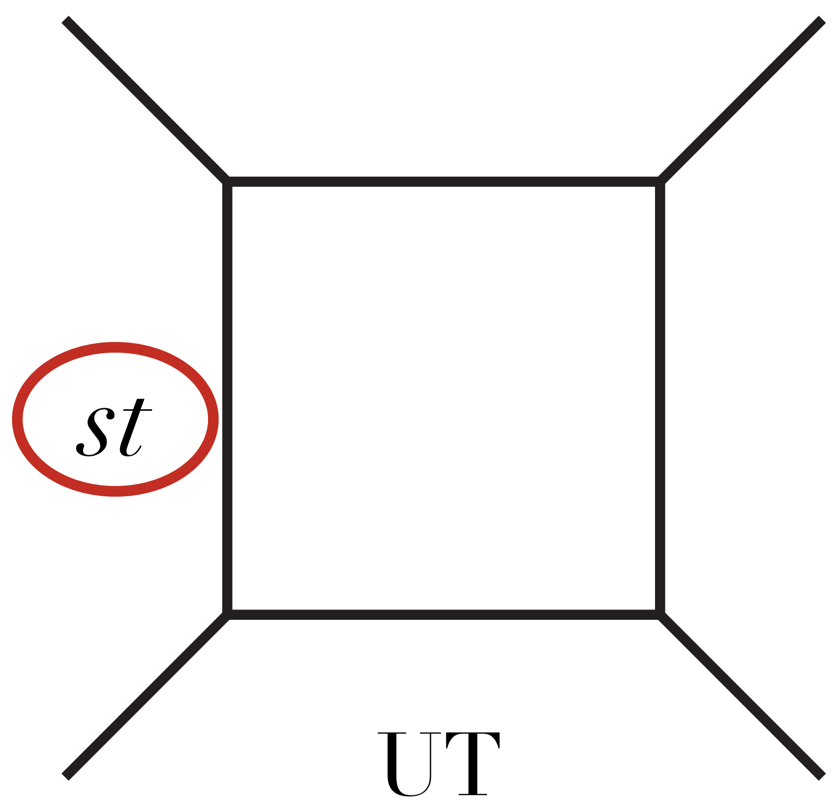
dlog integrand construction Henn, Smirnov, Wasser 2018 Chen, Jiang, Xu, Yang 2019

initial algorithm Dlapa, Henn, Kai, 2019

based on the transformation of a canonical differential equation

Lee's algorithm 2014 Lee, 2019

UT integral usually has constant residues



$$\xrightarrow{\text{residues in 4D}} \quad (-1, +1)$$

It is much easier to compute residues than Feynman integrals, so residue analysis can be a good approach for UT searching.

ideas from the study of N=4 SYM

Arkani-Hamed, Bourjaily, Cachazo, Trnka 2010

Use residues to find UT integrals: the reality

sometimes, it is necessary to consider D-dimensional residues

$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{1}{D_1^{\alpha_1} \ldots D_k^{\alpha_k}} = C(s_{ij}, L, E, D) \int dz_1 \ldots dz_k \frac{F^{\frac{D-L-E-1}{2}}}{z_1^{\alpha_1} \ldots z_k^{\alpha_k}}$$

Baikov parameterization
a duality of
Feynman parameterization

Chicherin, Gehrmann, Henn, Wasser, **YZ**, Zoia, 2019

Frequently, it is necessary to consider a complicated numerator as a UT candidate

$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{N}{D_1 \ldots D_k}$$

contains
loop momenta

$$N = \sum f_\alpha(s_{ij}) \times (\text{scalar product})^\alpha$$

UT numerator

Requirement for getting constant residues

or other constant vectors

$$\sum_{\alpha} f_a \times \mathrm{Res}\left(\frac{(\text{Scalar Product})^{\alpha}}{D_1 \ldots D_n}\right) = (1, 0, \ldots, 0, \ldots 0)$$

It would better to be polynomials in Mandelstam variables to avoid unphysical poles

If we solve the equation for f_α's above as a linear equation, the solutions generators are usually rational functions not polynomials!

We need the CAG method "Lift"

Lift

"Lift" means to find linear solutions with polynomial entries only

Example: $I = \langle y^2 - x^3 - x - 1, x + y + 2, x^2 + y^2 - 1 \rangle$, $\mathcal{Z}(I) = \emptyset$. By Hilberts' Nullstellensatz, $1 \in I$, i.e., there exist polynomials f_1, f_2, f_3

$$1 = \underline{f_1(x,y)}(y^2 - x^3 - x - 1) + \underline{f_2(x,y)}(x + y + 2) + \underline{f_3(x,y)}(x^2 + y^2 - 1)$$

But Hilbert did not say how to find such f_i's.

It can be solved by the "Lift" computation in CAG.

The key is to recursively pick up a pair in $\{y^2 - x^3 - x - 1, x + y + 2, x^2 + y^2 - 1\}$, eliminate the highest degree monomial (Groebner basis computation), until we get the number 1.

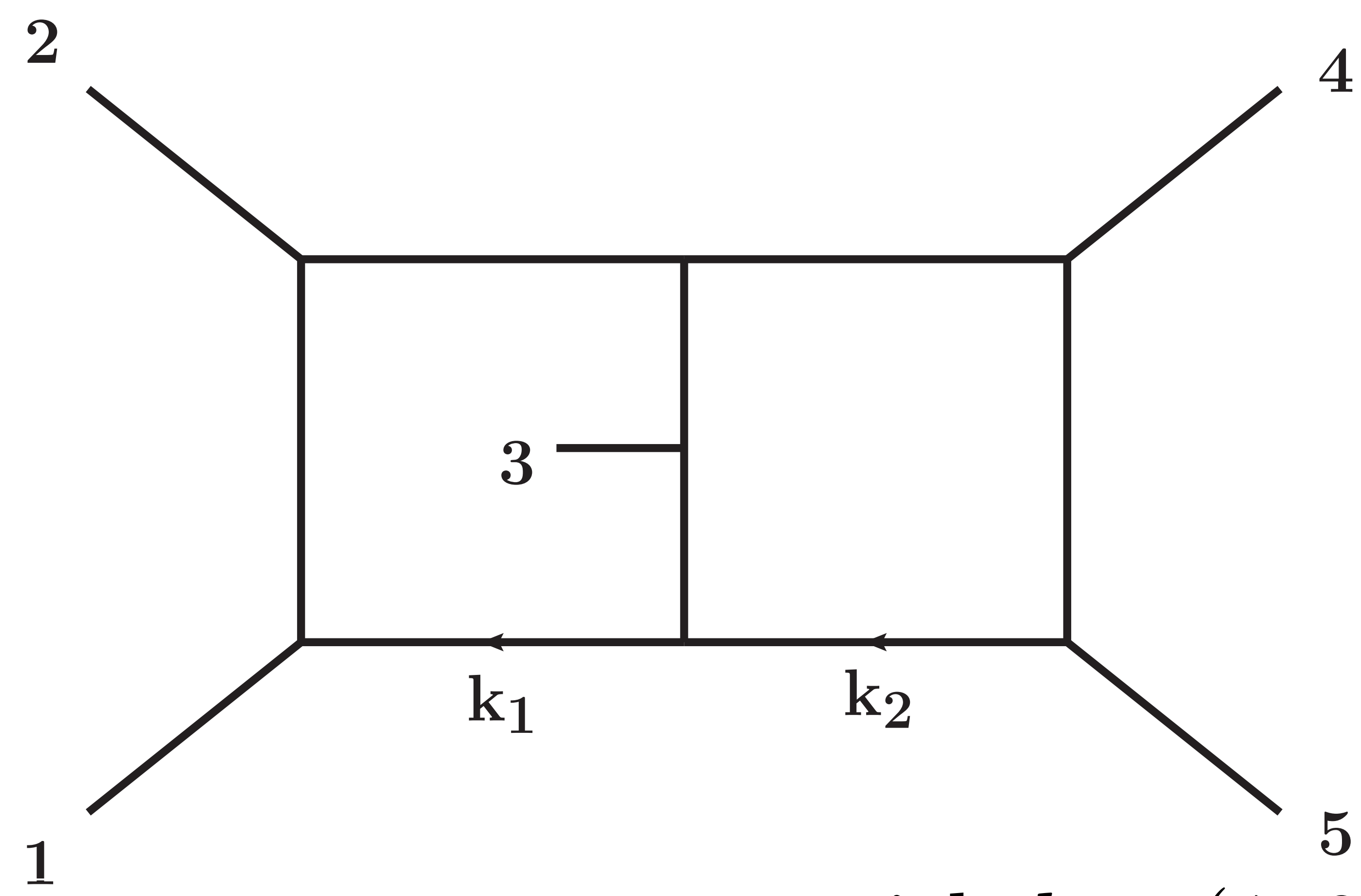
$$f_1 = -\frac{4}{3}(1 + y)$$

$$f_2 = \frac{1}{3}(2 - 5x - 3y + 2xy^2 + 2y^3)$$

$$f_3 = \frac{1}{3}(5 - 4x - 4xy - 2y^2)$$

Example

Chicherin, Gehrmann, Henn, Wasser, **YZ**, Zoia, 2019



8 poles, 10 possible scalar products

$$\sum_{\alpha=1}^{10} f_\alpha M_{\alpha\beta} = b_\beta, \quad \beta = 1, \ldots 8$$

with $b = (1,0,0,-1,0,0,0,0)^T$, SINGULAR provides the solution for f's,

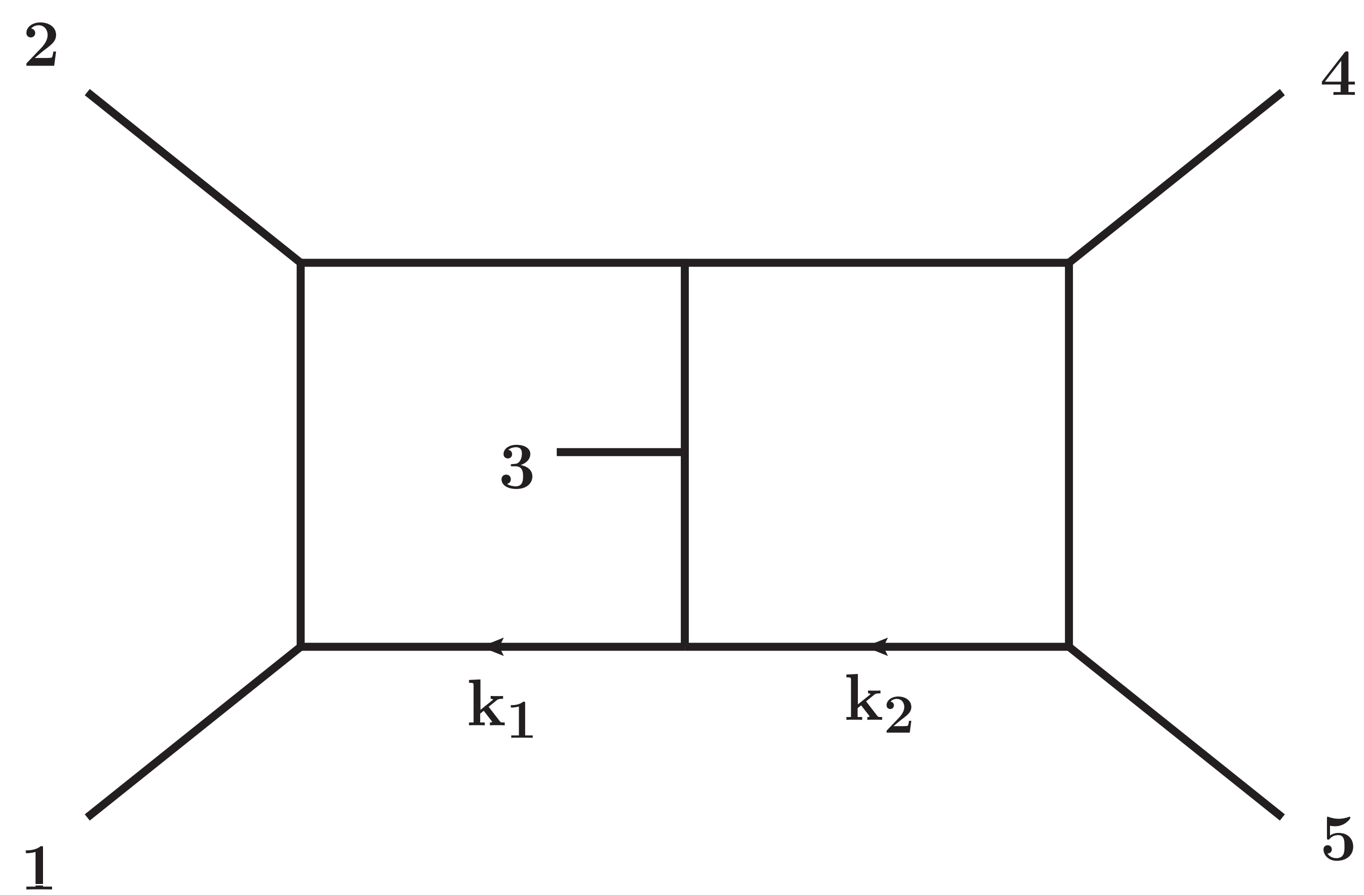
A computer algebra system for CAG

Find all UT integrals on this sector

$$f = \begin{pmatrix} 0 \\ 4s_{12}s_{23}(s_{12}-2s_{45}) \\ 4s_{12}s_{34}s_{45} \\ 4s_{12}s_{15}(s_{12}-s_{45}) \\ 0 \\ 0 \\ -4s_{12}s_{15} \\ -4s_{12}(s_{12}-s_{45}) \\ -4s_{12}(s_{23}+s_{45}) \\ -4s_{12}(s_{12}-s_{34}) \end{pmatrix}$$

Analytic Feynman integral computation example

Traditional way:



1.4 GB

$$\frac{\partial}{\partial x_i} I = A_i I$$

It took 3 months on UZH cluster to do the algebra ...

