Hypothesis Testing

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1 Motivation

As a successful theory, Standard Model has the capacity of describing various physical phenomena. But some experimental results indicate some new physics beyond the SM. And statistical method has been come up to find out whether some BSM are correct or not.

2 General Idea of the Statistic Method

Actually it's a hypothesis testing question. Suppose the null hypothesis to be "SM is right" (which is also called Background-Only), and denote it by H_0, while H_1 is "There are some new phenomena in this process"(which is called Background + Signal). And if new physics really exists, then H_0 should be rejected and accept H_1.

To check H_0, a proper statistical test (called λ) should be constructed and then obtain the observation value of λ(denoted by λ_{obs}) from experiment. Suppose H_0 is true, then λ should obey some known distributions. In statistical world, the events with small probabilities tend no to happen. So if the of observation value satisfies: Possibility($\lambda > \lambda_{obs}$) is rather small[1], it's reasonably to regard H_0 untrue and reject it.

From the analysis above, the role of Possibility($\lambda > \lambda_{obs}$) is of great importance, thus give it a name , called p-value. And one can also define another quantity corresponding to p-value: Significance Z: $p = 1 - \Phi(Z)$[2], which can describe the certainty of rejecting null hypothesis.

[1] The definition of "small" depends on the specific cases. And in general convention, possibility $< \alpha$ =0.05 can be called small.

[2] Φ is the standard Guaasian distribution function.

3 Example of Counting Numbers

Let consider such a case as an example: detector picked up n particles. The number of particles follows Possion distribution:

$$P_{n|B} = \frac{b^n}{n!} e^{-b} \tag{1}$$

$$P_{n|BS} = \frac{(b+s)^n}{n!} e^{-(b+s)} \tag{2}$$

the b and s here are expected numbers of particles under two hypothesis. Then what information can one gain?

First, construct the hypothesis:

$$\mathrm{H_0 : Background - Only}$$

$$\mathrm{H_1 : Background + Signal}$$

And then, select a statistic test λ. The λ we'll use here is defined by[3]:

$$\lambda = \frac{L_B}{L_{BS}} = \Pi_i \Big(\frac{b_i}{b_i + s_i} \Big)^n e^s \tag{3}$$

where L represents likelihood function.

Wilks theorem states that when $s + b \to$large, $Q = -2 \ln \lambda \sim \chi^2$. After some calculation, the distribution has the form:

$$F(Q) = \Phi(\sqrt{Q}) \tag{4}$$

According to the definition of p-value: $p = Possibility(Q > Q_{obs}) = 1 - \Phi(\sqrt{Q_{obs}})$, and in association with $p = 1 - \Phi(Z)$, thus,

$$Z \simeq \sqrt{Q_{obs}} = \sqrt{2 \Big(n \ln(1 + \frac{s}{b}) - s \Big)} \qquad b + s \to \text{large} \tag{5}$$

If the hypothesis H_1 tends to be accepted, then $\langle n \rangle$ should be b+s. Then

$$Z \simeq \sqrt{Q_{obs}} = \sqrt{2 \Big((b+s) \ln(1 + \frac{s}{b}) - s \Big)} \tag{6}$$

In particular, when $s \ll b$,

$$Z \simeq \sqrt{Q_{obs}} = \sqrt{2b} \sqrt{\Big((1+x) \ln(1+x) - x \Big)} \qquad x = s/b \tag{7}$$

$$= \sqrt{2b} \sqrt{(1+x)(x - \frac{1}{2} x^2 + o) - x} \tag{8}$$

$$\simeq \sqrt{2b} \frac{x}{\sqrt{2}} = \frac{s}{\sqrt{b}} \tag{9}$$

[3]the reason why use this statistic test actually is based on some mathematical theories which suggest this test containing some useful properties.

4 Further Story

In the slides of Wanyun Su's, the story of hypothesis testing doesn't end at Significance. So I am going to follow the idea to see the whole statistics story in the next time.

5 Appendix

5.1 Poisson Distribution

Suppose the probability of detecting each particle to be p, then the probability of detecting n particles should follow bernoulli distribution:

$$P_n = \mathbf{C}_N^n p^n (1-p)^{N-n} \qquad \text{N} \to \text{large} \tag{10}$$

$$= \frac{\lambda^n}{n!} e^{-\lambda} \tag{11}$$

where $\lambda = pN$=expected value of numbers.