Canonical differential equation and UT:

$$\tilde{I} = T(\epsilon) I, \quad \frac{\partial}{\partial x_i} \tilde{I} = \epsilon \tilde{A}_i \tilde{I}$$

Canonical
5 MB

Examples: 2-loop 5-point nonplanar DE

New method:

$$d\tilde{I}(s_{ij};\epsilon) = \epsilon\left(\sum_{k=1}^{31} a_k d\log W_k(s_{ij})\right)\tilde{I}(s_{ij};\epsilon)$$

31 (108,108) matrices with rational number entries, fitted numerically, 291 kB

symbol letters

$$\begin{aligned}
W_1 &= v_1\,, & W_6 &= v_3+v_4\,, & W_{11} &= v_1-v_4\,, & W_{16} &= v_1+v_2-v_4\,,\\
W_2 &= v_2\,, & W_7 &= v_4+v_5\,, & W_{12} &= v_2-v_5\,, & W_{17} &= v_2+v_3-v_5\,,\\
W_3 &= v_3\,, & W_8 &= v_5+v_1\,, & W_{13} &= v_3-v_1\,, & W_{18} &= v_3+v_4-v_1\,,\\
W_4 &= v_4\,, & W_9 &= v_1+v_2\,, & W_{14} &= v_4-v_2\,, & W_{19} &= v_4+v_5-v_2\,,\\
W_5 &= v_5\,, & W_{10} &= v_2+v_3\,, & W_{15} &= v_5-v_3\,, & W_{20} &= v_5+v_1-v_3\,,
\end{aligned}$$

$$v_1 = s_{12},\ v_2 = s_{23},\ v_3 = s_{34},\ v_4 = s_{45},\ v_5 = s_{15}$$

$$\begin{aligned}
W_{21} &= v_3+v_4-v_1-v_2\,,\\
W_{22} &= v_4+v_5-v_2-v_3\,,\\
W_{23} &= v_5+v_1-v_3-v_4\,,\\
W_{24} &= v_1+v_2-v_4-v_5\,,\\
W_{25} &= v_2+v_3-v_5-v_1\,,
\end{aligned}$$

only for nonplanar

$$\begin{aligned}
W_{26} &= \frac{v_1v_2-v_2v_3+v_3v_4-v_1v_5-v_4v_5-\sqrt{\Delta}}{v_1v_2-v_2v_3+v_3v_4-v_1v_5-v_4v_5+\sqrt{\Delta}}\,,\\
W_{27} &= \frac{-v_1v_2+v_2v_3-v_3v_4-v_1v_5+v_4v_5-\sqrt{\Delta}}{-v_1v_2+v_2v_3-v_3v_4-v_1v_5+v_4v_5+\sqrt{\Delta}}\,,\\
W_{28} &= \frac{-v_1v_2-v_2v_3+v_3v_4+v_1v_5-v_4v_5-\sqrt{\Delta}}{-v_1v_2-v_2v_3+v_3v_4+v_1v_5-v_4v_5+\sqrt{\Delta}}\,,\\
W_{29} &= \frac{v_1v_2-v_2v_3-v_3v_4-v_1v_5+v_4v_5-\sqrt{\Delta}}{v_1v_2-v_2v_3-v_3v_4-v_1v_5+v_4v_5+\sqrt{\Delta}}\,,\\
W_{30} &= \frac{-v_1v_2+v_2v_3-v_3v_4+v_1v_5-v_4v_5-\sqrt{\Delta}}{-v_1v_2+v_2v_3-v_3v_4+v_1v_5-v_4v_5+\sqrt{\Delta}}\,,
\end{aligned}$$

odd letters

$$W_{31} = \sqrt{\Delta}\,.$$

Boundary conditions are determined by the physical limits

Analytic Feynman integral: Solution

implemented in **Ginac**

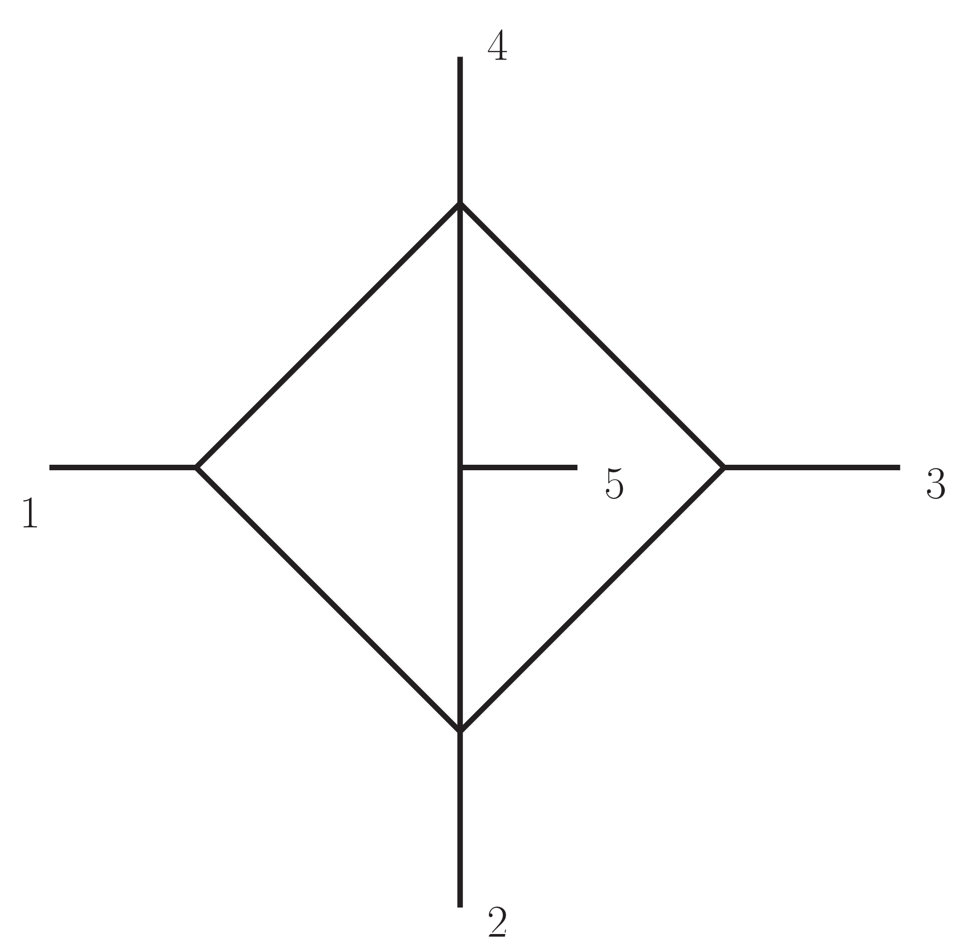
$$G(\underbrace{0,\ldots,0}_{k};z) = \frac{1}{k!}\left(\log z\right)^k, \qquad G(a_1,\ldots,a_k;z) = \int_0^z \frac{dt}{t-a_1} G(a_2,\ldots,a_k;t)$$

Chicherin, Gehrmann, Henn, Wasser, **YZ**, Zoia, 2019

All 2-loop 5-point massless integrals are analytically evaluated
Goncharov polylogarithm (or simpler logarithm) up to weight 4

A tiny example:

Caron-Huot, Chicherin, Henn, Peraro, **YZ**, Zoia 2020



$$I = \frac{1}{\epsilon^2} f^{(2)} + \frac{1}{\epsilon} f^{(3)} + f^{(4)} + \mathcal{O}(\epsilon)\,.$$

Leading part:

$$f^{(2)} = -3\Big[\mathrm{Li}_2\Big(\frac{1}{W_{27}}\Big) - \mathrm{Li}_2\Big(W_{27}\Big) + \mathrm{Li}_2\Big(\frac{1}{W_{28}}\Big) - \mathrm{Li}_2\Big(\frac{1}{W_{27}W_{28}}\Big) - \mathrm{Li}_2\Big(W_{28}\Big) + \mathrm{Li}_2\Big(W_{27}W_{28}\Big)\Big]\,.$$

Again, why analytic

2-loop 5-point massless integrals in the physical region

Analytic (our result)

with one CPU, GiNaC

~ minutes for one point, 10+ digits

Numeric (**pySecDec**)

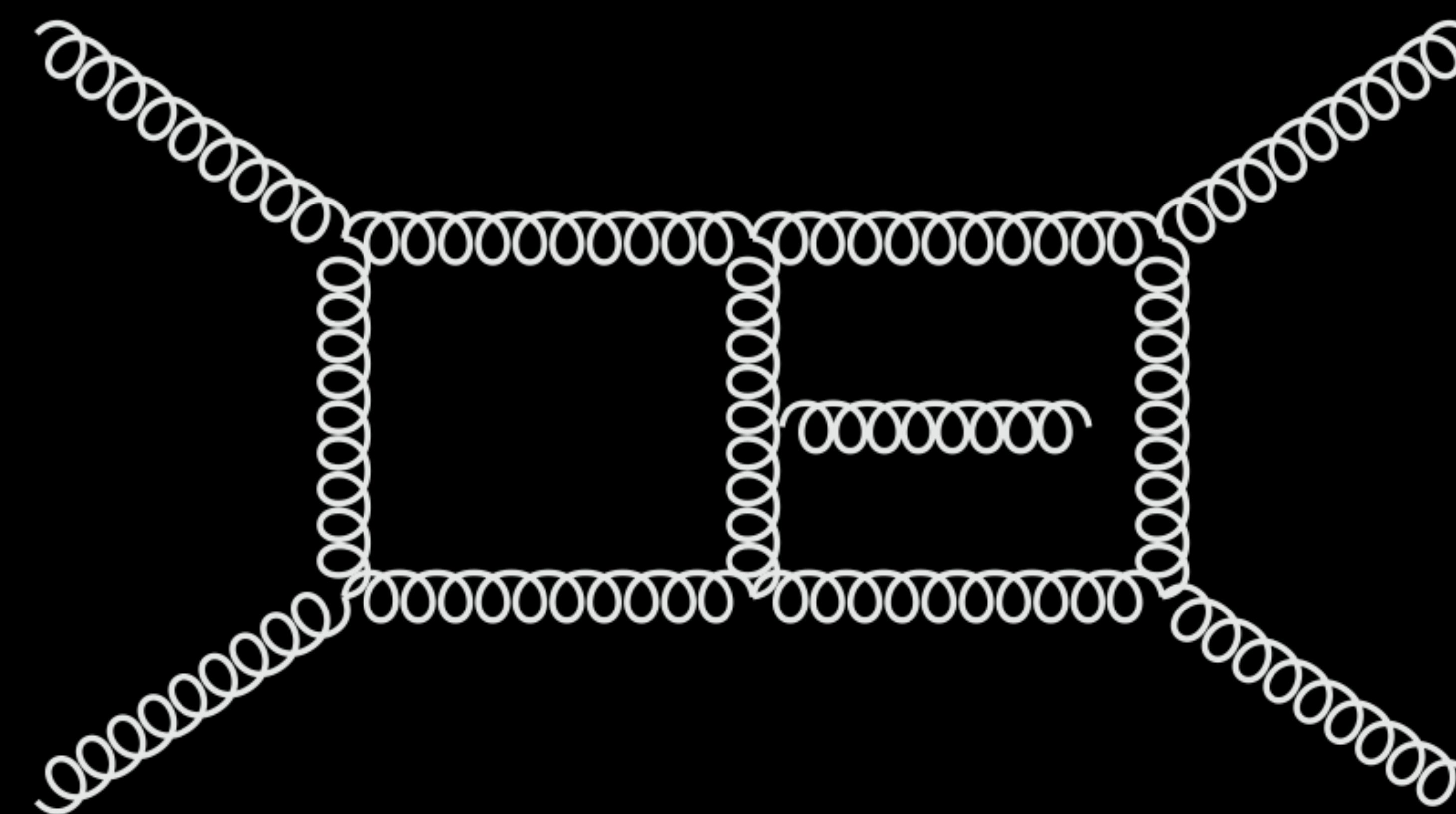
$\times 8$

NVIDIA Tesla V100 GPUs

1 week to get one numeric point
error estimated to be ~0.5%

Application of Computational Algebraic Geometry for IBP reductions

An Example



2loop 5point nonplanar massless integrals with the numerator degree up to deg 5

Analytic IBP reduction coefficients
can be obtained, but huge …

Look at one example, a 5-page long coefficient

$$
\begin{aligned}
&(-64\epsilon s_{15}^3 s_{45}^7-16 s_{15}^3 s_{45}^7+64\epsilon s_{34}^3 s_{45}^7+16 s_{34}^3 s_{45}^7-4\epsilon s_{12} s_{15}^2 s_{45}^7-s_{12} s_{15}^2 s_{45}^7\\
&-4\epsilon s_{12} s_{34}^2 s_{45}^7-s_{12} s_{34}^2 s_{45}^7-192\epsilon s_{15} s_{34}^2 s_{45}^7-48 s_{15} s_{34}^2 s_{45}^7-48\epsilon s_{23} s_{34}^2 s_{45}^7\\
&-12 s_{23} s_{34}^2 s_{45}^7-48\epsilon s_{15}^2 s_{23} s_{45}^7-12 s_{15}^2 s_{23} s_{45}^7+12\epsilon s_{12} s_{15} s_{23} s_{45}^7+3 s_{12} s_{15} s_{23} s_{45}^7\\
&+192\epsilon s_{15}^2 s_{34} s_{45}^7+48 s_{15}^2 s_{34} s_{45}^7+8\epsilon s_{12} s_{15} s_{34} s_{45}^7+2 s_{12} s_{15} s_{34} s_{45}^7\\
&-12\epsilon s_{12} s_{23} s_{34} s_{45}^7-3 s_{12} s_{23} s_{34} s_{45}^7+96\epsilon s_{15} s_{23} s_{34} s_{45}^7+24 s_{15} s_{23} s_{34} s_{45}^7\\
&-64\epsilon s_{15}^4 s_{45}^6-16 s_{15}^4 s_{45}^6+128\epsilon s_{34}^4 s_{45}^6+32 s_{34}^4 s_{45}^6+248\epsilon s_{12} s_{15}^3 s_{45}^6\\
&+62 s_{12} s_{15}^3 s_{45}^6-196\epsilon s_{12} s_{34}^3 s_{45}^6-49 s_{12} s_{34}^3 s_{45}^6-320\epsilon s_{15} s_{34}^3 s_{45}^6\\
&-80 s_{15} s_{34}^3 s_{45}^6-352\epsilon s_{23} s_{34}^3 s_{45}^6-76 s_{23} s_{34}^3 s_{45}^6+12\epsilon s_{12}^2 s_{15}^2 s_{45}^6\\
&+3 s_{12}^2 s_{15}^2 s_{45}^6-12\epsilon s_{12}^2 s_{23}^2 s_{45}^6-3 s_{12}^2 s_{23}^2 s_{45}^6+96\epsilon s_{15}^2 s_{23}^2 s_{45}^6\\
&+24 s_{15}^2 s_{23}^2 s_{45}^6-96\epsilon s_{12} s_{15} s_{23}^2 s_{45}^6-24 s_{12} s_{15} s_{23}^2 s_{45}^6+8\epsilon s_{12}^2 s_{34}^2 s_{45}^6\\
&+2 s_{12}^2 s_{34}^2 s_{45}^6+192\epsilon s_{15}^2 s_{34}^2 s_{45}^6+48 s_{15}^2 s_{34}^2 s_{45}^6+144\epsilon s_{23}^2 s_{34}^2 s_{45}^6\\
&+36 s_{23}^2 s_{34}^2 s_{45}^6+640\epsilon s_{12} s_{15} s_{34}^2 s_{45}^6+160 s_{12} s_{15} s_{34}^2 s_{45}^6+264\epsilon s_{12} s_{23} s_{34}^2 s_{45}^6\\
&+54 s_{12} s_{23} s_{34}^2 s_{45}^6+784\epsilon s_{15} s_{23} s_{34}^2 s_{45}^6+172 s_{15} s_{23} s_{34}^2 s_{45}^6+80\epsilon s_{15}^3 s_{23} s_{45}^6\\
&+20 s_{15}^3 s_{23} s_{45}^6+176\epsilon s_{12} s_{15}^2 s_{23} s_{45}^6+44 s_{12} s_{15}^2 s_{23} s_{45}^6-24\epsilon s_{12}^2 s_{15} s_{23} s_{45}^6\\
&-6 s_{12}^2 s_{15} s_{23} s_{45}^6+64\epsilon s_{15}^3 s_{34} s_{45}^6+16 s_{15}^3 s_{34} s_{45}^6-692\epsilon s_{12} s_{15}^2 s_{34} s_{45}^6\\
&-173 s_{12} s_{15}^2 s_{34} s_{45}^6-12\epsilon s_{12} s_{23}^2 s_{34} s_{45}^6-3 s_{12} s_{23}^2 s_{34} s_{45}^6-240\epsilon s_{15} s_{23}^2 s_{34} s_{45}^6\\
&-60 s_{15} s_{23}^2 s_{34} s_{45}^6-20\epsilon s_{12}^2 s_{15} s_{34} s_{45}^6-5 s_{12}^2 s_{15} s_{34} s_{45}^6+28\epsilon s_{12}^2 s_{23} s_{34} s_{45}^6\\
&+7 s_{12}^2 s_{23} s_{34} s_{45}^6-512\epsilon s_{15}^2 s_{23} s_{34} s_{45}^6-116 s_{15}^2 s_{23} s_{34} s_{45}^6-440\epsilon s_{12} s_{15} s_{23} s_{34} s_{45}^6\\
&-98 s_{12} s_{15} s_{23} s_{34} s_{45}^6+64\epsilon s_{34}^5 s_{45}^5+16 s_{34}^5 s_{45}^5+252\epsilon s_{12} s_{15}^4 s_{45}^5+63 s_{12} s_{15}^4 s_{45}^5\\
&-376\epsilon s_{12} s_{34}^4 s_{45}^5-94 s_{12} s_{34}^4 s_{45}^5-64\epsilon s_{15} s_{34}^4 s_{45}^5-16 s_{15} s_{34}^4 s_{45}^5-560\epsilon s_{23} s_{34}^4 s_{45}^5\\
&-116 s_{23} s_{34}^4 s_{45}^5-360\epsilon s_{12}^2 s_{15}^3 s_{45}^5-90 s_{12}^2 s_{15}^3 s_{45}^5+24\epsilon s_{12}^2 s_{23}^3 s_{45}^5\\
&+6 s_{12}^2 s_{23}^3 s_{45}^5-48\epsilon s_{15}^2 s_{23}^3 s_{45}^5-12 s_{15}^2 s_{23}^3 s_{45}^5+156\epsilon s_{12} s_{15} s_{23}^3 s_{45}^5\\
&+39 s_{12} s_{15} s_{23}^3 s_{45}^5+256\epsilon s_{12}^2 s_{34}^3 s_{45}^5+64 s_{12}^2 s_{34}^3 s_{45}^5-192\epsilon s_{15}^2 s_{34}^3 s_{45}^5\\
&-48 s_{15}^2 s_{34}^3 s_{45}^5+672\epsilon s_{23}^2 s_{34}^3 s_{45}^5+144 s_{23}^2 s_{34}^3 s_{45}^5+1056\epsilon s_{12} s_{15} s_{34}^3 s_{45}^5\\
&+264 s_{12} s_{15} s_{34}^3 s_{45}^5+1164\epsilon s_{12} s_{23} s_{34}^3 s_{45}^5+231 s_{12} s_{23} s_{34}^3 s_{45}^5+1088\epsilon s_{15} s_{23} s_{34}^3 s_{45}^5\\
&+236 s_{15} s_{23} s_{34}^3 s_{45}^5-12\epsilon s_{12}^3 s_{15}^2 s_{45}^5-3 s_{12}^3 s_{15}^2 s_{45}^5+32\epsilon s_{12}^3 s_{23}^2 s_{45}^5\\
&+8 s_{12}^3 s_{23}^2 s_{45}^5-32\epsilon s_{15}^3 s_{23}^2 s_{45}^5-8 s_{15}^3 s_{23}^2 s_{45}^5-368\epsilon s_{12} s_{15}^2 s_{23}^2 s_{45}^5\\
&-92 s_{12} s_{15}^2 s_{23}^2 s_{45}^5+236\epsilon s_{12}^2 s_{15} s_{23}^2 s_{45}^5+59 s_{12}^2 s_{15} s_{23}^2 s_{45}^5-4\epsilon s_{12}^3 s_{34}^2 s_{45}^5\\
&-s_{12}^3 s_{34}^2 s_{45}^5+320\epsilon s_{15}^3 s_{34}^2 s_{45}^5+80 s_{15}^3 s_{34}^2 s_{45}^5-144\epsilon s_{23}^3 s_{34}^2 s_{45}^5\\
&-36 s_{23}^3 s_{34}^2 s_{45}^5-732\epsilon s_{12} s_{15}^2 s_{34}^2 s_{45}^5-183 s_{12} s_{15}^2 s_{34}^2 s_{45}^5-832\epsilon s_{12} s_{23}^2 s_{34}^2 s_{45}^5\\
&-184 s_{12} s_{23}^2 s_{34}^2 s_{45}^5-1152\epsilon s_{15} s_{23}^2 s_{34}^2 s_{45}^5-240 s_{15} s_{23}^2 s_{34}^2 s_{45}^5\\
&-884\epsilon s_{12}^2 s_{15} s_{34}^2 s_{45}^5-221 s_{12}^2 s_{15} s_{34}^2 s_{45}^5-464\epsilon s_{12}^2 s_{23} s_{34}^2 s_{45}^5\\
&-80 s_{12}^2 s_{23} s_{34}^2 s_{45}^5-416\epsilon s_{15}^2 s_{23} s_{34}^2 s_{45}^5-104 s_{15}^2 s_{23} s_{34}^2 s_{45}^5\\
&-2208\epsilon s_{12} s_{15} s_{23} s_{34}^2 s_{45}^5-468 s_{12} s_{15} s_{23} s_{34}^2 s_{45}^5+80\epsilon s_{15}^4 s_{23} s_{45}^5\\
&+20 s_{15}^4 s_{23} s_{45}^5-264\epsilon s_{12} s_{15}^3 s_{23} s_{45}^5-66 s_{12} s_{15}^3 s_{23} s_{45}^5-220\epsilon s_{12}^2 s_{15}^2 s_{23} s_{45}^5\\
&-55 s_{12}^2 s_{15}^2 s_{23} s_{45}^5+4\epsilon s_{12}^3 s_{15} s_{23} s_{45}^5+s_{12}^3 s_{15} s_{23} s_{45}^5-128\epsilon s_{15}^4 s_{34} s_{45}^5\\
&-32 s_{15}^4 s_{34} s_{45}^5-200\epsilon s_{12} s_{15}^3 s_{34} s_{45}^5-50 s_{12} s_{15}^3 s_{34} s_{45}^5+60\epsilon s_{12} s_{23}^3 s_{34} s_{45}^5\\
&+15 s_{12} s_{23}^3 s_{34} s_{45}^5+192\epsilon s_{15} s_{23}^3 s_{34} s_{45}^5+48 s_{15} s_{23}^3 s_{34} s_{45}^5+988\epsilon s_{12}^2 s_{15}^2 s_{34} s_{45}^5\\
&+247 s_{12}^2 s_{15}^2 s_{34} s_{45}^5+128\epsilon s_{12}^2 s_{23}^2 s_{34} s_{45}^5+32 s_{12}^2 s_{23}^2 s_{34} s_{45}^5\\
&+512\epsilon s_{15}^2 s_{23}^2 s_{34} s_{45}^5+104 s_{15}^2 s_{23}^2 s_{34} s_{45}^5+796\epsilon s_{12} s_{15} s_{23}^2 s_{34} s_{45}^5\\
&+175 s_{12} s_{15} s_{23}^2 s_{34} s_{45}^5+16\epsilon s_{12}^3 s_{15} s_{34} s_{45}^5+4 s_{12}^3 s_{15} s_{34} s_{45}^5-20\epsilon s_{12}^3 s_{23} s_{34} s_{45}^5\\
&-5 s_{12}^3 s_{23} s_{34} s_{45}^5-192\epsilon s_{15}^3 s_{23} s_{34} s_{45}^5-36 s_{15}^3 s_{23} s_{34} s_{45}^5+1308\epsilon s_{12} s_{15}^2 s_{23} s_{34} s_{45}^5\\
&+303 s_{12} s_{15}^2 s_{23} s_{34} s_{45}^5+632\epsilon s_{12}^2 s_{15} s_{23} s_{34} s_{45}^5+122 s_{12}^2 s_{15} s_{23} s_{34} s_{45}^5\\
&-184\epsilon s_{12} s_{34}^5 s_{45}^4-46 s_{12} s_{34}^5 s_{45}^4+64\epsilon s_{15} s_{34}^5 s_{45}^4+16 s_{15} s_{34}^5 s_{45}^4-256\epsilon s_{23} s_{34}^5 s_{45}^4\\
&-52 s_{23} s_{34}^5 s_{45}^4-372\epsilon s_{12}^2 s_{15}^4 s_{45}^4-93 s_{12}^2 s_{15}^4 s_{45}^4-12\epsilon s_{12}^2 s_{23}^4 s_{45}^4\\
&-3 s_{12}^2 s_{23}^4 s_{45}^4-72\epsilon s_{12} s_{15} s_{23}^4 s_{45}^4-18 s_{12} s_{15} s_{23}^4 s_{45}^4+368\epsilon s_{12}^2 s_{34}^4 s_{45}^4
\end{aligned}
$$

page 1

Look at one example, a 5-page long coefficient

$$
\begin{aligned}
&+92s_{12}^2s_{34}^4s_{45}^4-192\epsilon s_{15}^2s_{34}^4s_{45}^4-48s_{15}^2s_{34}^4s_{45}^4+912\epsilon s_{23}^2s_{34}^4s_{45}^4\\
&+180s_{23}^2s_{34}^4s_{45}^4+240\epsilon s_{12}s_{15}s_{34}^4s_{45}^4+60s_{12}s_{15}s_{34}^4s_{45}^4+1520\epsilon s_{12}s_{23}s_{34}^4s_{45}^4\\
&+308s_{12}s_{23}s_{34}^4s_{45}^4+208\epsilon s_{15}s_{23}s_{34}^4s_{45}^4+52s_{15}s_{23}s_{34}^4s_{45}^4+232\epsilon s_{12}^3s_{15}^3s_{45}^4\\
&+58s_{12}^3s_{15}^3s_{45}^4-52\epsilon s_{12}^3s_{23}^3s_{45}^4-13s_{12}^3s_{23}^3s_{45}^4+148\epsilon s_{12}s_{15}^2s_{23}^3s_{45}^4\\
&+37s_{12}s_{15}^2s_{23}^3s_{45}^4-384\epsilon s_{12}^2s_{15}s_{23}^3s_{45}^4-96s_{12}^2s_{15}s_{23}^3s_{45}^4-124\epsilon s_{12}^3s_{34}^3s_{45}^4\\
&-31s_{12}^3s_{34}^3s_{45}^4+192\epsilon s_{15}^3s_{34}^3s_{45}^4+48s_{15}^3s_{34}^3s_{45}^4-544\epsilon s_{23}^3s_{34}^3s_{45}^4\\
&-124s_{23}^3s_{34}^3s_{45}^4+572\epsilon s_{12}s_{15}^2s_{34}^3s_{45}^4+143s_{12}s_{15}^2s_{34}^3s_{45}^4-2284\epsilon s_{12}s_{23}^2s_{34}^3s_{45}^4\\
&-463s_{12}s_{23}^2s_{34}^3s_{45}^4-1392\epsilon s_{15}s_{23}^2s_{34}^3s_{45}^4-264s_{15}s_{23}^2s_{34}^3s_{45}^4\\
&-1160\epsilon s_{12}^2s_{15}s_{34}^3s_{45}^4-290s_{12}^2s_{15}s_{34}^3s_{45}^4-1668\epsilon s_{12}^2s_{23}s_{34}^3s_{45}^4\\
&-321s_{12}^2s_{23}s_{34}^3s_{45}^4+512\epsilon s_{15}^2s_{23}s_{34}^3s_{45}^4+92s_{15}^2s_{23}s_{34}^3s_{45}^4\\
&-2708\epsilon s_{12}s_{15}s_{23}s_{34}^3s_{45}^4-605s_{12}s_{15}s_{23}s_{34}^3s_{45}^4+4\epsilon s_{12}^4s_{15}^2s_{45}^4+s_{12}^4s_{15}^2s_{45}^4\\
&-28\epsilon s_{12}^4s_{23}^2s_{45}^4-7s_{12}^4s_{23}^2s_{45}^4+176\epsilon s_{12}s_{15}^3s_{23}^2s_{45}^4+44s_{12}s_{15}^3s_{23}^2s_{45}^4\\
&+588\epsilon s_{12}^2s_{15}^2s_{23}^2s_{45}^4+147s_{12}^2s_{15}^2s_{23}^2s_{45}^4-156\epsilon s_{12}^3s_{15}s_{23}^2s_{45}^4\\
&-39s_{12}^3s_{15}s_{23}^2s_{45}^4-64\epsilon s_{15}^4s_{34}^2s_{45}^4-16s_{15}^4s_{34}^2s_{45}^4+48\epsilon s_{23}^4s_{34}^2s_{45}^4\\
&+12s_{23}^4s_{34}^2s_{45}^4-1128\epsilon s_{12}s_{15}^3s_{34}^2s_{45}^4-282s_{12}s_{15}^3s_{34}^2s_{45}^4+920\epsilon s_{12}s_{23}^3s_{34}^2s_{45}^4\\
&+218s_{12}s_{23}^3s_{34}^2s_{45}^4+720\epsilon s_{15}s_{23}^3s_{34}^2s_{45}^4+156s_{15}s_{23}^3s_{34}^2s_{45}^4\\
&+832\epsilon s_{12}^2s_{15}^2s_{34}^2s_{45}^4+208s_{12}^2s_{15}^2s_{34}^2s_{45}^4+1508\epsilon s_{12}^2s_{23}^2s_{34}^2s_{45}^4\\
&+317s_{12}^2s_{23}^2s_{34}^2s_{45}^4+416\epsilon s_{15}^2s_{23}^2s_{34}^2s_{45}^4+80s_{15}^2s_{23}^2s_{34}^2s_{45}^4\\
&+2628\epsilon s_{12}s_{15}s_{23}^2s_{34}^2s_{45}^4+501s_{12}s_{15}s_{23}^2s_{34}^2s_{45}^4+560\epsilon s_{12}^3s_{15}s_{34}^2s_{45}^4\\
&+140s_{12}^3s_{15}s_{34}^2s_{45}^4+460\epsilon s_{12}^3s_{23}s_{34}^2s_{45}^4+79s_{12}^3s_{23}s_{34}^2s_{45}^4\\
&-624\epsilon s_{15}^3s_{23}s_{34}^2s_{45}^4-132s_{15}^3s_{23}s_{34}^2s_{45}^4+1124\epsilon s_{12}s_{15}^2s_{23}s_{34}^2s_{45}^4\\
&+317s_{12}s_{15}^2s_{23}s_{34}^2s_{45}^4+2424\epsilon s_{12}^2s_{15}s_{23}s_{34}^2s_{45}^4+510s_{12}^2s_{15}s_{23}s_{34}^2s_{45}^4\\
&-252\epsilon s_{12}s_{15}^4s_{23}s_{45}^4-63s_{12}s_{15}^4s_{23}s_{45}^4+292\epsilon s_{12}^2s_{15}^3s_{23}s_{45}^4\\
&+73s_{12}^2s_{15}^3s_{23}s_{45}^4+80\epsilon s_{12}^3s_{15}^2s_{23}s_{45}^4+20s_{12}^3s_{15}^2s_{23}s_{45}^4+16\epsilon s_{12}^4s_{15}s_{23}s_{45}^4\\
&+4s_{12}^4s_{15}s_{23}s_{45}^4+500\epsilon s_{12}s_{15}^4s_{34}s_{45}^4+125s_{12}s_{15}^4s_{34}s_{45}^4-36\epsilon s_{12}s_{23}^4s_{34}s_{45}^4\\
&-9s_{12}s_{23}^4s_{34}s_{45}^4-48\epsilon s_{15}s_{23}^4s_{34}s_{45}^4-12s_{15}s_{23}^4s_{34}s_{45}^4+332\epsilon s_{12}^2s_{15}^3s_{34}s_{45}^4\\
&+83s_{12}^2s_{15}^3s_{34}s_{45}^4-324\epsilon s_{12}^2s_{23}^3s_{34}s_{45}^4-81s_{12}^2s_{23}^3s_{34}s_{45}^4\\
&-176\epsilon s_{15}^2s_{23}^3s_{34}s_{45}^4-32s_{15}^2s_{23}^3s_{34}s_{45}^4-400\epsilon s_{12}s_{15}s_{23}^3s_{34}s_{45}^4\\
&-88s_{12}s_{15}s_{23}^3s_{34}s_{45}^4-668\epsilon s_{12}^3s_{15}^2s_{34}s_{45}^4-167s_{12}^3s_{15}^2s_{34}s_{45}^4\\
&-228\epsilon s_{12}^3s_{23}^2s_{34}s_{45}^4-57s_{12}^3s_{23}^2s_{34}s_{45}^4+64\epsilon s_{15}^3s_{23}^2s_{34}s_{45}^4+4s_{15}^3s_{23}^2s_{34}s_{45}^4\\
&-1068\epsilon s_{12}s_{15}^2s_{23}^2s_{34}s_{45}^4-219s_{12}s_{15}^2s_{23}^2s_{34}s_{45}^4-664\epsilon s_{12}^2s_{15}s_{23}^2s_{34}s_{45}^4\\
&-106s_{12}^2s_{15}s_{23}^2s_{34}s_{45}^4-4\epsilon s_{12}^4s_{15}s_{34}s_{45}^4-s_{12}^4s_{15}s_{34}s_{45}^4+4\epsilon s_{12}^4s_{23}s_{34}s_{45}^4\\
&+s_{12}^4s_{23}s_{34}s_{45}^4+160\epsilon s_{15}^4s_{23}s_{34}s_{45}^4+40s_{15}^4s_{23}s_{34}s_{45}^4+316\epsilon s_{12}s_{15}^3s_{23}s_{34}s_{45}^4\\
&+43s_{12}s_{15}^3s_{23}s_{34}s_{45}^4-1228\epsilon s_{12}^2s_{15}^2s_{23}s_{34}s_{45}^4-307s_{12}^2s_{15}^2s_{23}s_{34}s_{45}^4\\
&-300\epsilon s_{12}^3s_{15}s_{23}s_{34}s_{45}^4-39s_{12}^3s_{15}s_{23}s_{34}s_{45}^4+120\epsilon s_{12}^2s_{34}^5s_{45}^3\\
&+30s_{12}^2s_{34}^5s_{45}^3+384\epsilon s_{23}^2s_{34}^5s_{45}^3+72s_{23}^2s_{34}^5s_{45}^3-184\epsilon s_{12}s_{15}s_{34}^5s_{45}^3\\
&-46s_{12}s_{15}s_{34}^5s_{45}^3+632\epsilon s_{12}s_{23}s_{34}^5s_{45}^3+134s_{12}s_{23}s_{34}^5s_{45}^3-192\epsilon s_{15}s_{23}s_{34}^5s_{45}^3\\
&-36s_{15}s_{23}s_{34}^5s_{45}^3+244\epsilon s_{12}^3s_{15}^4s_{45}^3+61s_{12}^3s_{15}^4s_{45}^3+4\epsilon s_{12}^3s_{23}^4s_{45}^3\\
&+s_{12}^3s_{23}^4s_{45}^3+124\epsilon s_{12}^2s_{15}s_{23}^4s_{45}^3+31s_{12}^2s_{15}s_{23}^4s_{45}^3-120\epsilon s_{12}^3s_{34}^4s_{45}^3\\
&-30s_{12}^3s_{34}^4s_{45}^3-656\epsilon s_{23}^3s_{34}^4s_{45}^3-140s_{23}^3s_{34}^4s_{45}^3+616\epsilon s_{12}s_{15}^2s_{34}^4s_{45}^3\\
&+154s_{12}s_{15}^2s_{34}^4s_{45}^3-2256\epsilon s_{12}s_{23}^2s_{34}^4s_{45}^3-444s_{12}s_{23}^2s_{34}^4s_{45}^3\\
&-288\epsilon s_{15}s_{23}^2s_{34}^4s_{45}^3-48s_{15}s_{23}^2s_{34}^4s_{45}^3-176\epsilon s_{12}^2s_{15}s_{34}^4s_{45}^3\\
&-44s_{12}^2s_{15}s_{34}^4s_{45}^3-1512\epsilon s_{12}^2s_{23}s_{34}^4s_{45}^3-306s_{12}^2s_{23}s_{34}^4s_{45}^3\\
&+464\epsilon s_{15}^2s_{23}s_{34}^4s_{45}^3+92s_{15}^2s_{23}s_{34}^4s_{45}^3-504\epsilon s_{12}s_{15}s_{23}s_{34}^4s_{45}^3\\
&-150s_{12}s_{15}s_{23}s_{34}^4s_{45}^3-56\epsilon s_{12}^4s_{15}^3s_{45}^3-14s_{12}^4s_{15}^3s_{45}^3+52\epsilon s_{12}^4s_{23}^3s_{45}^3\\
&+13s_{12}^4s_{23}^3s_{45}^3-112\epsilon s_{12}^2s_{15}^2s_{23}^3s_{45}^3-28s_{12}^2s_{15}^2s_{23}^3s_{45}^3+360\epsilon s_{12}^3s_{15}s_{23}^3s_{45}^3
\end{aligned}
$$

page 2

Look at one example, a 5-page long coefficient

$$
\begin{aligned}
&+90s_{12}^3s_{15}s_{23}^3s_{45}^3+160\epsilon s_{23}^4s_{34}^3s_{45}^3+40s_{23}^4s_{34}^3s_{45}^3-680\epsilon s_{12}s_{15}^3s_{34}^3s_{45}^3\\
&-170s_{12}s_{15}^3s_{34}^3s_{45}^3+1940\epsilon s_{12}s_{23}^3s_{34}^3s_{45}^3+437s_{12}s_{23}^3s_{34}^3s_{45}^3\\
&+800\epsilon s_{15}s_{23}^3s_{34}^3s_{45}^3+152s_{15}s_{23}^3s_{34}^3s_{45}^3-688\epsilon s_{12}^2s_{15}^2s_{34}^3s_{45}^3\\
&-172s_{12}^2s_{15}^2s_{34}^3s_{45}^3+3184\epsilon s_{12}^2s_{23}^2s_{34}^3s_{45}^3+652s_{12}^2s_{23}^2s_{34}^3s_{45}^3\\
&-320\epsilon s_{15}^2s_{23}^2s_{34}^3s_{45}^3-56s_{15}^2s_{23}^2s_{34}^3s_{45}^3+2480\epsilon s_{12}s_{15}s_{23}^2s_{34}^3s_{45}^3\\
&+452s_{12}s_{15}s_{23}^2s_{34}^3s_{45}^3+544\epsilon s_{12}^3s_{15}s_{34}^3s_{45}^3+136s_{12}^3s_{15}s_{34}^3s_{45}^3\\
&+1032\epsilon s_{12}^3s_{23}s_{34}^3s_{45}^3+198s_{12}^3s_{23}s_{34}^3s_{45}^3-352\epsilon s_{15}^3s_{23}s_{34}^3s_{45}^3\\
&-76s_{15}^3s_{23}s_{34}^3s_{45}^3-1072\epsilon s_{12}s_{15}^2s_{23}s_{34}^3s_{45}^3-160s_{12}s_{15}^2s_{23}s_{34}^3s_{45}^3\\
&+1808\epsilon s_{12}^2s_{15}s_{23}s_{34}^3s_{45}^3+428s_{12}^2s_{15}s_{23}s_{34}^3s_{45}^3+8\epsilon s_{12}^5s_{23}^2s_{45}^3\\
&+2s_{12}^5s_{23}^2s_{45}^3-276\epsilon s_{12}^2s_{15}^3s_{23}^2s_{45}^3-69s_{12}^2s_{15}^3s_{23}^2s_{45}^3-496\epsilon s_{12}^3s_{15}^2s_{23}^2s_{45}^3\\
&-124s_{12}^3s_{15}^2s_{23}^2s_{45}^3-32\epsilon s_{12}^4s_{15}s_{23}^2s_{45}^3-8s_{12}^4s_{15}s_{23}^2s_{45}^3+248\epsilon s_{12}s_{15}^4s_{34}^2s_{45}^3\\
&+62s_{12}s_{15}^4s_{34}^2s_{45}^3-348\epsilon s_{12}s_{23}^4s_{34}^2s_{45}^3-87s_{12}s_{23}^4s_{34}^2s_{45}^3\\
&-160\epsilon s_{15}s_{23}^4s_{34}^2s_{45}^3-40s_{15}s_{23}^4s_{34}^2s_{45}^3+1420\epsilon s_{12}^2s_{15}^3s_{34}^2s_{45}^3\\
&+355s_{12}^2s_{15}^3s_{34}^2s_{45}^3-1688\epsilon s_{12}^2s_{23}^3s_{34}^2s_{45}^3-398s_{12}^2s_{23}^3s_{34}^2s_{45}^3\\
&-144\epsilon s_{15}^2s_{23}^3s_{34}^2s_{45}^3-12s_{15}^2s_{23}^3s_{34}^2s_{45}^3-1336\epsilon s_{12}s_{15}s_{23}^3s_{34}^2s_{45}^3\\
&-262s_{12}s_{15}s_{23}^3s_{34}^2s_{45}^3-348\epsilon s_{12}^3s_{15}^2s_{34}^2s_{45}^3-87s_{12}^3s_{15}^2s_{34}^2s_{45}^3\\
&-1536\epsilon s_{12}^3s_{23}^2s_{34}^2s_{45}^3-336s_{12}^3s_{23}^2s_{34}^2s_{45}^3+224\epsilon s_{15}^3s_{23}^2s_{34}^2s_{45}^3\\
&+32s_{15}^3s_{23}^2s_{34}^2s_{45}^3-808\epsilon s_{12}s_{15}^2s_{23}^2s_{34}^2s_{45}^3-178s_{12}s_{15}^2s_{23}^2s_{34}^2s_{45}^3\\
&-2064\epsilon s_{12}^2s_{15}s_{23}^2s_{34}^2s_{45}^3-348s_{12}^2s_{15}s_{23}^2s_{34}^2s_{45}^3-124\epsilon s_{12}^4s_{15}s_{34}^2s_{45}^3\\
&-31s_{12}^4s_{15}s_{34}^2s_{45}^3-212\epsilon s_{12}^4s_{23}s_{34}^2s_{45}^3-41s_{12}^4s_{23}s_{34}^2s_{45}^3+80\epsilon s_{15}^4s_{23}s_{34}^2s_{45}^3\\
&+20s_{15}^4s_{23}s_{34}^2s_{45}^3+1428\epsilon s_{12}s_{15}^3s_{23}s_{34}^2s_{45}^3+297s_{12}s_{15}^3s_{23}s_{34}^2s_{45}^3\\
&-588\epsilon s_{12}^2s_{15}^2s_{23}s_{34}^2s_{45}^3-231s_{12}^2s_{15}^2s_{23}s_{34}^2s_{45}^3-676\epsilon s_{12}^3s_{15}s_{23}s_{34}^2s_{45}^3\\
&-133s_{12}^3s_{15}s_{23}s_{34}^2s_{45}^3+264\epsilon s_{12}^2s_{15}^4s_{23}s_{45}^3+66s_{12}^2s_{15}^4s_{23}s_{45}^3\\
&-112\epsilon s_{12}^3s_{15}^3s_{23}s_{45}^3-28s_{12}^3s_{15}^3s_{23}s_{45}^3+36\epsilon s_{12}^4s_{15}^2s_{23}s_{45}^3+9s_{12}^4s_{15}^2s_{23}s_{45}^3\\
&-8\epsilon s_{12}^5s_{15}s_{23}s_{45}^3-2s_{12}^5s_{15}s_{23}s_{45}^3-676\epsilon s_{12}^2s_{15}^4s_{34}s_{45}^3-169s_{12}^2s_{15}^4s_{34}s_{45}^3\\
&+184\epsilon s_{12}^2s_{23}^4s_{34}s_{45}^3+46s_{12}^2s_{23}^4s_{34}s_{45}^3+68\epsilon s_{12}s_{15}s_{23}^4s_{34}s_{45}^3\\
&+17s_{12}s_{15}s_{23}^4s_{34}s_{45}^3-320\epsilon s_{12}^3s_{15}^3s_{34}s_{45}^3-80s_{12}^3s_{15}^3s_{34}s_{45}^3\\
&+412\epsilon s_{12}^3s_{23}^3s_{34}s_{45}^3+103s_{12}^3s_{23}^3s_{34}s_{45}^3+148\epsilon s_{12}s_{15}^2s_{23}^3s_{34}s_{45}^3\\
&+13s_{12}s_{15}^2s_{23}^3s_{34}s_{45}^3-116\epsilon s_{12}^2s_{15}s_{23}^3s_{34}s_{45}^3-53s_{12}^2s_{15}s_{23}^3s_{34}s_{45}^3\\
&+180\epsilon s_{12}^4s_{15}^2s_{34}s_{45}^3+45s_{12}^4s_{15}^2s_{34}s_{45}^3+216\epsilon s_{12}^4s_{23}^2s_{34}s_{45}^3+54s_{12}^4s_{23}^2s_{34}s_{45}^3\\
&+268\epsilon s_{12}s_{15}^3s_{23}^2s_{34}s_{45}^3+91s_{12}s_{15}^3s_{23}^2s_{34}s_{45}^3+1072\epsilon s_{12}^2s_{15}^2s_{23}^2s_{34}s_{45}^3\\
&+244s_{12}^2s_{15}^2s_{23}^2s_{34}s_{45}^3+72\epsilon s_{12}^3s_{15}s_{23}^2s_{34}s_{45}^3-30s_{12}^3s_{15}s_{23}^2s_{34}s_{45}^3\\
&-484\epsilon s_{12}s_{15}^4s_{23}s_{34}s_{45}^3-121s_{12}s_{15}^4s_{23}s_{34}s_{45}^3-160\epsilon s_{12}^2s_{15}^3s_{23}s_{34}s_{45}^3\\
&-4s_{12}^2s_{15}^3s_{23}s_{34}s_{45}^3+444\epsilon s_{12}^3s_{15}^2s_{23}s_{34}s_{45}^3+135s_{12}^3s_{15}^2s_{23}s_{34}s_{45}^3\\
&-92\epsilon s_{12}^4s_{15}s_{23}s_{34}s_{45}^3-35s_{12}^4s_{15}s_{23}s_{34}s_{45}^3-256\epsilon s_{23}^3s_{34}^5s_{45}^2\\
&-52s_{23}^3s_{34}^5s_{45}^2-792\epsilon s_{12}s_{23}^2s_{34}^5s_{45}^2-162s_{12}s_{23}^2s_{34}^5s_{45}^2+192\epsilon s_{15}s_{23}^2s_{34}^5s_{45}^2\\
&+36s_{15}s_{23}^2s_{34}^5s_{45}^2+120\epsilon s_{12}^2s_{15}s_{34}^5s_{45}^2+30s_{12}^2s_{15}s_{34}^5s_{45}^2-336\epsilon s_{12}^2s_{23}s_{34}^5s_{45}^2\\
&-72s_{12}^2s_{23}s_{34}^5s_{45}^2+448\epsilon s_{12}s_{15}s_{23}s_{34}^5s_{45}^2+88s_{12}s_{15}s_{23}s_{34}^5s_{45}^2-60\epsilon s_{12}^4s_{15}^4s_{45}^2\\
&-15s_{12}^4s_{15}^4s_{45}^2+8\epsilon s_{12}^4s_{23}^4s_{45}^2+2s_{12}^4s_{23}^4s_{45}^2-52\epsilon s_{12}^3s_{15}s_{23}^4s_{45}^2\\
&-13s_{12}^3s_{15}s_{23}^4s_{45}^2+176\epsilon s_{23}^4s_{34}^4s_{45}^2+44s_{23}^4s_{34}^4s_{45}^2+1520\epsilon s_{12}s_{23}^3s_{34}^4s_{45}^2\\
&+332s_{12}s_{23}^3s_{34}^4s_{45}^2+208\epsilon s_{15}s_{23}^3s_{34}^4s_{45}^2+28s_{15}s_{23}^3s_{34}^4s_{45}^2\\
&-544\epsilon s_{12}^2s_{15}^2s_{34}^4s_{45}^2-136s_{12}^2s_{15}^2s_{34}^4s_{45}^2+2144\epsilon s_{12}^2s_{23}^2s_{34}^4s_{45}^2\\
&+440s_{12}^2s_{23}^2s_{34}^4s_{45}^2-320\epsilon s_{15}^2s_{23}^2s_{34}^4s_{45}^2-56s_{15}^2s_{23}^2s_{34}^4s_{45}^2\\
&+400\epsilon s_{12}s_{15}s_{23}^2s_{34}^4s_{45}^2+76s_{12}s_{15}s_{23}^2s_{34}^4s_{45}^2+552\epsilon s_{12}^3s_{23}s_{34}^4s_{45}^2\\
&+114s_{12}^3s_{23}s_{34}^4s_{45}^2-1064\epsilon s_{12}s_{15}^2s_{23}s_{34}^4s_{45}^2-218s_{12}s_{15}^2s_{23}s_{34}^4s_{45}^2\\
&-192\epsilon s_{12}^2s_{15}s_{23}s_{34}^4s_{45}^2-24\epsilon s_{12}^5s_{23}^3s_{45}^2-6s_{12}^5s_{23}^3s_{45}^2+12\epsilon s_{12}^3s_{15}^2s_{23}^3s_{45}^2
\end{aligned}
$$

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Look at one example, a 5-page long coefficient

$$
\begin{aligned}
&+3s_{12}^{3}s_{15}^{2}s_{23}^{3}s_{45}^{2}-132\epsilon s_{12}^{4}s_{15}s_{23}^{3}s_{45}^{2}-33s_{12}^{4}s_{15}s_{23}^{3}s_{45}^{2}-624\epsilon s_{12}s_{23}^{4}s_{34}^{3}s_{45}^{2}\\
&-156s_{12}s_{23}^{4}s_{34}^{3}s_{45}^{2}-176\epsilon s_{15}s_{23}^{4}s_{34}^{3}s_{45}^{2}-44s_{15}s_{23}^{4}s_{34}^{3}s_{45}^{2}\\
&+728\epsilon s_{12}^{2}s_{15}^{3}s_{34}^{3}s_{45}^{2}+182s_{12}^{2}s_{15}^{3}s_{34}^{3}s_{45}^{2}-2500\epsilon s_{12}^{2}s_{23}^{3}s_{34}^{3}s_{45}^{2}\\
&-577s_{12}^{2}s_{23}^{3}s_{34}^{3}s_{45}^{2}+48\epsilon s_{15}^{2}s_{23}^{3}s_{34}^{3}s_{45}^{2}+24s_{15}^{2}s_{23}^{3}s_{34}^{3}s_{45}^{2}\\
&-1028\epsilon s_{12}s_{15}s_{23}^{3}s_{34}^{3}s_{45}^{2}-173s_{12}s_{15}s_{23}^{3}s_{34}^{3}s_{45}^{2}+428\epsilon s_{12}^{3}s_{15}^{2}s_{34}^{3}s_{45}^{2}\\
&+107s_{12}^{3}s_{15}^{2}s_{34}^{3}s_{45}^{2}-1980\epsilon s_{12}^{3}s_{23}^{2}s_{34}^{3}s_{45}^{2}-423s_{12}^{3}s_{23}^{2}s_{34}^{3}s_{45}^{2}\\
&+128\epsilon s_{15}^{3}s_{23}^{2}s_{34}^{3}s_{45}^{2}+20s_{15}^{3}s_{23}^{2}s_{34}^{3}s_{45}^{2}+356\epsilon s_{12}s_{15}^{2}s_{23}^{2}s_{34}^{3}s_{45}^{2}\\
&+41s_{12}s_{15}^{2}s_{23}^{2}s_{34}^{3}s_{45}^{2}-748\epsilon s_{12}^{2}s_{15}s_{23}^{2}s_{34}^{3}s_{45}^{2}-79s_{12}^{2}s_{15}s_{23}^{2}s_{34}^{3}s_{45}^{2}\\
&-120\epsilon s_{12}^{4}s_{15}s_{34}^{3}s_{45}^{2}-30s_{12}^{4}s_{15}s_{34}^{3}s_{45}^{2}-216\epsilon s_{12}^{4}s_{23}s_{34}^{3}s_{45}^{2}-42s_{12}^{4}s_{23}s_{34}^{3}s_{45}^{2}\\
&+848\epsilon s_{12}s_{15}^{3}s_{23}s_{34}^{3}s_{45}^{2}+188s_{12}s_{15}^{3}s_{23}s_{34}^{3}s_{45}^{2}+876\epsilon s_{12}^{2}s_{15}^{2}s_{23}s_{34}^{3}s_{45}^{2}\\
&+111s_{12}^{2}s_{15}^{2}s_{23}s_{34}^{3}s_{45}^{2}+20\epsilon s_{12}^{3}s_{15}s_{23}s_{34}^{3}s_{45}^{2}-19s_{12}^{3}s_{15}s_{23}s_{34}^{3}s_{45}^{2}\\
&+132\epsilon s_{12}^{3}s_{15}^{3}s_{23}^{2}s_{45}^{2}+33s_{12}^{3}s_{15}^{3}s_{23}^{2}s_{45}^{2}+180\epsilon s_{12}^{4}s_{15}^{2}s_{23}^{2}s_{45}^{2}\\
&+45s_{12}^{4}s_{15}^{2}s_{23}^{2}s_{45}^{2}+48\epsilon s_{12}^{5}s_{15}s_{23}^{2}s_{45}^{2}+12s_{12}^{5}s_{15}s_{23}^{2}s_{45}^{2}\\
&-304\epsilon s_{12}^{2}s_{15}^{4}s_{34}^{2}s_{45}^{2}-76s_{12}^{2}s_{15}^{4}s_{34}^{2}s_{45}^{2}+668\epsilon s_{12}^{2}s_{23}^{4}s_{34}^{2}s_{45}^{2}\\
&+167s_{12}^{2}s_{23}^{4}s_{34}^{2}s_{45}^{2}+388\epsilon s_{12}s_{15}s_{23}^{4}s_{34}^{2}s_{45}^{2}+97s_{12}s_{15}s_{23}^{4}s_{34}^{2}s_{45}^{2}\\
&-792\epsilon s_{12}^{3}s_{15}^{3}s_{34}^{2}s_{45}^{2}-198s_{12}^{3}s_{15}^{3}s_{34}^{2}s_{45}^{2}+1536\epsilon s_{12}^{3}s_{23}^{3}s_{34}^{2}s_{45}^{2}\\
&+372s_{12}^{3}s_{23}^{3}s_{34}^{2}s_{45}^{2}-128\epsilon s_{12}s_{15}^{2}s_{23}^{3}s_{34}^{2}s_{45}^{2}-68s_{12}s_{15}^{2}s_{23}^{3}s_{34}^{2}s_{45}^{2}\\
&+528\epsilon s_{12}^{2}s_{15}s_{23}^{3}s_{34}^{2}s_{45}^{2}+72s_{12}^{2}s_{15}s_{23}^{3}s_{34}^{2}s_{45}^{2}+56\epsilon s_{12}^{4}s_{15}^{2}s_{34}^{2}s_{45}^{2}\\
&+14s_{12}^{4}s_{15}^{2}s_{34}^{2}s_{45}^{2}+732\epsilon s_{12}^{4}s_{23}^{2}s_{34}^{2}s_{45}^{2}+171s_{12}^{4}s_{23}^{2}s_{34}^{2}s_{45}^{2}\\
&-28\epsilon s_{12}s_{15}^{3}s_{23}^{2}s_{34}^{2}s_{45}^{2}+29s_{12}s_{15}^{3}s_{23}^{2}s_{34}^{2}s_{45}^{2}+224\epsilon s_{12}^{2}s_{15}^{2}s_{23}^{2}s_{34}^{2}s_{45}^{2}\\
&+56s_{12}^{2}s_{15}^{2}s_{23}^{2}s_{34}^{2}s_{45}^{2}-240\epsilon s_{12}^{3}s_{15}s_{23}^{2}s_{34}^{2}s_{45}^{2}-132s_{12}^{3}s_{15}s_{23}^{2}s_{34}^{2}s_{45}^{2}\\
&-232\epsilon s_{12}s_{15}^{4}s_{23}s_{34}^{2}s_{45}^{2}-58s_{12}s_{15}^{4}s_{23}s_{34}^{2}s_{45}^{2}-828\epsilon s_{12}^{2}s_{15}^{3}s_{23}s_{34}^{2}s_{45}^{2}\\
&-159s_{12}^{2}s_{15}^{3}s_{23}s_{34}^{2}s_{45}^{2}+192\epsilon s_{12}^{3}s_{15}^{2}s_{23}s_{34}^{2}s_{45}^{2}+108s_{12}^{3}s_{15}^{2}s_{23}s_{34}^{2}s_{45}^{2}\\
&-380\epsilon s_{12}^{4}s_{15}s_{23}s_{34}^{2}s_{45}^{2}-95s_{12}^{4}s_{15}s_{23}s_{34}^{2}s_{45}^{2}-92\epsilon s_{12}^{3}s_{15}^{4}s_{23}s_{45}^{2}\\
&-23s_{12}^{3}s_{15}^{4}s_{23}s_{45}^{2}+4\epsilon s_{12}^{4}s_{15}^{3}s_{23}s_{45}^{2}+s_{12}^{4}s_{15}^{3}s_{23}s_{45}^{2}-24\epsilon s_{12}^{5}s_{15}^{2}s_{23}s_{45}^{2}\\
&-6s_{12}^{5}s_{15}^{2}s_{23}s_{45}^{2}+364\epsilon s_{12}^{3}s_{15}^{4}s_{34}s_{45}^{2}+91s_{12}^{3}s_{15}^{4}s_{34}s_{45}^{2}-228\epsilon s_{12}^{3}s_{23}^{4}s_{34}s_{45}^{2}\\
&-57s_{12}^{3}s_{23}^{4}s_{34}s_{45}^{2}-40\epsilon s_{12}^{2}s_{15}s_{23}^{4}s_{34}s_{45}^{2}-10s_{12}^{2}s_{15}s_{23}^{4}s_{34}s_{45}^{2}\\
&+124\epsilon s_{12}^{4}s_{15}^{3}s_{34}s_{45}^{2}+31s_{12}^{4}s_{15}^{3}s_{34}s_{45}^{2}-276\epsilon s_{12}^{4}s_{23}^{3}s_{34}s_{45}^{2}-69s_{12}^{4}s_{23}^{3}s_{34}s_{45}^{2}\\
&+160\epsilon s_{12}^{2}s_{15}^{2}s_{23}^{3}s_{34}s_{45}^{2}+52s_{12}^{2}s_{15}^{2}s_{23}^{3}s_{34}s_{45}^{2}+448\epsilon s_{12}^{3}s_{15}s_{23}^{3}s_{34}s_{45}^{2}\\
&+124s_{12}^{3}s_{15}s_{23}^{3}s_{34}s_{45}^{2}-104\epsilon s_{12}^{5}s_{23}^{2}s_{34}s_{45}^{2}-26s_{12}^{5}s_{23}^{2}s_{34}s_{45}^{2}\\
&-536\epsilon s_{12}^{2}s_{15}^{3}s_{23}^{2}s_{34}s_{45}^{2}-146s_{12}^{2}s_{15}^{3}s_{23}^{2}s_{34}s_{45}^{2}-504\epsilon s_{12}^{3}s_{15}^{2}s_{23}^{2}s_{34}s_{45}^{2}\\
&-126s_{12}^{3}s_{15}^{2}s_{23}^{2}s_{34}s_{45}^{2}+252\epsilon s_{12}^{4}s_{15}s_{23}^{2}s_{34}s_{45}^{2}+75s_{12}^{4}s_{15}s_{23}^{2}s_{34}s_{45}^{2}\\
&+416\epsilon s_{12}^{2}s_{15}^{4}s_{23}s_{34}s_{45}^{2}+104s_{12}^{2}s_{15}^{4}s_{23}s_{34}s_{45}^{2}-80\epsilon s_{12}^{3}s_{15}^{3}s_{23}s_{34}s_{45}^{2}\\
&-32s_{12}^{3}s_{15}^{3}s_{23}s_{34}s_{45}^{2}-100\epsilon s_{12}^{4}s_{15}^{2}s_{23}s_{34}s_{45}^{2}-37s_{12}^{4}s_{15}^{2}s_{23}s_{34}s_{45}^{2}\\
&+104\epsilon s_{12}^{5}s_{15}s_{23}s_{34}s_{45}^{2}+26s_{12}^{5}s_{15}s_{23}s_{34}s_{45}^{2}+64\epsilon s_{23}^{4}s_{34}^{5}s_{45}+16s_{23}^{4}s_{34}^{5}s_{45}\\
&+440\epsilon s_{12}s_{23}^{3}s_{34}^{5}s_{45}+98s_{12}s_{23}^{3}s_{34}^{5}s_{45}-64\epsilon s_{15}s_{23}^{3}s_{34}^{5}s_{45}-16s_{15}s_{23}^{3}s_{34}^{5}s_{45}\\
&+312\epsilon s_{12}^{2}s_{23}^{2}s_{34}^{5}s_{45}+66s_{12}^{2}s_{23}^{2}s_{34}^{5}s_{45}-344\epsilon s_{12}s_{15}s_{23}^{2}s_{34}^{5}s_{45}\\
&-74s_{12}s_{15}s_{23}^{2}s_{34}^{5}s_{45}-216\epsilon s_{12}^{2}s_{15}s_{23}s_{34}^{5}s_{45}-42s_{12}^{2}s_{15}s_{23}s_{34}^{5}s_{45}\\
&-408\epsilon s_{12}s_{23}^{4}s_{34}^{4}s_{45}-102s_{12}s_{23}^{4}s_{34}^{4}s_{45}-64\epsilon s_{15}s_{23}^{4}s_{34}^{4}s_{45}-16s_{15}s_{23}^{4}s_{34}^{4}s_{45}\\
&-1288\epsilon s_{12}^{2}s_{23}^{3}s_{34}^{4}s_{45}-298s_{12}^{2}s_{23}^{3}s_{34}^{4}s_{45}+64\epsilon s_{15}^{2}s_{23}^{3}s_{34}^{4}s_{45}\\
&+16s_{15}^{2}s_{23}^{3}s_{34}^{4}s_{45}-152\epsilon s_{12}s_{15}s_{23}^{3}s_{34}^{4}s_{45}-14s_{12}s_{15}s_{23}^{3}s_{34}^{4}s_{45}\\
&+120\epsilon s_{12}^{3}s_{15}^{2}s_{34}^{4}s_{45}+30s_{12}^{3}s_{15}^{2}s_{34}^{4}s_{45}-720\epsilon s_{12}^{3}s_{23}^{2}s_{34}^{4}s_{45}\\
&-156s_{12}^{3}s_{23}^{2}s_{34}^{4}s_{45}+464\epsilon s_{12}s_{15}^{2}s_{23}^{2}s_{34}^{4}s_{45}+92s_{12}s_{15}^{2}s_{23}^{2}s_{34}^{4}s_{45}\\
&+576\epsilon s_{12}^{2}s_{15}s_{23}^{2}s_{34}^{4}s_{45}+144s_{12}^{2}s_{15}s_{23}^{2}s_{34}^{4}s_{45}+424\epsilon s_{12}^{2}s_{15}^{2}s_{23}s_{34}^{4}s_{45}\\
&+82s_{12}^{2}s_{15}^{2}s_{23}s_{34}^{4}s_{45}+408\epsilon s_{12}^{3}s_{15}s_{23}s_{34}^{4}s_{45}+78s_{12}^{3}s_{15}s_{23}s_{34}^{4}s_{45}\\
&+744\epsilon s_{12}^{2}s_{23}^{4}s_{34}^{3}s_{45}+186s_{12}^{2}s_{23}^{4}s_{34}^{3}s_{45}+344\epsilon s_{12}s_{15}s_{23}^{4}s_{34}^{3}s_{45}
\end{aligned}
$$

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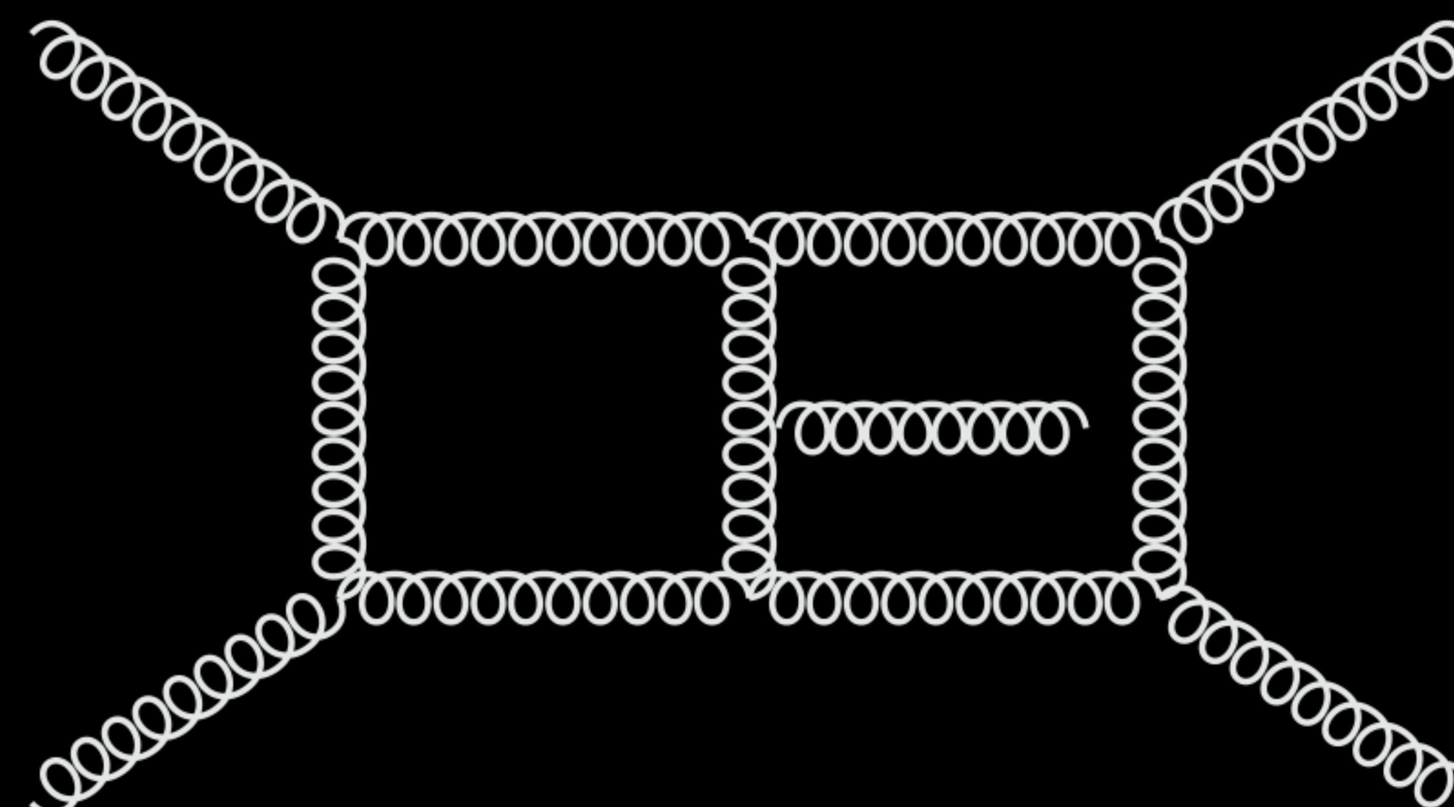
Look at one example, a 5-page long coefficient

$$
\begin{aligned}
&+86s_{12}s_{15}s_{23}^{4}s_{34}^{3}s_{45}-240\epsilon s_{12}^{3}s_{15}^{3}s_{34}^{3}s_{45}-60s_{12}^{3}s_{15}^{3}s_{34}^{3}s_{45}\\
&+1344\epsilon s_{12}^{3}s_{23}^{3}s_{34}^{3}s_{45}+324s_{12}^{3}s_{23}^{3}s_{34}^{3}s_{45}-224\epsilon s_{12}s_{15}^{2}s_{23}^{3}s_{34}^{3}s_{45}\\
&-68s_{12}s_{15}^{2}s_{23}^{3}s_{34}^{3}s_{45}+36\epsilon s_{12}^{2}s_{15}s_{23}^{3}s_{34}^{3}s_{45}-27s_{12}^{2}s_{15}s_{23}^{3}s_{34}^{3}s_{45}\\
&-120\epsilon s_{12}^{4}s_{15}^{2}s_{34}^{3}s_{45}-30s_{12}^{4}s_{15}^{2}s_{34}^{3}s_{45}+504\epsilon s_{12}^{4}s_{23}^{2}s_{34}^{3}s_{45}\\
&+114s_{12}^{4}s_{23}^{2}s_{34}^{3}s_{45}-120\epsilon s_{12}s_{15}^{3}s_{23}^{2}s_{34}^{3}s_{45}-18s_{12}s_{15}^{3}s_{23}^{2}s_{34}^{3}s_{45}\\
&-324\epsilon s_{12}^{2}s_{15}^{2}s_{23}^{2}s_{34}^{3}s_{45}-57s_{12}^{2}s_{15}^{2}s_{23}^{2}s_{34}^{3}s_{45}-788\epsilon s_{12}^{3}s_{15}s_{23}^{2}s_{34}^{3}s_{45}\\
&-221s_{12}^{3}s_{15}s_{23}^{2}s_{34}^{3}s_{45}-360\epsilon s_{12}^{2}s_{15}^{3}s_{23}s_{34}^{3}s_{45}-78s_{12}^{2}s_{15}^{3}s_{23}s_{34}^{3}s_{45}\\
&-124\epsilon s_{12}^{3}s_{15}^{2}s_{23}s_{34}^{3}s_{45}+5s_{12}^{3}s_{15}^{2}s_{23}s_{34}^{3}s_{45}-288\epsilon s_{12}^{4}s_{15}s_{23}s_{34}^{3}s_{45}\\
&-60s_{12}^{4}s_{15}s_{23}s_{34}^{3}s_{45}+120\epsilon s_{12}^{3}s_{15}^{4}s_{34}^{2}s_{45}+30s_{12}^{3}s_{15}^{4}s_{34}^{2}s_{45}\\
&-520\epsilon s_{12}^{3}s_{23}^{4}s_{34}^{2}s_{45}-130s_{12}^{3}s_{23}^{4}s_{34}^{2}s_{45}-340\epsilon s_{12}^{2}s_{15}s_{23}^{4}s_{34}^{2}s_{45}\\
&-85s_{12}^{2}s_{15}s_{23}^{4}s_{34}^{2}s_{45}+180\epsilon s_{12}^{4}s_{15}^{3}s_{34}^{2}s_{45}+45s_{12}^{4}s_{15}^{3}s_{34}^{2}s_{45}\\
&-584\epsilon s_{12}^{4}s_{23}^{3}s_{34}^{2}s_{45}-146s_{12}^{4}s_{23}^{3}s_{34}^{2}s_{45}+528\epsilon s_{12}^{2}s_{15}^{2}s_{23}^{3}s_{34}^{2}s_{45}\\
&+144s_{12}^{2}s_{15}^{2}s_{23}^{3}s_{34}^{2}s_{45}+424\epsilon s_{12}^{3}s_{15}s_{23}^{3}s_{34}^{2}s_{45}+118s_{12}^{3}s_{15}s_{23}^{3}s_{34}^{2}s_{45}\\
&-96\epsilon s_{12}^{5}s_{23}^{2}s_{34}^{2}s_{45}-24s_{12}^{5}s_{23}^{2}s_{34}^{2}s_{45}-340\epsilon s_{12}^{2}s_{15}^{3}s_{23}^{2}s_{34}^{2}s_{45}\\
&-97s_{12}^{2}s_{15}^{3}s_{23}^{2}s_{34}^{2}s_{45}+8\epsilon s_{12}^{3}s_{15}^{2}s_{23}^{2}s_{34}^{2}s_{45}+2s_{12}^{3}s_{15}^{2}s_{23}^{2}s_{34}^{2}s_{45}\\
&+732\epsilon s_{12}^{4}s_{15}s_{23}^{2}s_{34}^{2}s_{45}+195s_{12}^{4}s_{15}s_{23}^{2}s_{34}^{2}s_{45}+152\epsilon s_{12}^{2}s_{15}^{4}s_{23}s_{34}^{2}s_{45}\\
&+38s_{12}^{2}s_{15}^{4}s_{23}s_{34}^{2}s_{45}-32\epsilon s_{12}^{3}s_{15}^{3}s_{23}s_{34}^{2}s_{45}-20s_{12}^{3}s_{15}^{3}s_{23}s_{34}^{2}s_{45}\\
&-328\epsilon s_{12}^{4}s_{15}^{2}s_{23}s_{34}^{2}s_{45}-94s_{12}^{4}s_{15}^{2}s_{23}s_{34}^{2}s_{45}+96\epsilon s_{12}^{5}s_{15}s_{23}s_{34}^{2}s_{45}\\
&+24s_{12}^{5}s_{15}s_{23}s_{34}^{2}s_{45}-60\epsilon s_{12}^{4}s_{15}^{4}s_{34}s_{45}-15s_{12}^{4}s_{15}^{4}s_{34}s_{45}+120\epsilon s_{12}^{4}s_{23}^{4}s_{34}s_{45}\\
&+30s_{12}^{4}s_{23}^{4}s_{34}s_{45}+60\epsilon s_{12}^{3}s_{15}s_{23}^{4}s_{34}s_{45}+15s_{12}^{3}s_{15}s_{23}^{4}s_{34}s_{45}+88\epsilon s_{12}^{5}s_{23}^{3}s_{34}s_{45}\\
&+22s_{12}^{5}s_{23}^{3}s_{34}s_{45}-212\epsilon s_{12}^{3}s_{15}^{2}s_{23}^{3}s_{34}s_{45}-53s_{12}^{3}s_{15}^{2}s_{23}^{3}s_{34}s_{45}\\
&-244\epsilon s_{12}^{4}s_{15}s_{23}^{3}s_{34}s_{45}-61s_{12}^{4}s_{15}s_{23}^{3}s_{34}s_{45}+244\epsilon s_{12}^{3}s_{15}^{3}s_{23}^{2}s_{34}s_{45}\\
&+61s_{12}^{3}s_{15}^{3}s_{23}^{2}s_{34}s_{45}+68\epsilon s_{12}^{4}s_{15}^{2}s_{23}^{2}s_{34}s_{45}+17s_{12}^{4}s_{15}^{2}s_{23}^{2}s_{34}s_{45}\\
&-176\epsilon s_{12}^{5}s_{15}s_{23}^{2}s_{34}s_{45}-44s_{12}^{5}s_{15}s_{23}^{2}s_{34}s_{45}-92\epsilon s_{12}^{3}s_{15}^{4}s_{23}s_{34}s_{45}\\
&-23s_{12}^{3}s_{15}^{4}s_{23}s_{34}s_{45}+116\epsilon s_{12}^{4}s_{15}^{3}s_{23}s_{34}s_{45}+29s_{12}^{4}s_{15}^{3}s_{23}s_{34}s_{45}\\
&+88\epsilon s_{12}^{5}s_{15}^{2}s_{23}s_{34}s_{45}+22s_{12}^{5}s_{15}^{2}s_{23}s_{34}s_{45}-96\epsilon s_{12}s_{23}^{4}s_{34}^{5}-24s_{12}s_{23}^{4}s_{34}^{5}\\
&-96\epsilon s_{12}^{2}s_{23}^{3}s_{34}^{5}-24s_{12}^{2}s_{23}^{3}s_{34}^{5}+96\epsilon s_{12}s_{15}s_{23}^{3}s_{34}^{5}+24s_{12}s_{15}s_{23}^{3}s_{34}^{5}\\
&+96\epsilon s_{12}^{2}s_{15}s_{23}^{2}s_{34}^{5}+24s_{12}^{2}s_{15}s_{23}^{2}s_{34}^{5}+288\epsilon s_{12}^{2}s_{23}^{4}s_{34}^{4}+72s_{12}^{2}s_{23}^{4}s_{34}^{4}\\
&+96\epsilon s_{12}s_{15}s_{23}^{4}s_{34}^{4}+24s_{12}s_{15}s_{23}^{4}s_{34}^{4}+288\epsilon s_{12}^{3}s_{23}^{3}s_{34}^{4}+72s_{12}^{3}s_{23}^{3}s_{34}^{4}\\
&-96\epsilon s_{12}s_{15}^{2}s_{23}^{3}s_{34}^{4}-24s_{12}s_{15}^{2}s_{23}^{3}s_{34}^{4}-288\epsilon s_{12}^{2}s_{15}s_{23}^{3}s_{34}^{4}-72s_{12}^{2}s_{15}s_{23}^{3}s_{34}^{4}\\
&-384\epsilon s_{12}^{3}s_{15}s_{23}^{2}s_{34}^{4}-96s_{12}^{3}s_{15}s_{23}^{2}s_{34}^{4}+96\epsilon s_{12}^{3}s_{15}^{2}s_{23}s_{34}^{4}+24s_{12}^{3}s_{15}^{2}s_{23}s_{34}^{4}\\
&-288\epsilon s_{12}^{3}s_{23}^{4}s_{34}^{3}-72s_{12}^{3}s_{23}^{4}s_{34}^{3}-192\epsilon s_{12}^{2}s_{15}s_{23}^{4}s_{34}^{3}-48s_{12}^{2}s_{15}s_{23}^{4}s_{34}^{3}\\
&-288\epsilon s_{12}^{4}s_{23}^{3}s_{34}^{3}-72s_{12}^{4}s_{23}^{3}s_{34}^{3}+288\epsilon s_{12}^{2}s_{15}^{2}s_{23}^{3}s_{34}^{3}+72s_{12}^{2}s_{15}^{2}s_{23}^{3}s_{34}^{3}\\
&+288\epsilon s_{12}^{3}s_{15}s_{23}^{3}s_{34}^{3}+72s_{12}^{3}s_{15}s_{23}^{3}s_{34}^{3}-96\epsilon s_{12}^{2}s_{15}^{3}s_{23}^{2}s_{34}^{3}\\
&-24s_{12}^{2}s_{15}^{3}s_{23}^{2}s_{34}^{3}+96\epsilon s_{12}^{3}s_{15}^{2}s_{23}^{2}s_{34}^{3}+24s_{12}^{3}s_{15}^{2}s_{23}^{2}s_{34}^{3}\\
&+480\epsilon s_{12}^{4}s_{15}s_{23}^{2}s_{34}^{3}+120s_{12}^{4}s_{15}s_{23}^{2}s_{34}^{3}-96\epsilon s_{12}^{3}s_{15}^{3}s_{23}s_{34}^{3}\\
&-24s_{12}^{3}s_{15}^{3}s_{23}s_{34}^{3}-192\epsilon s_{12}^{4}s_{15}^{2}s_{23}s_{34}^{3}-48s_{12}^{4}s_{15}^{2}s_{23}s_{34}^{3}+96\epsilon s_{12}^{4}s_{23}^{4}s_{34}^{2}\\
&+24s_{12}^{4}s_{23}^{4}s_{34}^{2}+96\epsilon s_{12}^{3}s_{15}s_{23}^{4}s_{34}^{2}+24s_{12}^{3}s_{15}s_{23}^{4}s_{34}^{2}+96\epsilon s_{12}^{5}s_{23}^{3}s_{34}^{2}\\
&+24s_{12}^{5}s_{23}^{3}s_{34}^{2}-192\epsilon s_{12}^{3}s_{15}^{2}s_{23}^{3}s_{34}^{2}-48s_{12}^{3}s_{15}^{2}s_{23}^{3}s_{34}^{2}-96\epsilon s_{12}^{4}s_{15}s_{23}^{3}s_{34}^{2}\\
&-24s_{12}^{4}s_{15}s_{23}^{3}s_{34}^{2}+96\epsilon s_{12}^{3}s_{15}^{3}s_{23}^{2}s_{34}^{2}+24s_{12}^{3}s_{15}^{3}s_{23}^{2}s_{34}^{2}\\
&-96\epsilon s_{12}^{4}s_{15}^{2}s_{23}^{2}s_{34}^{2}-24s_{12}^{4}s_{15}^{2}s_{23}^{2}s_{34}^{2}-192\epsilon s_{12}^{5}s_{15}s_{23}^{2}s_{34}^{2}\\
&-48s_{12}^{5}s_{15}s_{23}^{2}s_{34}^{2}+96\epsilon s_{12}^{4}s_{15}^{3}s_{23}s_{34}^{2}+24s_{12}^{4}s_{15}^{3}s_{23}s_{34}^{2}+96\epsilon s_{12}^{5}s_{15}^{2}s_{23}s_{34}^{2}\\
&+24s_{12}^{5}s_{15}^{2}s_{23}s_{34}^{2})/(8(4\epsilon+1)s_{12}s_{23}s_{45}^{2}(s_{12}-s_{45})(s_{34}+s_{45})(s_{12}+s_{15}-s_{34})(s_{12}+s_{23}-s_{45})\\
&(s_{12}-s_{34}-s_{45})(-s_{15}+s_{23}+s_{34})(-s_{15}+s_{23}-s_{45}))
\end{aligned}
$$

Using our algorithm, this coefficient is simplified to

$$
\begin{aligned}
= \; & \frac{3s_{23}s_{34}}{2(4\epsilon+1)s_{12}s_{45}(-s_{15}+s_{23}+s_{34})} - \frac{3s_{34}}{2(4\epsilon+1)s_{12}s_{45}} + \frac{15{s_{15}}^2-15s_{15}s_{34}}{8s_{23}s_{45}(-s_{12}-s_{15}+s_{34})} \\
& + \frac{s_{23}{s_{34}}^2}{{s_{45}}^2(s_{45}-s_{12})(-s_{15}+s_{23}+s_{34})} - \frac{2s_{23}{s_{34}}^2}{s_{12}{s_{45}}^2(-s_{15}+s_{23}+s_{34})} + \frac{s_{23}s_{34}+{s_{34}}^2}{s_{45}(s_{45}-s_{12})(-s_{15}+s_{23}+s_{34})} \\
& - \frac{11s_{23}s_{34}}{2s_{12}s_{45}(-s_{15}+s_{23}+s_{34})} - \frac{15s_{15}s_{34}}{8s_{23}s_{45}(-s_{12}+s_{34}+s_{45})} + \frac{s_{15}-s_{23}-s_{34}}{s_{45}(-s_{12}-s_{23}+s_{45})} + \frac{2s_{15}-2s_{34}}{s_{12}s_{23}} \\
& + \frac{15s_{15}-15s_{34}}{8s_{23}(-s_{12}-s_{15}+s_{34})} - \frac{7s_{23}}{2s_{12}(-s_{15}+s_{23}+s_{34})} - \frac{15s_{23}}{4(-s_{12}-s_{15}+s_{34})(-s_{15}+s_{23}+s_{34})} \\
& - \frac{s_{15}}{2s_{12}(-s_{12}-s_{23}+s_{45})} + \frac{s_{23}-s_{45}}{2s_{12}(s_{15}-s_{23}+s_{45})} + \frac{15s_{15}}{8s_{45}(-s_{12}-s_{15}+s_{34})} \\
& + \frac{15}{4(-s_{12}-s_{15}+s_{34})} + \frac{7s_{34}}{4s_{23}(-s_{12}-s_{23}+s_{45})} - \frac{5s_{34}}{4(s_{45}-s_{12})(-s_{12}-s_{23}+s_{45})} \\
& - \frac{15s_{34}}{8s_{23}(-s_{12}+s_{34}+s_{45})} + \frac{1}{2(-s_{12}-s_{23}+s_{45})} + \frac{4s_{34}}{s_{12}s_{45}} - \frac{11s_{34}}{4s_{45}(s_{45}-s_{12})} \\
& - \frac{15}{8(-s_{12}+s_{34}+s_{45})} + \frac{5}{4(s_{45}-s_{12})} + \frac{4}{s_{12}} - \frac{3{s_{34}}^2}{{s_{45}}^2(-s_{15}+s_{23}+s_{34})} \\
& + \frac{s_{34}}{4s_{45}(-s_{15}+s_{23}+s_{34})} + \frac{s_{45}}{2(s_{34}+s_{45})(s_{15}-s_{23}+s_{45})} + \frac{3s_{34}}{{s_{45}}^2} - \frac{1}{2(s_{34}+s_{45})} - \frac{1}{4s_{45}}
\end{aligned}
$$

It is ~16 times compression !



IBP coefficients size

$$\sim 20 \text{ GB} \quad \Longrightarrow 190 \text{ MB}$$

two orders of magnitude simplification

Boehm, Wittmann, Xu, Wu and **YZ** 2020
Bendle, Boehm, Heymann, Ma, Rahn, Wittman, Ristau, Wu, YZ 2021

To simplify IBP reduction coefficients

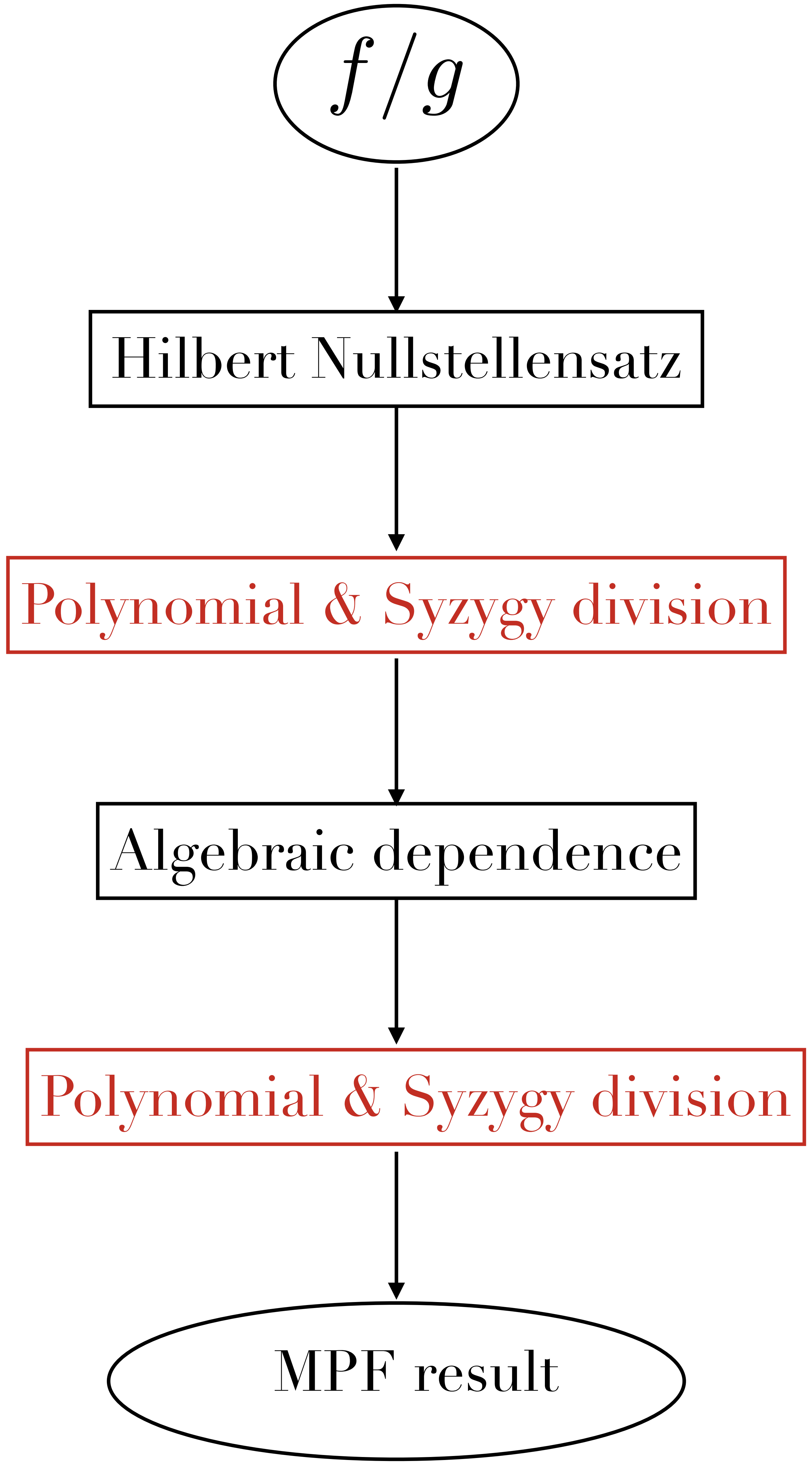
Boehm, Wittmann, Xu, Wu and **YZ** 2020

observation:
the coefficients do not have
unphysical poles

- Use a UT basis for IBP reduction
- use a CAG based method to decompose the reduction coefficients

multivariate partial fraction
from computational
algebraic geometry

Multivariate partial fraction from Algebraic Geometry



Improved Leinartas algorithm

black — Orignial Leinartas

Red — Our improved Leinartas' algorithm

our package:
(now it is a standard library of the software Singular)

can simplify not only IBP reduction coefficients but also general rational functions in scattering amplitude computations.

https://raw.githubusercontent.com/Singular/Singular/spielwiese/Singular/LIB/pfd.lib

A key point

Polynomial division

$$\frac{f}{g} = \frac{f}{q_1^{e_1}\cdots q_m^{e_m}}$$

$q_1, \cdots, q_m$ have common zero point & algebraic independent

$$f = \sum_{i=1}^{m} b_i q_i + r \quad \Rightarrow \quad \frac{f}{q_1^{e_1}\cdots q_m^{e_m}} = \frac{r}{q_1^{e_1}\cdots q_m^{e_m}} + \sum_{i=q}^{m} \frac{b_i}{q_i^{e_i-1}\prod_{j=1,j\neq i}^{m} q_j^{e_j}}$$

unique, low degree

non-unique, degree out of control

Syzygy division to significantly simplify the quotients

$$\sum_{i=1}^{m} a_i^{(\alpha)} q_i = 0$$

syzygy generators

$$(b_1, \ldots, b_m) = \sum_{\alpha} \cancel{P_\alpha \times (a_1^{(\alpha)}, \ldots, a_m^{(\alpha)})} + (B_1, \ldots B_m)$$

remove the syzygy part and thus simplify the quotients

Summary

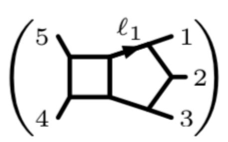
- Significant progress of analytic Feynman evaluation
- Canonical differential equation
- Computational algebraic geometry applications:

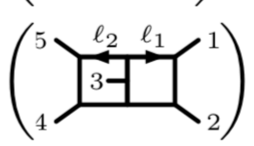
  Lift method to find UT integrals
  Syzygy method to simplify IBP reduction coefficients

Vielen Dank!

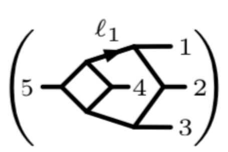
2-loop 5-point +++++ pure-YM amplitude

Badger, Frellesvig, **YZ**, 2013
Badger, Mogull, Ochirov, O'Connell 2015

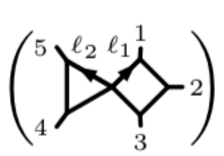
$\Delta_{431} = \Delta\Big($  $\Big) = -\frac{s_{12}s_{23}s_{45}F_1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle \mathrm{tr}_5}\left(\mathrm{tr}_+(1345)(\ell_1+p_5)^2 + s_{15}s_{34}s_{45}\right),$

$\Delta_{332} = \Delta\Big($  $\Big) = \frac{s_{12}s_{45}F_1}{4\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle \mathrm{tr}_5}$

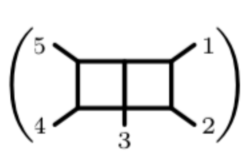
$$\times\Big(s_{23}\mathrm{tr}_+(1345)\big(2s_{12} - 4\,\ell_1\cdot(p_5-p_4) + 2(\ell_1-\ell_2)\cdot p_3\big) - s_{34}\mathrm{tr}_+(1235)\big(2s_{45} - 4\,\ell_2\cdot(p_1-p_2) - 2(\ell_1-\ell_2)\cdot p_3\big) - 4s_{23}s_{34}s_{15}(\ell_1-\ell_2)\cdot p_3\Big),$$

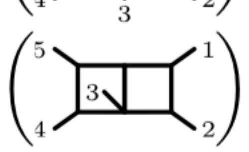
$\Delta_{422} = \Delta\Big($  $\Big) = -\frac{s_{12}s_{23}s_{45}F_1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle \mathrm{tr}_5}$

$$\times\left(\mathrm{tr}_+(1345)\left(\ell_1\cdot(p_5-p_4) - \frac{s_{45}}{2}\right) + s_{15}s_{34}s_{45}\right).$$

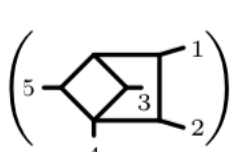
$\Delta_{430} = \Delta\Big($  $\Big) = -\frac{s_{12}\mathrm{tr}_+(1345)}{2\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle s_{13}}(2(\ell_1\cdot\omega_{123}) + s_{23})$

$$\times\left(F_2 + F_3\frac{(\ell_1+\ell_2)^2 + s_{45}}{s_{45}}\right),$$

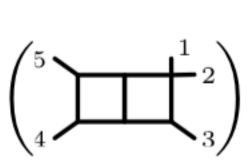
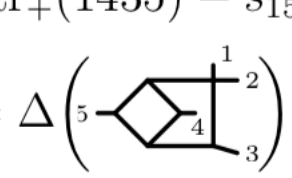
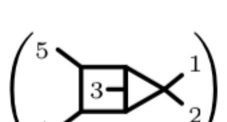
$\Delta_{331;5L_1} = \Delta\Big($  $\Big) = \frac{s_{12}s_{23}s_{34}s_{45}s_{51}F_1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle \mathrm{tr}_5},$

$\Delta_{331;5L_2} = \Delta\Big($  $\Big) = -\frac{s_{12}s_{45}F_1}{4\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle \mathrm{tr}_5}$

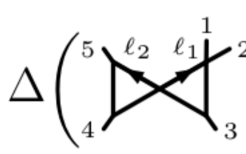
$$\times\big(2s_{23}s_{34}s_{15} - s_{23}\mathrm{tr}_+(1345) + s_{34}\mathrm{tr}_+(1235)\big),$$

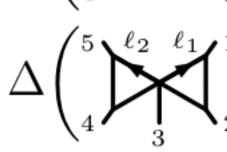
$\Delta_{322;5L_1} = \Delta\Big($  $\Big) = -\frac{s_{12}F_1}{2\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle \mathrm{tr}_5}$

$$\times\big(s_{23}s_{45}\mathrm{tr}_+(1435) - s_{15}s_{34}\mathrm{tr}_+(2453)\big),$$

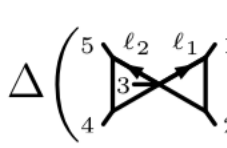
$\Delta_{331;M_1} = \Delta\Big($  $\Big) = \Delta_{322;M_1} = \Delta\Big($  $\Big) = \Delta_{232;M_1} = \Delta\Big($  $\Big)$

$$= -\frac{s_{34}s_{45}^2\mathrm{tr}_+(1235)F_1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle \mathrm{tr}_5},$$

$\Delta_{330;M_1} = \Delta\Big($  $\Big) = -\frac{(s_{45}-s_{12})\mathrm{tr}_+(1345)}{2\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle s_{13}}\left(F_2 + F_3\frac{(\ell_1+\ell_2)^2 + s_{45}}{s_{45}}\right),$

$\Delta_{330;5L_1} = \Delta\Big($  $\Big) = -\frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$

$$\times\Bigg\{\frac{1}{2}\left(\mathrm{tr}_+(1245) - \frac{\mathrm{tr}_+(1345)\mathrm{tr}_+(1235)}{s_{13}s_{35}}\right)\times\left(F_2 + F_3\frac{4(\ell_1\cdot p_3)(\ell_2\cdot p_3) + (\ell_1+\ell_2)^2(s_{12}+s_{45}) + s_{12}s_{45}}{s_{12}s_{45}}\right)$$
$$+ F_3\Bigg[(\ell_1+\ell_2)^2 s_{15} + \mathrm{tr}_+(1235)\left(\frac{(\ell_1+\ell_2)^2}{2s_{35}} - \frac{\ell_1\cdot p_3}{s_{12}}\left(1 + \frac{2(\ell_2\cdot\omega_{543})}{s_{35}} + \frac{s_{12}-s_{45}}{s_{35}s_{45}}(\ell_2-p_5)^2\right)\right)$$
$$+ \mathrm{tr}_+(1345)\left(\frac{(\ell_1+\ell_2)^2}{2s_{13}} - \frac{\ell_2\cdot p_3}{s_{45}}\left(1 + \frac{2(\ell_1\cdot\omega_{123})}{s_{13}} + \frac{s_{45}-s_{12}}{s_{12}s_{13}}(\ell_1-p_1)^2\right)\right)\Bigg]\Bigg\},$$

$\Delta_{330;5L_2} = \Delta\Big($  $\Big) = \frac{F_3}{2\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle s_{12}}$

$$\times\Bigg((s_{45}-s_{12})\mathrm{tr}_+(1245) - \left(\mathrm{tr}_+(1245) - \frac{\mathrm{tr}_+(1345)\mathrm{tr}_+(1235)}{s_{13}s_{35}}\right)2(\ell_1\cdot p_3)$$
$$- \frac{s_{45}\mathrm{tr}_+(1235)}{s_{35}}\left(2(\ell_2\cdot\omega_{543}) + \frac{s_{12}-s_{45}}{s_{45}}(\ell_2-p_5)^2\right) + \frac{s_{12}\mathrm{tr}_+(1345)}{s_{13}}\left(2(\ell_1\cdot\omega_{123}) + \frac{s_{45}-s_{12}}{s_{12}}(\ell_1-p_1)^2\right)\Bigg).$$

2-loop 5-point +++++ pure-YM amplitude

numerator degree-5 IBP needed (impossible by current analytic IBP method)
indirect finite-field fitting for the amplitude (after IBP) is applicable

All weight-3, weight-4 part of the amplitude cancels out

$$\mathcal{H}^{(2)} = \sum_{S_5/S_{T_1}} T_1\, \mathcal{H}_1^{(2)} + \sum_{S_5/S_{T_{13}}} T_{13}\, \mathcal{H}_{13}^{(2)}$$

$$I_{123;45} = \mathrm{Li}_2\left(1 - s_{12}/s_{45}\right) + \mathrm{Li}_2\left(1 - s_{23}/s_{45}\right) + \log^2\left(s_{12}/s_{23}\right) + \pi^2/6\,.$$

$$\mathcal{H}_1^{(2,0)} = \sum_{S_{T_1}} \left\{ -\kappa \frac{[45]^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} I_{123;45} + \kappa^2 \frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} \left[5\, s_{12}s_{23} + s_{12}s_{34} + \frac{\mathrm{tr}_+^2(1245)}{s_{12}s_{45}} \right] \right\},$$

$$\mathcal{H}_{13}^{(2,1)} = \sum_{S_{T_{13}}} \left\{ \kappa \frac{[15]^2}{\langle 23\rangle\langle 34\rangle\langle 42\rangle} \left[I_{234;15} + I_{243;15} - I_{324;15} - 4\, I_{345;12} - 4\, I_{354;12} - 4\, I_{435;12} \right] \right.$$
$$\left. - 6\,\kappa^2 \left[\frac{s_{23}\, \mathrm{tr}_-(1345)}{s_{34}\,\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} - \frac{3}{2} \frac{[12]^2}{\langle 34\rangle\langle 45\rangle\langle 53\rangle} \right] \right\},$$

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Infrared structure

(Catani's dipole formula 98)

$$A(s_{ij},\epsilon) = \mathbf{Z}(s_{ij},\epsilon) A^f(s_{ij},\epsilon) \qquad \mathbf{Z}(s_{ij},\epsilon) = \exp g^2 \left(\frac{\mathbf{D}_0}{2\epsilon^2} - \frac{\mathbf{D}}{2\epsilon} \right)$$

$$\mathbf{D}_0 = \sum_{i \neq j} \vec{\mathbf{T}}_i \cdot \vec{\mathbf{T}}_j \,, \quad \mathbf{D} = \sum_{i \neq j} \vec{\mathbf{T}}_i \cdot \vec{\mathbf{T}}_j \, \log\left(-\frac{s_{ij}}{\mu^2} \right) ,$$

$\mathbf{T}_i$ is the adjoint action of $su(N_c)$ Lie algebra.

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